A GENERALIZATION OF THE CLASSICAL LAGUERRE POLYNOMIALS: ASYMPTOTIC PROPERTIES AND ZEROS. [1]

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Key words and phrases: Laguerre polynomials, discrete measures, zeros.
AMS (MOS) subject classification:**33C25.**

Abstract

We consider a modification of the gamma distribution by adding a discrete measure supported in the point $x = 0$. We study some properties of the polynomials orthogonal with respect to such measures [1]. In particular, we deduce the second order differential equation and the three term recurrence relation which such polynomials satisfy as well as, for large n, the behaviour of their zeros.

§1 Introduction.

In 1940, H. L. Krall [14] obtained three new classes of polynomials orthogonal with respect to measures which are not absolutely continuous with respect to the Lebesgue measure. In fact, his study is related to an extension of the very well known characterization of classical orthogonal polynomials by S. Bochner. This kind of measures was not considered in [22].

In the above mentioned paper by H. L. Krall, the resulting polynomials satisfy a fourth order differential equation and the corresponding measures are, respectively,

1. Laguerre-type case: $d\mu = e^{-x}\,dx + M\delta(x) \qquad M > 0, \qquad supp\,\mu = \mathbb{R}^+$,

2. Legendre-type case: $d\mu = \dfrac{\alpha}{2}\,dx + \dfrac{\delta(x-1)}{2} + \dfrac{\delta(x+1)}{2} \quad \alpha > 0, \quad supp\,\mu = [-1, 1]$,

3. Jacobi-type case: $d\mu = (1-x)^{\alpha}\,dx + M\delta(x) \quad \alpha > -1, \quad M > 0, \quad supp\,\mu = [0, 1]$.

A new approach to this subject was presented in [13].

The analysis of properties of polynomials orthogonal with respect to a perturbation of a measure via the addition of mass points was introduced by P.Nevai [18] when asymptotic properties of the new polynomials have been considered. In particular, he proved the dependence

[1] October 29, 1996.

of such properties in terms of the location of the mass points with respect to the support of the measure. Particular emphasis was given to measures supported in $[-1,1]$ and satisfying some extra conditions in terms of the parameters of the three term recurrence relation which the corresponding sequence of orthogonal polynomials satisfy.

The analysis of algebraic properties for such polynomials attracted the interest of several researchers (see [6] for positive Borel measures and [16] for a more general situation). From the point of view of differential equations see [17].

When two mass points are considered, the difficulties increase as shows [8]. An interesting application for the addition of two mass points at ± 1 to the Jacobi weight function was analyzed in [12].

The study of the modification of a measure via the derivatives of a delta Dirac measure is intimately related with approximation theory (see [9] for the bounded case and [15] for the unbounded one). In fact, the denominators $Q_n(x)$ of the main diagonal sequence for Padé approximants of Stieltjes type meromorphic functions

$$\int \frac{d\mu(x)}{z-x} + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{i,j}\frac{i!}{(z-c_j)^{i+1}} \qquad A_{N_j,j} \neq 0,$$

satisfy orthogonal relations

$$\int p(x)Q_n(x)d\mu(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{i,j}(p(z)Q_n(z))^{(i)}_{z=c_j} = 0, \tag{1}$$

where $p(x)$ is a polynomial of degree at most $n-1$. His study has known an increasing interest during the last five years because their connection with spectral methods for boundary value problems of fourth order differential equations [4].

The first approach to such a kind of modifications of a moment linear functional is [3]. In particular, necessary and sufficient conditions for the existence of a sequence of polynomials orthogonal with respect to such a linear functional are obtained. Furthermore, an extensive study for the new orthogonal polynomials was performed when the initial functional is semi-classical.

More recently (see [1]), the authors have analyzed a generalization of the classical Laguerre polynomials by addition of the derivative of the Dirac measure at $x = 0$ to the Laguerre measure. In particular, the hypergeometric character of these polynomials was proved. In [1], we consider a particular case of (1) ($m = 1$ and $N_1 = 1$) and we analyze the corresponding polynomials when μ is the Laguerre measure. In other direction Koekoek and Meijer [10] (see also [11]) have considered some special inner products of Sobolev-type as

$$< p, q >= \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + \sum_{k=0}^{N} M_k p^{(k)}(0)q^{(k)}(0). \tag{2}$$

Our case is very different with respect to this one. In fact, if $\{M_k\}_{k=0}^N$ are non-negative, then the above bilinear form is positive-definite. If we consider the bilinear form associated with the functional $\mathcal{U}$ (see formula (4) from below),

$$(p, q) = \int_0^\infty p(x)q(x)x^\alpha e^{-x} dx + \sum_{k=0}^N M_k (p(z)q(z))_{z=0}^{(k)}, \tag{3}$$

this bilinear form is not positive definite and, even, in general is not quasi-definite. This means that the monic orthogonal polynomial sequence does not exists for all values of M_k. Furthermore, the present paper constitutes a second step in the study started in [1].

In section 2 we deduce the relative asymptotics of our polynomials with respect to the Laguerre polynomials as well as the ratio of the corresponding norms.

In section 3 our main aim is concentrated in the location of the zeros of these new orthogonal polynomials. We deduce that, for n large enough, they are real and simple, $n-1$ of them are positive and the other one is negative and is attracted by the end of the support with order $O(n^{-\frac{\alpha}{2}-2})$. The technical methods used for the analysis of the zeros are the ones presented in [10]-[11]. But, the kind of orthogonal polynomials in such two situations are very different. In our case, we deal with orthogonality with respect to a linear functional. This functional leads to an inner product (3) which is, in general, not positive definite, and zeros appear as eigenvalues of Jacobi matrices. In [10]-[11], there is not a linear functional which induces the considered positive definite inner product (2). Moreover, their zeros are not eigenvalues of a Jacobi Matrices because of, according to the properties of the *shift* operator in our case is symmetric with respect to the induced inner product (3) but in [10]-[11] is not symmetric with respect to the inner product (2).

In section 4, we determine explicitly a second order differential equation for such polynomials, which is a consequence, from a theoretical point of view, of the fact that the linear functional $\mathcal{U}$ is semiclassical [3]. As a consequence, using the techniques developed in [5] and [2],[25], we deduce in section 5 some moments of the distribution of zeros, as well as it semiclassical WKB density. Finally, in section 6 we obtain the parameters of the three term recurrence relation as well as an asymptotic estimate of them.

§2 Some asymptotic formulas.

