Second order difference equations for certain families of "discrete" polynomials.

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Abstract

In this paper we will consider two algorithms which allow us to obtain second order linear difference equations for certain families of polynomials. The corresponding algorithms can be implemented in any computer algebra system in order to obtain explicit expressions of the coefficients of the difference equations.

1 Introduction

In this paper we will describe two algorithms which allow us to find the second order linear difference equation (SODE) which satisfy certain families of polynomials. We will consider two different cases. The first case appears when a polynomial $\widetilde{P}_n$ is given in terms of a known polynomial set $\{P_n\}$ by

$$q(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_n(x+1), \tag{1.1}$$

where q, a and b are some known functions which, in general, depend on n, and P_n is a polynomial for which it is known a SODE of the form

$$\begin{aligned} &\sigma(x;n)\triangle\nabla P_n(x) + \tau(x;n)\triangle P_n(x) + \lambda(x;n)P_n(x) = 0, \\ &\triangle F(x) = F(x+1) - F(x), \quad \nabla F(x) = F(x) - F(x-1), \end{aligned} \tag{1.2}$$

where σ, τ and λ are also given functions. The above Eq. can be written in the following equivalent form

$$\begin{aligned} &\sigma(x;n)P_n(x-1) - \phi(x;n)P_n(x) + \psi(x;n)P_n(x+1) = 0\ , \\ &\psi(x;n) = \sigma(x;n) + \tau(x;n), \qquad \phi(x;n) = 2\sigma(x;n) + \tau(x;n) - \lambda(x;n). \end{aligned} \tag{1.3}$$

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For our purpose it is convenient also to rewrite (1.3) in the form

$$\Sigma(x;n) \triangle^2 P_n(x) + \mathrm{T}(x;n) \triangle P_n(x) + \Lambda(x;n) P_n(x) = 0, \tag{1.4}$$

where now,

$$\Sigma(x;n) = \psi(x+1;n), \quad T(x;n) = 2\psi(x+1;n) - \phi(x+1;n),$$

$$\Lambda(x;n) = \sigma(x+1;n) + \psi(x+1;n) - \phi(x+1;n).$$

An example of polynomials $\widetilde{P}_n$ satisfying (1.1) are the *Krall-type* discrete orthogonal polynomials, firstly studied in [1, 2, 3, 4, 6, 7]. These polynomials are orthogonal with respect to an inner product (p,q) defined by the bilinear form $(x_{i+1} = x_i + 1)$

$$(p,q) = \sum_{x_i=a}^{b-1} p(x_i)\, q(x_i) \rho(x_i) + \sum_{k=1}^{N} A_k p(y_k)\, q(y_k), \qquad A_k \geq 0.$$

In other words, these polynomials are obtained via the addition of delta Dirac measures to a positive definite weight function ρ [11, 14]. A special emphasis was given to the case when ρ is the classical *"discrete"* weight function corresponding to the classical "discrete" [12] polynomials of Hahn [3], Meixner [1, 2, 6], Kravchuk [2] and Charlier [2, 7] for which the coefficients σ, τ and λ in (1.2) are polynomials such that σ and τ do not depend of n, λ is a constant, degree$(\sigma) \leq 2$ and degree$(\tau) \equiv 1$. In all these cases, when one or two delta Dirac measures ($N = 1$ or 2) are added to a weight function corresponding to the Hahn, Meixner, Kravchuk and Charlier polynomials at the ends of the interval of orthogonality, it is possible to find for the resulting polynomials an expression similar to (1.1) where $\widetilde{P}_n$ are the new (Krall-type) polynomials and P_n are the classical ones [4].

The second case appears when the polynomial $\widetilde{P}_n$ is given by

$$q(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_{n-1}(x), \tag{1.5}$$

where q, a and b are some known functions, in general depending on n but now the family $\{P_n\}$ satisfies the following two "*difference-recurrence*" relations

$$\sigma(x;n) \triangle P_n(x) = \tau(x;n)P_n(x) + \gamma(x;n)P_{n-1}(x), \tag{1.6}$$

and

$$\bar{\sigma}(x;n) \triangle P_n(x) = \bar{\tau}(x;n)P_n(x) + \bar{\gamma}(x;n)P_{n+1}(x), \tag{1.7}$$

being σ, $\bar{\sigma}$, τ, $\bar{\tau}$, γ and $\bar{\gamma}$ known functions (they should not coincide with the functions given in (1.2).) If the polynomial set $\{P_n\}$ satisfies a three-term recurrence relation,

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \tag{1.8}$$

then, from the Eq. (1.6) we can easily derive the Eq. (1.7). Here we will suppose that the both equations, (1.6) and (1.7) are known.