In [1] we began the study of polynomials, orthogonal with respect to the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{U}, P > \; = \int_0^\infty P(x) \; x^\alpha e^{-x} dx + M_0 P(0) + M_1 P'(0), \quad M_0 \geq 0, M_1 \geq 0, \alpha > -1. \tag{4}$$

For large n we deduced that the monic polynomials $L_n^{\alpha,M_0,M_1}(x)$, orthogonal with respect to the functional (4), exist for all the values of the masses M_0 and M_1. Furthermore we get the following expression for these generalized Laguerre polynomials

$$L_n^{\alpha,M_0,M_1}(x) = L_n^\alpha(x) + A_1 (L_n^\alpha)'(x) + A_2 (L_n^\alpha)''(x), \tag{5}$$

where

$$A_1 = \frac{(-1)^n[M_0 L_n^{\alpha,M_0,M_1}(0) + M_1 (L_n^{\alpha,M_0,M_1})'(0)]}{\Gamma(\alpha+1)n!} - \frac{(-1)^n M_1 (n-1) L_n^{\alpha,M_0,M_1}(0)}{\Gamma(\alpha+2)n!}, \qquad (6)$$

$$A_2 = -\frac{(-1)^n M_1 L_n^{\alpha,M_0,M_1}(0)}{\Gamma(\alpha+2)n!}. \qquad (7)$$

and $L_n^{\alpha,M_0,M_1}(0)$ and $(L_n^{\alpha,M_0,M_1})'(0)$ are given by

$$L_n^{\alpha,M_0,M_1}(0) = \frac{\begin{vmatrix} (-1)^n n!\binom{n+\alpha}{n} & \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} \\ -(-1)^n n!\binom{n+\alpha}{n-1} & 1 - \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{M_0}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} - \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} & \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} \\ -\frac{M_0}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} + \frac{M_1}{\Gamma(\alpha+2)}\frac{n(\alpha+2)-\alpha-1}{(n-2)}\binom{n+\alpha}{n-3} & 1 - \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} \end{vmatrix}}, \qquad (8)$$

and

$$(L_n^{\alpha,M_0,M_1})'(0) =$$

$$= \frac{\begin{vmatrix} 1 + \frac{M_0}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} - \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} & (-1)^n n!\binom{n+\alpha}{n} \\ -\frac{M_0}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} + \frac{M_1}{\Gamma(\alpha+2)}\frac{n(\alpha+2)-\alpha-1}{(n-2)}\binom{n+\alpha}{n-3} & -(-1)^n n!\binom{n+\alpha}{n-1} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{M_0}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} - \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} & \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} \\ -\frac{M_0}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} + \frac{M_1}{\Gamma(\alpha+2)}\frac{n(\alpha+2)-\alpha-1}{(n-2)}\binom{n+\alpha}{n-3} & 1 - \frac{M_1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-2} \end{vmatrix}}. \qquad (9)$$

Denoting by $d_{\alpha,n}^{M_0,M_1}$ the denominator in the formulas (8) and (9)

$$d_{\alpha,n}^{M_0,M_1} = 1 + \frac{M_0\,(\alpha+1)_n}{(n-1)!\,\Gamma(\alpha+2)} - \frac{2\,M_1\,(\alpha+1)_n}{(n-2)!\,\Gamma(\alpha+3)} - $$

$$- \frac{{M_1}^2\,(n+\alpha+1)\,(\alpha+1)_n{}^2}{(\alpha+1)\,(\alpha+3)\,(n-2)!\,(n-1)!\,\Gamma(\alpha+3)^2},$$

we obtain

$$L_n^{\alpha,M_0,M_1}(0) = \frac{(-1)^n\,(\alpha+1)_n}{d_{\alpha,n}^{M_0,M_1}} \left(1 + \frac{M_1\,n^2\,(\alpha+1)_n}{(\alpha+1)\,n!\,\Gamma(\alpha+2)} - \frac{M_1\,(\alpha+1)_n}{(n-2)!\,\Gamma(\alpha+3)}\right),$$

and

$$(L_n^{\alpha,M_0,M_1})'(0) = \frac{(-1)^n\,(\alpha+1)_n}{d_{\alpha,n}^{M_0,M_1}} \left(-\frac{M_0\,(\alpha+1)_n}{(n-1)!\,\Gamma(\alpha+3)} + \frac{M_1\,(n+\alpha+1)\,(\alpha+1)_n}{(\alpha+1)\,(n-2)!\,\Gamma(\alpha+4)}\right).$$

Here $(\alpha+1)_n$ is the *shifted factorial* or Pochhammer symbol defined by

$$(a)_0 := 1, (a)_k := a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Now if we use the asymptotic formula for the Gamma function [20] (formula 8.16 page 88)

$$\Gamma(x) \sim e^{-x} x^x \sqrt{\frac{2\pi}{x}}, \qquad x >> 1, x \in \mathbb{R},$$

where the symbol $a(x) \sim b(x)$ means, as usual, $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 1$. Then, for large n we obtain the asymptotic behavior for the denominator $d_{\alpha,n}^{M_0,M_1}$

$$d_{\alpha,n}^{M_0,M_1} \sim -\frac{M_1^2(n-1)(n+\alpha+1)n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(\alpha+4)} \tag{10}$$

Thus for any fixed M_0 and M_1 and for sufficiently large n, the denominator could be taken as $O(-n^{2\alpha+4})$, i.e., for large n we can guarantee the existence of the polynomials for all values of the masses M_0 and M_1.

Using (10) we obtain, for large n, the following asymptotic formulas

$$L_n^{\alpha,M_0,M_1}(0) \sim \frac{(-1)^{n+1}\Gamma(\alpha+4)(n-2)!}{M_1} \tag{11}$$

and

$$(L_n^{\alpha,M_0,M_1})'(0) \sim \frac{(-1)^{n+1}\Gamma(\alpha+3)(n-1)!}{M_1}. \tag{12}$$

From the above two expressions and using (6) and (7) we find

$$A_1 \sim \frac{2(\alpha+2)}{n} > 0, \qquad A_2 \sim \frac{(\alpha+2)(\alpha+3)}{n(n-1)} > 0, \qquad \forall n >> 1, n \in \mathbb{N}, \alpha > -1. \tag{13}$$

Let us now to study the asymptotic behavior of the ratio $\frac{L_n^{\alpha,M_0,M_1}(z)}{L_n^\alpha(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Notice that (5) can be rewritten

$$\frac{L_n^{\alpha,M_0,M_1}(z)}{L_n^\alpha(z)} = 1 + A_1 \frac{(L_n^{\alpha,M_0,M_1})'(z)}{L_n^\alpha(z)} + A_2 \frac{(L_n^{\alpha,M_0,M_1})''(z)}{L_n^\alpha(z)}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{14}$$

Using the asymptotic formula for the classical Laguerre polynomials (see [22], Thm. 8.22.3, and [23], section 4.6) for the ratio $\frac{1}{\sqrt{n}} \frac{L_n^\alpha)'(z)}{L_n^\alpha(z)}$ then

$$\frac{1}{\sqrt{n}} \frac{(L_n^\alpha)'(z)}{L_n^\alpha(z)} = \frac{-1}{\sqrt{z}} \left\{1 + \frac{1}{\sqrt{n}}[C_1(\alpha+1,z) - C_1(\alpha,z) - \sqrt{-z}]\right\} + o\left(\frac{1}{\sqrt{n}}\right)$$

holds, where $C_1(\alpha,z) = \frac{1}{4\sqrt{-z}}\left(-3z + \frac{1}{3}z^2 + \frac{1}{4} - \alpha^2\right)$ (see [23], Eq. (4.2.6) page 133).

Analogously,

$$\frac{1}{n} \frac{(L_n^\alpha)''(z)}{L_n^\alpha(z)} = \frac{-1}{z} \left\{1 + \frac{1}{\sqrt{n}}[C_1(\alpha+2,z) - C_1(\alpha,z) - 2\sqrt{-z}]\right\} + o\left(\frac{1}{\sqrt{n}}\right).$$

holds. Using the above two formulas as well as formula (14) we deduce the relative asymptotic relation for the generalized polynomials $L_n^{\alpha,M_0,M_1}(z)$ which is valid for all $z \in \mathbb{C} \setminus \mathbb{R}$. In fact we have proved the following

Theorem 1 *Let $L_n^{\alpha,M_0,M_1}(z)$ be our generalized polynomials and $L_n^\alpha(z)$ the Laguerre polynomials. Then $\forall z \in \mathbb{C} \setminus \mathbb{R}$*

$$\frac{L_n^{\alpha,M_0,M_1}(z)}{L_n^\alpha(z)} = 1+\frac{2\alpha+4}{\sqrt{-n\,z}}\left\{1+\frac{1}{\sqrt{n}}[C_1(\alpha+1,z)-C_1(\alpha,z)-\sqrt{-z}]\right\}+\frac{(\alpha+2)(\alpha+3)}{-n\,z}+o\left(\frac{1}{n}\right).$$

Finally, we provide an asymptotic formula for the ratio of the norms of the perturbed polynomials $\tilde{d}_n^2$ and the classical ones d_n^2.

Using the orthogonality of the generalized polynomials with respect to the functional $\mathcal{U}$ as well as the orthogonality of the classical Laguerre polynomials we have

$$\begin{aligned}\tilde{d}_n^2 = &< \mathcal{U}, (L_n^{\alpha,M_0,M_1})^2 > \; = \; < \mathcal{U}, L_n^{\alpha,M_0,M_1}\, L_n^\alpha >= \\ = &\; d_n^2 + M_0 L_n^{\alpha,M_0,M_1}(0)\, L_n^\alpha(0) + M_1 (L_n^{\alpha,M_0,M_1})'(0) L_n^\alpha(0) + M_1 L_n^{\alpha,M_0,M_1}(0)(L_n^\alpha)'(0).\end{aligned}$$

Taking into account that

$$L_n^\alpha(0) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \quad (L_n^\alpha)'(0) = \frac{n(-1)^{n-1}\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)}, \quad d_n^2 = n!\Gamma(n+\alpha+1)$$

and using formulas (11) and (12) we obtain

$$\frac{\tilde{d}_n^2}{d_n^2} = 1+\frac{(\alpha+2)(2n+\alpha+1)}{n(n-1)}+o\left(\frac{1}{n}\right) \sim 1+\frac{2(\alpha+2)}{n}.$$

§3 The zeros of $L_n^{\alpha,M_0,M_1}(x)$.

In this section we will study the zeros of the above generalized polynomials. We use the same methods as in [10]-[11] (see also [7] and [22]). It is known that the polynomials $\{P_n(x)\}_{n=0}^\infty$ which are orthogonal on an interval with respect to a positive weight function have the property that $P_n(x)$ has n real and simple zeros which are located in the interior of the interval of orthogonality. Since our functional is not positive definite we may no expect that the polynomials $L_n^{\alpha,M_0,M_1}(x)$ have real and simple zeros. However they have the following property

Theorem 2 *For sufficiently large n, the orthogonal polynomial $L_n^{\alpha,M_0,M_1}(x)$ has n real and simple zeros, $n-1$ of them are nonnegative and only one zero lies in $(-\infty,0)$.*

<u>Proof:</u> We will assume that $M_0>0$, $M_1>0$ and take $n>>1$. Since $M_0>0$, $M_1>0$ and $sign(L_n^{\alpha,M_0,M_1}(0)) = sign((L_n^{\alpha,M_0,M_1})'(0))$ and different from zero (see (11) and (12)), the polynomial changes its sign on $(0,\infty)$ at least one time. It follows from the fact that

$$< \mathcal{U}\,,\, L_n^{\alpha,M_0,M_1}(x) \; >= \int_0^\infty L_n^{\alpha,M_0,M_1}(x) x^\alpha e^{-x} dx + M_0 L_n^{\alpha,M_0,M_1}(0) + M_1 (L_n^{\alpha,M_0,M_1})'(0) = 0,$$

and if the sign of $L_n^{\alpha,M_0,M_1}(x)$ was constant in $(0,\infty)$ we will deduce that the last expression is the sum of three quantities of the same sign, which yields a contradiction. Let $x_1, x_2, ..., x_k$ be those positive zeros which have odd multiplicity. Define $p(x)$ such that

$$p(x) = (x-x_1)(x-x_2)\ldots(x-x_k).$$

Then $p(x)L_n^{\alpha,M_0,M_1}(x) \geq 0, \ \ \forall x \geq 0$. Now define $h(x)$

$$h(x) = (x + d)p(x) = (x + d)(x - x_1)(x - x_2)\dots(x - x_k),$$

in such a way that $h'(0) = 0$. Then

$$0 = h'(0) = dp'(0) + p(0), \qquad d = -\frac{p(0)}{p'(0)}.$$

Taking into account that

$$\frac{p'(0)}{p(0)} = -\left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k}\right) < 0,$$

we have $d > 0$. Hence $h(x)L_n^{\alpha,M_0,M_1}(x) \geq 0$, for all $x \geq 0$, and then $< \mathcal{U}\,,\, h\,L_n^{\alpha,M_0,M_1} >> 0$. This implies that $degree[h(x)] \geq n$ and then $degree[p(x)] \geq n-1$, i.e., $k \geq n-1$. As a consequence, all the positive zeros of $L_n^{\alpha,M_0,M_1}(x)$ are simple and at most one zero is negative. This immediately implies that all zeros are real.

To prove that for sufficiently large n the polynomial has one negative zero we notice that for even n, $L_n^{\alpha,M_0,M_1}(0) < 0$, as well as $L_n^{\alpha,M_0,M_1}(x) > 0$ for all $x < -N_1$ if $N_1 > 0$ is sufficiently large. Analogously, for odd n, $L_n^{\alpha,M_0,M_1}(0) > 0$ and $L_n^{\alpha,M_0,M_1}(x) < 0$ for all $x < -N_2$ if $N_2 > 0$ is sufficiently large. Then in some negative values of x the polynomial changes its sign. This implies that the polynomial $L_n^{\alpha,M_0,M_1}(x)$ has exactly one zero in $(-\infty, 0)$ and $n-1$ zeros in $(0, \infty)$. ■

We will denote the zeros of $L_n^{\alpha,M_0,M_1}(x)$ as $x_{n,1} < 0 < x_{n,2} < \dots < x_{n,n}$ Let us now examine the negative zero $x_{n,1}$ in more detail.

Theorem 3 *For sufficiently large n*

1. *The negative zero $x_{n,1}$ of the polynomial $L_n^{\alpha,M_0,M_1}(x)$ is bounded from below*

$$-\frac{2M_1}{M_0} < x_{n,1} < 0. \tag{15}$$

2. *The following inequalities hold*

   (a)

$$\frac{1}{2} < -x_{n,1}\left(\frac{1}{x_{n,2}} + \frac{1}{x_{n,3}} + \dots + \frac{1}{x_{n,n}}\right) < 1, \tag{16}$$

   (b)

$$-x_{n,1} < x_{n,2}. \tag{17}$$

Proof: Let $x_{n,2} < \dots < x_{n,n}$ be the positive zeros of $L_n^{\alpha,M_0,M_1}(x)$. Define $r(x)$

$$r(x) = (x - x_{n,2})(x - x_{n,3})\dots(x - x_{n,n}).$$

Then $L_n^{\alpha,M_0,M_1}(x) = r(x)(x - x_{n,1})$, where $x_{n,1} < 0$ and

$$\frac{r'(0)}{r(0)} = -\left(\frac{1}{x_{n,2}} + \frac{1}{x_{n,3}} + \ldots + \frac{1}{x_{n,n}}\right) < 0, \tag{18}$$

Since $degree[r(x)] = n - 1$ we have $< \mathcal{U}\,, r(x)L_n^{\alpha,M_0,M_1}(x) >= 0$, i.e.

$$\int_0^\infty r(x)^2(x - x_n)x^\alpha e^{-x}dx - M_0 r(0)^2 x_{n,1} + M_1 r(0)^2 - 2M_1 r(0)r'(0)x_{n,1} = 0.$$

Since the integral is positive we must have

$$-M_0 r(0)^2 x_{n,1} + M_1 r(0)^2 - 2M_1 r(0)r'(0)x_{n,1} < 0.$$

Using that $r(0)$ and $r'(0)$ have opposite signs, as well as $x_{n,1}r(0)r'(0) = |x_{n,1}||r(0)r'(0)| > 0$ we obtain

$$M_1 r(0)^2 < 2M_1|r(0)r'(0)||x_{n,1}| \qquad \text{and} \qquad M_0 r(0)^2|x_{n,1}| < 2M_1|r(0)r'(0)||x_{n,1}|.$$

Then we have

$$\frac{1}{2} < |x_{n,1}|\left(\frac{1}{x_{n,2}} + \frac{1}{x_{n,3}} + \ldots + \frac{1}{x_{n,n}}\right) \qquad \text{and} \qquad \frac{M_0}{2M_1} < \frac{1}{x_{n,2}} + \frac{1}{x_{n,3}} + \ldots + \frac{1}{x_{n,n}}. \tag{19}$$

Now, using (11) and (12),

$$\frac{(L_n^{\alpha,M_0,M_1})'(0)}{L_n^{\alpha,M_0,M_1}(0)} = \frac{r(0) - r'(0)x_{n,1}}{-x_{n,1}r(0)} = \frac{n-1}{\alpha+3} > 0 \qquad \text{and} \qquad 1 - |x_{n,1}|\left|\frac{r'(0)}{r(0)}\right| > 0.$$

Thus

$$|x_{n,1}|\left(\frac{1}{x_{n,2}} + \frac{1}{x_{n,3}} + \ldots + \frac{1}{x_{n,n}}\right) < 1.$$

Combining this inequality with (19), (16) holds. The second inequality (17) is a simple consequence of the first one (16). Notice that

$$\frac{M_0}{2M_1}|x_{n,1}| < |x_{n,1}|\left(\frac{1}{x_{n,2}} + \frac{1}{x_{n,3}} + \ldots + \frac{1}{x_{n,n}}\right) < 1, \tag{20}$$

hence $|x_{n,1}| < \dfrac{2M_1}{M_0}$. This prove (15) and therefore our theorem. ■

Theorem 4 *For n sufficiently large* $x_{n,1} = O(n^{-\frac{\alpha}{2}-2})$.

Proof: Firstly, we introduce the polynomial $\tilde{L}_n^{\alpha,M_0,M_1}(x) = \dfrac{(-1)^n}{n!}L_n^{\alpha,M_0,M_1}(x)$.

From Taylor's Theorem we have for $x < 0$, $x < \xi < 0$

$$\tilde{L}_n^{\alpha,M_0,M_1}(x) = \tilde{L}_n^{\alpha,M_0,M_1}(0) + x(\tilde{L}_n^{\alpha,M_0,M_1})'(0) + \frac{x^2}{2}(\tilde{L}_n^{\alpha,M_0,M_1})''(0) + \frac{x^3}{6}(\tilde{L}_n^{\alpha,M_0,M_1})'''(\xi) \tag{21}$$

As a consequence of the Rolle's Theorem every zero of $(\tilde{L}_n^{\alpha,M_0,M_1})'(x)$ lies between two consecutive zeros of the polynomial $\tilde{L}_n^{\alpha,M_0,M_1}(x)$. From (12) we conclude that $(\tilde{L}_n^{\alpha,M_0,M_1})'(0) < 0$. Thus all the zeros of $(\tilde{L}_n^{\alpha,M_0,M_1})'(x)$ are positive and it is a positive and increasing function for negative x ($(\tilde{L}_n^{\alpha,M_0,M_1})'(x)$ goes from $-\infty$, when $x \to -\infty$ to 0 in some $x > 0$, the first minimum). In the same way we have that $(\tilde{L}_n^{\alpha,M_0,M_1})''(x) > 0$ and decreasing function for $x < 0$ (the polynomial $\tilde{L}_n^{\alpha,M_0,M_1}(x)$ is a convex upward function for negative x and have his first inflection point in some positive value of x). This implies that $(\tilde{L}_n^{\alpha,M_0,M_1})'''(x) < 0$ for $x < 0$. Then, from (11), (12) and (5) we obtain

$$\tilde{L}_n^{\alpha,M_0,M_1}(x) > ax^2 + bx + c, \qquad \text{for } x < 0, \tag{22}$$

where

$$a = \frac{1}{2}(\tilde{L}_n^{\alpha,M_0,M_1})''(0) \sim \frac{n^{\alpha+2}}{\Gamma(\alpha+5)}, \qquad b = (\tilde{L}_n^{\alpha,M_0,M_1})'(0) \sim -\frac{\Gamma(\alpha+3)}{M_1\, n},$$

and

$$c = \tilde{L}_n^{\alpha,M_0,M_1}(0) \sim -\frac{\Gamma(\alpha+4)}{M_1\, n(n-1)}.$$

According with (22) the negative zero $x_{n,1}$ lies between the two roots $\tilde{x}_-$ and $\tilde{x}_+$ of ax^2+bx+c. These two roots are equal to

$$\tilde{x}_\pm = \frac{\Gamma(\alpha+3)\Gamma(\alpha+5)}{2M_1 n^{\alpha+3}}\left[1 \pm \sqrt{1 + \frac{4(\alpha+3)M_1^2 n^{\alpha+3}}{\Gamma(\alpha+3)\Gamma(\alpha+5)(n-1)}}\right].$$

When $n \to \infty$

$$\tilde{x}_\pm \sim \pm\sqrt{\frac{\Gamma(\alpha+4)\Gamma(\alpha+5)}{n^{\alpha+4}}}.$$

Since $\tilde{x}_- < x_{n,1} < \tilde{x}_+$ then $x_{n,1} = O(n^{-\frac{\alpha}{2}-2})$. ■

§4 A second order differential equation for $L_n^{\alpha,M_0,M_1}(x)$.

In this section we will obtain a second order differential equation satisfied by the polynomials $L_n^{\alpha,M_0,M_1}(x)$.

Theorem 5 *The generalized polynomials $L_n^{\alpha,M_0,M_1}(x)$ satisfy a second order differential equation*

$$x\tilde{\sigma}(x)\frac{d^2}{dx^2}L_n^{\alpha,M_0,M_1}(x) + \tilde{\tau}(x)\frac{d}{dx}L_n^{\alpha,M_0,M_1}(x) + n\tilde{\lambda}_n(x)L_n^{\alpha,M_0,M_1}(x) = 0 \tag{23}$$

where $\tilde{\sigma}(x)$ and $\tilde{\lambda}_n(x)$ are polynomials of degree 2 and $\tilde{\tau}(x)$ is a polynomial of degree 3. More precisely

$$\begin{aligned}\tilde{\sigma}(x) = \;& 2\,A_2 + 3\,\alpha\,A_2 + \alpha^2\,A_2 - 2\,A_1\,A_2\,n - \alpha\,A_1\,A_2\,n - {A_2}^2\,n + {A_2}^2\,n^2 + \\ & + [-A_1\,x - \alpha\,A_1 - 2\,A_2 - 2\,\alpha\,A_2 + {A_1}^2\,n - 2\,A_2\,n + A_1\,A_2\,n]\,x + \\ & + [1 + A_1 + A_2]\,x^2,\end{aligned} \tag{24}$$

$$\tilde{\tau}(x) = \begin{aligned}[t] & 6\,A_2 + 11\,\alpha\,A_2 + 6\,\alpha^2\,A_2 + \alpha^3\,A_2 - 6\,A_1\,A_2\,n - 5\,\alpha\,A_1\,A_2\,n - \alpha^2\,A_1\,A_2\,n - \\ & -3\,{A_2}^2\,n - \alpha\,{A_2}^2\,n + 3\,{A_2}^2\,n^2 + \alpha\,{A_2}^2\,n^2 + [-2\,A_1 - 3\,\alpha\,A_1 - \alpha^2\,A_1 - \\ & -6\,A_2 - 9\,\alpha\,A_2 - 3\,\alpha^2\,A_2 + 2\,{A_1}^2\,n + \alpha\,{A_1}^2\,n - 4\,A_2\,n - 2\,\alpha\,A_2\,n + \\ & +4\,A_1\,A_2\,n + {A_2}^2\,n + 2\alpha A_1 A_2 n - {A_2}^2\,n^2]\,x + [1 + \alpha + 2\,A_1 + 2\,\alpha\,A_1 + \\ & +3\,A_2 + 3\,\alpha\,A_2 - {A_1}^2\,n + 2\,A_2\,n - A_1\,A_2\,n\,]\,x^2 - [1 + A_1 + A_2]\,x^3, \end{aligned} \tag{25}$$

$$\tilde{\lambda}_n(x) = \begin{aligned}[t] & 6\,A_2 + 5\,\alpha\,A_2 + \alpha^2\,A_2 + 3\,A_1\,A_2 + \alpha\,A_1\,A_2 + 2\,{A_2}^2 - 3\,A_1\,A_2\,n - \\ & -\alpha\,A_1\,A_2\,n - 3\,{A_2}^2\,n + {A_2}^2\,n^2 + [-2\,A_1 - \alpha\,A_1 - {A_1}^2 - 2\,A_2 - \\ & -2\,\alpha\,A_2 - A_1\,A_2 + {A_1}^2\,n - 2\,A_2\,n + A_1\,A_2\,n]\,x + [1 + A_1 + A_2]\,x^2. \end{aligned} \tag{26}$$

Proof: We will start from the differential equation for the classical Laguerre polynomials

$$x\frac{d^2}{dx^2}L_n^\alpha(x) + (\alpha + 1 - x)\frac{d}{dx}L_n^\alpha(x) + nL_n^\alpha(x) = 0. \tag{27}$$

If we take derivatives in this equation we obtain

$$\begin{aligned} & x\frac{d^3}{dx^3}L_n^\alpha(x) = -(\alpha + 2 - x)\frac{d^2}{dx^2}L_n^\alpha(x) - (n-1)\frac{d}{dx}L_n^\alpha(x) \qquad \text{and} \\ & x\frac{d^4}{dx^4}L_n^\alpha(x) = -(\alpha + 3 - x)\frac{d^3}{dx^3}L_n^\alpha(x) - (n-2)\frac{d^2}{dx^2}L_n^\alpha(x) \end{aligned} \tag{28}$$

If we multiply the first one by x and use (27) we can express $x^2\frac{d^3}{dx^3}L_n^\alpha(x)$ in terms of the classical Laguerre polynomials and their first derivatives

$$x^2\frac{d^3}{dx^3}L_n^\alpha(x) = n(\alpha + 2 - x)L_n^\alpha(x) + [(\alpha + 1 - x)(\alpha + 2 - x) - x(n-1)]\frac{d}{dx}L_n^\alpha(x). \tag{29}$$

In a similar way, if we multiply the second expression in (28) by x^2, from (27) and (29) we obtain

$$\begin{aligned} x^3\frac{d^4}{dx^4}L_n^\alpha(x) = {} & [xn(n-2) - n(\alpha + 2 - x)(\alpha + 3 - x)]L_n^\alpha(x) + \{x(n-2)(\alpha + 1 - x) - \\ & -(\alpha + 3 - x)[(\alpha + 1 - x)(\alpha + 2 - x) - x(n-1)]\}\frac{d}{dx}L_n^\alpha(x). \end{aligned} \tag{30}$$

Now from (5), (29) and (30)

$$\begin{aligned} & xL_n^{\alpha,M_0,M_1}(x) = a(x)L_n^\alpha(x) + b(x)\frac{d}{dx}L_n^\alpha(x), \\ & x^2\frac{d}{dx}L_n^{\alpha,M_0,M_1}(x) = c(x)L_n^\alpha(x) + d(x)\frac{d}{dx}L_n^\alpha(x), \\ & x^3\frac{d^2}{dx^2}L_n^{\alpha,M_0,M_1}(x) = e(x)L_n^\alpha(x) + f(x)\frac{d}{dx}L_n^\alpha(x) \end{aligned}$$

where

$$
\begin{aligned}
a(x) &= (x - nA_2),\\
b(x) &= xA_1 - (\alpha+1-x)A_2\\
c(x) &= -xnA_1 + n(\alpha+2-x)A_2,\\
d(x) &= x(x-(n-1)A_2) - (\alpha+1-x)[xA_1 - (\alpha+2-x)A_2]\\
e(x) &= -nx^2 + xn(n-2)A_2 + nx(\alpha+2-x)A_1 - n(\alpha+2-x)(\alpha+3-x)A_2\\
f(x) &= -x^2(\alpha+1-x) + x(\alpha+1-x)(n-2)A_2+\\
&\qquad +[(\alpha+1-x)(\alpha+2-x) - x(n-1)][xA_1 - (\alpha+3-x)A_2].
\end{aligned}
$$

From the above formulas

$$
\begin{vmatrix}
xL_n^{\alpha,M_0,M_1}(x) & a(x) & b(x)\\
x^2\dfrac{d}{dx}L_n^{\alpha,M_0,M_1}(x) & c(x) & d(x)\\
x^3\dfrac{d^2}{dx^2}L_n^{\alpha,M_0,M_1}(x) & e(x) & f(x)
\end{vmatrix}
=
\begin{vmatrix}
L_n^{\alpha,M_0,M_1}(x) & a(x) & b(x)\\
x\dfrac{d}{dx}L_n^{\alpha,M_0,M_1}(x) & c(x) & d(x)\\
x^2\dfrac{d^2}{dx^2}L_n^{\alpha,M_0,M_1}(x) & e(x) & f(x)
\end{vmatrix}
= 0. \qquad (31)
$$

Now, doing some algebraic straightforward calculation we obtain that $a(x)d(x) - c(x)b(x) = x\tilde{\sigma}(x)$, $b(x)e(x) - a(x)f(x) = x\tilde{\tau}(x)$ and $c(x)f(x) - e(x)d(x) = nx^2\tilde{\lambda}_n(x)$, where $\tilde{\sigma}(x)$ and $\tilde{\lambda}_n(x)$ are polynomials of degree 2, explicitly given in (24) and (26), and $\tilde{\tau}(x)$ is a polynomial of degree 3, whose explicit expression is given in (25). Then, expanding the determinant (31) by the first column and dividing by x^2, the theorem follows. ■

§5 The moments μ_r of the zeros of $L_n^{\alpha,M_0,M_1}(x)$. WKB method.

In this section we will study the moments of the distribution of zeros using a general method presented in [5]. Notice that we will study the polynomial of large degree, since this is a necessary and sufficient condition for the polynomial $L_n^{\alpha,M_0,M_1}(x)$ to be defined (see [1] and section 2 of the present work). This method presented in [5] allows us to compute the moments μ_r of the distribution of zeros $\rho_n(x)$ around the origin

$$
\rho_n = \frac{1}{n}\sum_{i=1}^{n}\delta(x - x_{n,i}), \qquad \mu_r = \frac{1}{n}y_r = \frac{1}{n}\sum_{i=1}^{n}x_{n,i}^r.
$$

Buendía, Dehesa and Gálvez [5] have obtained a general formula to find these quantities (see [5], section II, Eq.(11) and (13) page 226). We will apply these two formulas to obtain the general expression for the moments μ_1 and μ_2, but firstly, let us introduce some notations.

We will rewrite the differential equation (23) in the form

$$
x\tilde{\sigma}(x)\frac{d^2}{dx^2}L_n^{\alpha,M_0,M_1}(x) + \tilde{\tau}(x)\frac{d}{dx}L_n^{\alpha,M_0,M_1}(x) + n\tilde{\lambda}_n(x)L_n^{\alpha,M_0,M_1}(x) = 0
$$

where now

$$
\begin{aligned}
x\tilde{\sigma}(x) &= a_0^{(2)} + a_1^{(2)}x + a_2^{(2)}x^2 + a_3^{(2)}x^3,\\
\tilde{\tau}(x) &= a_0^{(1)} + a_1^{(1)}x + a_2^{(1)}x^2 + a_3^{(1)}x^3,\\
n\tilde{\lambda}_n(x) &= a_0^{(0)} + a_1^{(0)}x + a_2^{(0)}x^2.
\end{aligned}
$$

Here the values $a_j^{(i)}$ can be found from (24), (25) and (26) in a straightforward way. Let $\xi_0 = 1$ and $q = max\{degree(x\tilde{\sigma}) - 2, degree(\tilde{\tau}) - 1, degree(n\tilde{\lambda}_n)\} = 2$. Then from [5], (section II, Eq.(11) and (13) page 226)

$$\xi_1 = y_1, \qquad \xi_2 = \frac{y_1^2 - y_2}{2}. \tag{32}$$

and

$$\xi_s = -\frac{\sum_{m=1}^{s} (-1)^m \xi_{s-m} \sum_{i=0}^{2} \frac{(n-s+m)!}{(n-s+m-i)!} a_{i+q-m}^{(i)}}{\sum_{i=0}^{2} \frac{(n-s)!}{(n-s-i)!} a_{i+q}^{(i)}}. \tag{33}$$

In general $\xi_k = \frac{(-1)^k}{k!} \mathcal{Y}_k(-y_1, -y_2, -2y_3, ..., -(k-1)!k_n)$ where $\mathcal{Y}_k$-symbols denote the well known Bell polynomials of the number theory [21].

Let us now to apply these general formulas to obtain the first two central moments μ_1 and μ_2 for our polynomials. Equation (33) give

$$\xi_1 = (\alpha - A_1 + n)\, n$$

and

$$\xi_2 = \frac{(-1+n)\, n\, (-\alpha + \alpha^2 - 2\,\alpha\, A_1 + 2\, A_2 - n + 2\,\alpha\, n - 2\, A_1\, n + n^2)}{2}.$$

Together with (32) this leads us to

$$\mu_1 = n + \alpha - A_1$$

and

$$\mu_2 = \quad -\alpha + \alpha^2 - 2\,\alpha\, A_1 + 2\, A_2 + \left(-1 + 3\,\alpha - 2\, A_1 + {A_1}^2 - 2\, A_2\right)\, n + 2\, n^2.$$

The asymptotic behavior of these two moments is

$$\mu_1 \sim n + O(1) \text{ and } \mu_2 \sim 2n^2 + O(n).$$

Notice that equation (33) and relation $\xi_k = \frac{(-1)^k}{k!} \mathcal{Y}_k(-y_1, -y_2, -2y_3, ..., -(k-1)!y_k)$ provide us a general method to obtain all the moments $\mu_r = \frac{1}{n} y_r$, but it is hightly non-linear and cumbersome. This is a reason to apply it in order to obtain only the moment of low order.

Next, we will analyze the so-called semiclassical or WKB approximation (see [2],[25] and references contained therein). Denoting the zeros of $L_n^{\alpha, M_0, M_1}(x)$ by $\{x_{n,k}\}_{k=1}^n$ we can define its distribution function as

$$\rho_n(x) = \frac{1}{n} \sum_{k=1}^{n} \delta(x - x_{n,k}). \tag{34}$$

We will use the method presented in [25] in order to obtain the WKB density of zeros, which is an approximate analytical expression for the density of zeros for the solutions of any linear second order differential equation with polynomial coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{35}$$

The main result is established in the following

Theorem 6 *Let $S(x)$ and $\epsilon(x)$ be the functions*

$$S(x) = \frac{1}{4a_2^2}\{2a_2(2a_0 - a_1') + a_1(2a_2' - a_1)\}, \tag{36}$$

$$\epsilon(x) = \frac{1}{4[S(x)]^2}\left\{\frac{5[S'(x)]^2}{4[S(x)]} - S''(x)\right\} = \frac{P(x,n)}{Q(x,n)}, \tag{37}$$

where $P(x,n)$ and $Q(x,n)$ are polynomials in x as well as in n. If the condition $\epsilon(x) << 1$ holds, then the semiclassical or WKB density of zeros of the solutions of (35) is given by

$$\rho_{WKB}(x) = \frac{1}{\pi}\sqrt{S(x)}, \qquad x \in I \subseteq \mathbb{R}, \tag{38}$$

in every interval I where the function $S(x)$ is positive.

The proof of this Theorem can be found in [2],[25].

Now we can apply this result to our differential equation (23). Using the coefficients of the equation (23) and taking into account that

$$\begin{aligned} x\tilde{\sigma}(x) &= a_2(x) = a_0^{(2)} + a_1^{(2)}x + a_2^{(2)}x^2 + a_3^{(2)}x^3, \\ \tilde{\tau}(x) &= a_1(x) = a_0^{(1)} + a_1^{(1)}x + a_2^{(1)}x^2 + a_3^{(1)}x^3, \\ n\tilde{\lambda}_n(x) &= a_0(x) = a_0^{(0)} + a_1^{(0)}x + a_2^{(0)}x^2, \end{aligned}$$

we obtain that for sufficiently large n, $\epsilon(x) \sim n^{-1}$. Then we can apply the above Theorem in order to find the corresponding WKB density of zeros of the polynomials $L_n^{\alpha,M_0,M_1}(x)$. The computations are very long and cumbersome. For this reason we provide a little program using *Mathematica* [24] and some graphics representation for the $\rho_{WKB}(x)$ function (see Appendix I). In figure 1 we represent the WKB density of zeros for our generalized Laguerre polynomials. We have used formula (13) and ploted the Density function for different values (from top to bottom) $n = 10000, 5000, 1000, 100$ and $\alpha = 0$. In figure 2 we represent the same function for the Classical Laguerre polynomials.

§6 A Three Term Recurrence Relation for $L_n^{\alpha,M_0,M_1}(x)$.

It is known that all polynomials, orthogonal on an interval with respect to a positive weight function satisfy a three term recurrence relation (TTRR). Koekoek and Meijer [10] (see also [11]) have considered some special inner products of Sobolev-type as

$$< p,q >= \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \sum_{k=0}^{N} M_k p^{(k)}(0)q^{(k)}(0), \qquad M_k \geq 0. \tag{39}$$

Notice, that, for such an inner product $< x\,,\, x > \neq < 1\,,\, x^2 >$. This implies that the polynomials orthogonal with respect to (39) may no satisfy a TTRR. In fact they satisfy a $(2N+3)$-term recurrence relation [11]. In our case (4), if the three consecutive polynomials $L_{n-1}^{\alpha,M_0,M_1}(x)$, $L_n^{\alpha,M_0,M_1}(x)$ and $L_{n+1}^{\alpha,M_0,M_1}(x)$ exist, then we will prove that they satisfy a certain three term recurrence relation.

Theorem 7 *For n large enough the generalized polynomials $L_n^{\alpha,M_0,M_1}(x)$ satisfy a three term recurrence relation*

$$xL_n^{\alpha,M_0,M_1}(x) = L_{n+1}^{\alpha,M_0,M_1}(x) + \tilde{\beta}_n^{M_0,M_1} L_n^{\alpha,M_0,M_1}(x) + \tilde{\gamma}_n^{M_0,M_1} L_{n-1}^{\alpha,M_0,M_1}(x).$$

Proof: Since $xL_n^{\alpha,M_0,M_1}(x)$ is a polynomial of degree $n+1$, for n large enough, we have

$$xL_n^{\alpha,M_0,M_1}(x) = L_{n+1}^{\alpha,M_0,M_1}(x) + \tilde{\beta}_n^{M_0,M_1} L_n^{\alpha,M_0,M_1}(x) + \tilde{\gamma}_n^{M_0,M_1} L_{n-1}^{\alpha,M_0,M_1}(x) + \sum_{k=0}^{n-2} C_k^n x^k,$$

where C_k^n, $k = 1, 2, ..., n-2$, are real coefficients. Taking the indefinite inner product $(\cdot,\cdot)$ associated with the functional $\mathcal{U}$

$$(\, p\,,\, q\,) \;\; = \int_0^\infty p(x)q(x) \;\; x^\alpha e^{-x} dx + M_0 p(0)q(0) + M_1 (p(x)q(x))'|_{x=0} \;\; , \alpha > -1, \qquad (40)$$

multiplying by x^m on both sides of the above expression, and using the orthogonality of the $L_n^{\alpha,M_0,M_1}(x)$ we find

$$0 = (L_n^{\alpha,M_0,M_1},\, x^{m+1}) = \sum_{k=0}^{n-2} C_k^n (1, x^{m+k}), \qquad m = 0, 1, ..., n-2.$$

Since the determinant of the the above linear system in C_k^n is different from zero (it coincide with the Gram determinant of order $n-1$ for the indefinite inner product (40), see [22], Section **2.2** pages 25-28) then we obtain that all $C_k^n = 0$ for all $k = 0, 1, ..., n-2$. Thus,

$$xL_n^{\alpha,M_0,M_1}(x) = C_{n+1}^n L_{n+1}^{\alpha,M_0,M_1}(x) + C_n^n L_n^{\alpha,M_0,M_1}(x) + C_{n-1}^n L_{n-1}^{\alpha,M_0,M_1}(x). \qquad (41)$$

Let us to obtain the coefficients C_{n-1}^n, C_n^n and C_{n+1}^n in (41). Firstly, comparing the coefficients x^{n+1} in (41) we find $C_{n+1}^n = 1$. Let b_n^A be the coefficient of x^{n-1} in the expansion $L_n^{\alpha,M_0,M_1}(x) = x^n + b_n^A x^{n-1} + ...$ Then, comparing the coefficients of x^n in the two sides of (41) we find $\tilde{\beta}_n^{M_0,M_1} = C_n^n = b_n^A - b_{n+1}^A$. To calculate $\tilde{\gamma}_n^{M_0,M_1} = C_{n-1}^n$ it is sufficient to evaluate (41) in $x = 0$ and remark that $L_n^{\alpha,M_0,M_1}(0) \neq 0$ $(n >> 1)$.

In order to obtain a general expression for the coefficient $\tilde{\beta}_n^{M_0,M_1}$ we can use the representation formula (5) for the generalized polynomials

$$L_n^{\alpha,M_0,M_1}(x) = L_n^\alpha(x) + A_1(n)(L_n^\alpha)'(x) + A_2(n)(L_n^\alpha)''(x).$$

Notice that A_1 and A_2 depend on n. Doing some algebraic calculations we find that $b_n^A = b_n + nA_1(n)$, where b_n denotes the coefficient of the $n-1$ power for the classical monic Laguerre polynomials

$$L_n^\alpha(x) = x^n - n(n+\alpha)x^{n-1} + \ldots, \text{ i.e., } b_n = -n(n+\alpha).$$

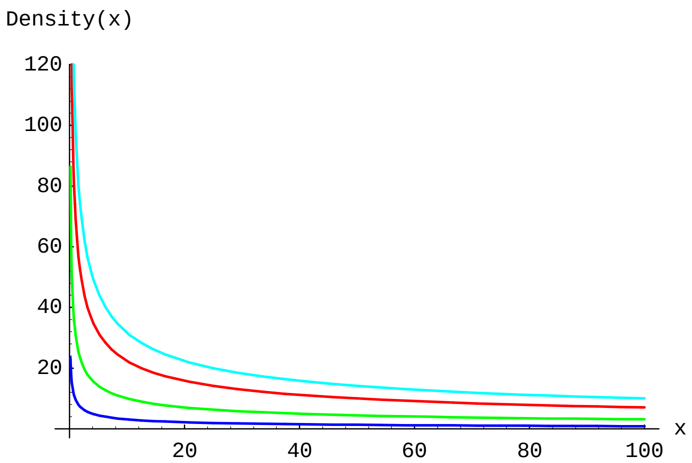

Figure 1: WKB Density of zeros of $L_n^{0,M_0,M_1}(x)$.

Using these formulas and the main data for Laguerre polynomials ([19], page 11, table 1.1) we obtain the following TTRR coefficients for generalized polynomials

$$\tilde{\beta}_n^{M_0,M_1} = 2n+\alpha+1+nA_1(n)-(n+1)A_1(n+1),$$

and

$$\tilde{\gamma}_n^{M_0,M_1} = -\frac{L_{n+1}^{\alpha,M_0,M_1}(0)}{L_{n-1}^{\alpha,M_0,M_1}(0)} - \tilde{\beta}_n^{M_0,M_1}\frac{L_{n}^{\alpha,M_0,M_1}(0)}{L_{n-1}^{\alpha,M_0,M_1}(0)}. \qquad \blacksquare$$

Using (13) for sufficiently large n

$$\tilde{\beta}_n^{M_0,M_1} \sim 2n+\alpha+1, \qquad \tilde{\gamma}_n^{M_0,M_1} \sim (n-2)(n+\alpha+2),$$

i.e., $\tilde{\beta}_n^{M_0,M_1} \sim \beta_n$ and $\tilde{\gamma}_n^{M_0,M_1} \sim \gamma_n - 2(\alpha+2)$. Here β_n and γ_n denote the coefficients of the TTRR for the classical monic Laguerre polynomials $xL_n^{\alpha}(x) = L_{n+1}^{\alpha}(x) + \beta_n L_{n+1}^{\alpha}(x) + \gamma_n L_{n+1}^{\alpha}(x)$, and they are equal to $\beta_n = 2n+\alpha+1$ and $\gamma_n = n(n+\alpha)$, respectively.

Appendix I.

Let us to obtain the coefficient of the second order differential equation (23) and the $S(x)$ (36) and the $\rho_{WKB}(x)$ function (34) by using *Mathematica*.

Firstly we will obtain the coefficients $\tilde{\sigma}(x)$, $\tilde{\tau}(x)$ and $\tilde{\lambda}_n(x)$ of the second order differential equation (23). Using formula (31) we find

```
a=(x - n A2);
b=x A1-( alpha+1-x) A2;
c=-x n A1+n(alpha+2-x)A2;
d=x(x-n A2+A2)-(alpha+1-x)(x A1-(alpha+2-x)A2);
```

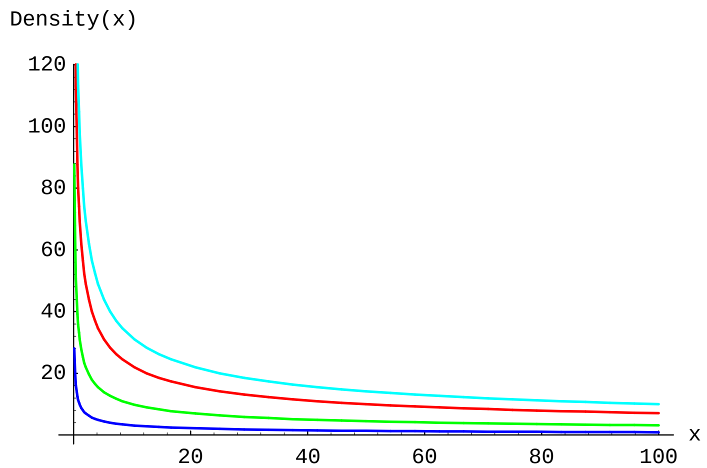

Figure 2: WKB Density of zeros of $L_n^0(x)$.

```
e=-n x^2+x(n-2)n A2 + n x(alpha+2-x)A1-n(alpha+2-x)(alpha+3-x)A2 );
f=(-x^2(alpha+1-x)+x(alpha+1-x)(n-2)A2)+
     ((alpha+1-x)(alpha+2-x)-x(n-1))(x A1-(alpha+3-x)A2);

sigma=Simplify[Expand[a d - c b]];
tau=Simplify[Expand[e b - a f]/x]
lambda=Simplify[Expand[c f - e d ]/x^2];
```

Now from (36) we find

```
dsig=D[sigma,x];
dt=D[tau1,x];
s[x_]:=(2 sigma(2 lambda-dt)+tau(2 dsig-tau))/(4 sigma^2)
num=Simplify[Numerator[s[x]]];
den=Simplify[Denominator[s[x]]];
Density[x_]=Sqrt[num/den]
```

Finally we plot the function $Density[x]$ for sufficiently large n for the generalized polynomials and for the classical ones (see Figure 1 and 2).

ACKNOWLEDGEMENTS:

The research of the two authors was supported by Comisión Interministerial de Ciencia y Tecnología (CICYT) of Spain under grant PB 93-0228-C02-01 and by INTAS. The authors thank to Professor Guillermo López Lagomasino (Universidad de la Habana, Cuba) for its useful suggestions and remarks as well as to the referees for their constructive criticism about the presentation of this manuscript.

References

[1] R. Álvarez-Nodarse and F. Marcellán : *A Generalization of the Classical Laguerre Polynomials.* Rend. Circ. Mat. Palermo. Serie II, **44**, (1995), 315-329.

[2] E.R. Arriola, A. Zarzo and J.S. Dehesa: *Spectral Properties of the biconfluent Heun differential equation.* J.Comput.Appl.Math. **37** (1991), 161-169.

[3] S. L. Belmehdi and F. Marcellán: *Orthogonal polynomials associated to some modifications of a linear functional.* Appl. Analysis **46**(1), (1992), 1-24.

[4] C. Bernardi and Y. Maday: *Some spectral approximation of onedimensional fourth order problems.* **Progress in Approximation Theory**. P. Nevai and A. Pinkus Editors. Academic Press, San Diego, 1991, 43-116.

[5] E. Buendía, S. J. Dehesa and F. J. Gálvez : *The distribution of the zeros of the polynomial eigenfunctions of ordinary differential operators of arbitrary order.* In **Orthogonal Polynomials and their Applications.** M.Alfaro et.al. Eds. Lecture Notes in Mathematics. **Vol. 1329**, Springer-Verlag, Berlin, 1988, 222-235.

[6] T. S. Chihara: *Orthogonal polynomials and measures with end point masses.* Rocky Mount. J. of Math.,**15**(3),(1985),705-719.

[7] T.S. Chihara: *An Introduction to Orthogonal Polynomials.* (Gordon and Breach, New York, 1978).

[8] N. Draïdi and P. Maroni. *Sur l'adjonction de deux masses de Dirac à une forme régulière quelconque.* In **Polinomios ortogonales y sus aplicaciones.** A. Cachafeiro and E. Godoy Eds. Actas del V Simposium. Universidad de Santiago. Vigo 1988, 83-90.

[9] A.A. Gonchar:*On the convergence of Padé approximants for some classes of meromorphic functions.* Mat. Sb. **97** (139), (1975), 607–629.

[10] R. Koekoek and H.G. Meijer: *A generalization of Laguerre polynomials.* SIAM J. Mat. Anal. **24** (3), (1993), 768-782.

[11] R. Koekoek *Generalizations of the classical Laguerre Polynomials and some q-analogues.* Doctoral Dissertation, Delft University of Technology. 1990.

[12] T. H. Koornwinder: *Orthogonal polynomials with weight function* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$. Canad.Math.Bull, **27** (2), (1984), 205-214.

[13] A. M. Krall: *Orthogonal Polynomials satisfying fourth order differential equations.* Proc. Roy. Soc. Edinburgh, **87**, (1981), 271-288.

[14] H. L. Krall : *On Orthogonal Polynomials satisfying a certain fourth order differential equation.* The Pennsylvania State College Bulletin, **6**, (1940), 1-24.

[15] G. López : *Convergence of Padé approximants of Stieltjes type meromorphic functions and comparative asymptotic for orthogonal polynomials.* Math. USSR Sb., **64**, (1989), 207-227.

[16] F. Marcellán and P. Maroni: *Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique.* Ann. Mat. Pura ed Appl., **IV**, CLXII, (1992), 1-22.

[17] F. Marcellán and A. Ronveaux : *Differential equations for classical type orthogonal polynomials.* Canad. Math. Bull. **32**, (4), (1989), 404-411.

[18] P. Nevai: *Orthogonal Polynomials.* Memoirs of the Amer. Math. Soc.**213**, Providence, Rhode Island, 1979.

[19] A.F.Nikiforov, S.K.Suslov, V.B.Uvarov: **Orthogonal Polynomials of Discrete Variables.** Springer Series in Computational Physics. Springer-Verlag, Berlin, 1991.

[20] F. W. J. Olver: **Asymptotics and Special Functions.** Academic Press Inc., New York, 1974.

[21] J. Riordan: **An Introduction to Combinatorial Analysis.** Wiley, New York, 1958.

[22] G. Szegö: **Orthogonal Polynomials.** Amer. Math. Soc. Colloq. Publ., **23**, Amer. Math. Soc., Providence, RI, 1975 (4th edition).

[23] W. Van Assche **Asymptotics for Orthogonal Polynomials.** Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1987.

[24] S. Wolfram: **Mathematica. A system for doing Mathematics by Computer**. Addison-Wesley Publishing Co. The Advanced Book Program, 1991.

[25] A.Zarzo and J.S. Dehesa: *Spectral Properties of solutions of hypergeometric-type differential equations.* J.Comput.Appl.Math. **50** (1994), 613-623.