Examples of the case (1.5) are the quasi-orthogonal polynomials [8, 9, 10, 13]. They are defined by the expression

$$\widetilde{P}_n(x) = AP_n(x) + BP_{n-1}(x), \tag{1.9}$$

being A, B non-vanishing constants. It is known [9] that, if $\{P_n\}$ is a sequence of polynomials orthogonal with respect to a measure $d\mu$ supported in the real line, i.e.,

$$(P_n, P_m) = \int P_n(x)P_m(x)d\mu(x) = 0 \quad \forall n \neq m,$$

then, the corresponding quasi-orthogonal polynomials $\widetilde{P}_n(x) = AP_n(x) + BP_{n-1}(x)$ satisfy the relation

$$(\widetilde{P}_n, \widetilde{P}_m) = 0, \quad \text{if} \quad |m-n| > 1,$$

i.e., they constitute a quasi-orthogonal family of order 1 [9]. The quasi-orthogonal polynomials have been used to obtain quadrature and interpolating formulas (see e.g. [9, 15]).

We are interested to find the second order difference equation

$$\tilde{\sigma}(x;n) \bigtriangleup \bigtriangledown \tilde{P}_n(x) + \tilde{\tau}(x;n) \bigtriangleup \tilde{P}_n(x) + \tilde{\lambda}(x;n)\tilde{P}_n(x) = 0, \tag{1.10}$$

which satisfy these new polynomials $\widetilde{P}_n$ defined in (1.1) and (1.5), finding, explicitly, the coefficients $\widetilde{\sigma}$, $\widetilde{\tau}$ and $\widetilde{\lambda}$, respectively.

The paper is structured as follows. In sections 2 and 3 we describe two algorithms which allow us to find the SODE for the polynomials $\widetilde{P}_n$ defined by (1.1) (section 2) and for the polynomials defined by (1.5) (section 3). Finally, in section 4, two illustrative examples are worked out.

2 Second order linear difference equation for Krall-type polynomials.

Here we will describe an algorithm for finding the second order linear difference equation which satisfy the polynomial $\widetilde{P}_n$ defined by (1.1). We will prove the following theorem.

Theorem 2.1 *Suppose that the polynomials $\{\widetilde{P}_n\}$ are defined by (1.1) where the polynomial P_n is a solution of a SODE of the form (1.3). Then $\{\widetilde{P}_n\}$ satisfy a SODE of the form*

$$\tilde{\sigma}(x;n) \bigtriangleup \bigtriangledown \tilde{P}_n(x) + \tilde{\tau}(x;n) \bigtriangleup \tilde{P}_n(x) + \tilde{\lambda}(x;n)\tilde{P}_n(x) = 0, \tag{2.1}$$

where $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{\lambda}$ are given explicitly in (2.7).

Proof. We start from the fact that the family of polynomials $\{\widetilde{P}_n\}$ is expressed in terms of the other one $\{P_n\}$, which is a solution of the SODE (1.4), by formula (1.1)

$$q(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_n(x+1). \tag{2.2}$$

The idea is the following: Firstly, we write $\tilde{P}_n(x+1)$ and $\tilde{P}_n(x-1)$ in terms of the classical ones. To do this we evaluate (2.2) in $x \pm 1$ and then we use (1.3) to substitute the values $P_n(x-1)$ and $P_n(x+2)$. So, we obtain

$$\begin{aligned} &r(x;n)\tilde{P}_n(x+1) = c(x;n)P_n(x) + d(x;n)P_n(x+1), \\ &c(x;n) = -\sigma(x+1;n)\, b(x+1;n), \\ &d(x;n) = a(x+1;n)\psi(x+1;n) + b(x+1;n)\phi(x+1;n), \end{aligned} \tag{2.3}$$

and

$$s(x;n)\tilde{P}_n(x-1) = e(x;n)P_n(x) + f(x;n)P_n(x+1),$$

$$e(x;n) = \sigma(x;n)b(x-1;n) + a(x-1;n)\phi(x;n), \tag{2.4}$$

$$f(x;n) = -a(x-1;n)\psi(x;n).$$

Then, Eqs. (2.2–2.4) yield

$$\begin{vmatrix} q(x;n)\tilde{P}_n(x) & a(x;n) & b(x;n) \\ r(x;n)\tilde{P}_n(x+1) & c(x;n) & d(x;n) \\ s(x;n)\tilde{P}_n(x-1) & e(x;n) & f(x;n) \end{vmatrix} = 0\,, \tag{2.5}$$

where the functions q, a and b are given (1.1) as well as c, d, e, f, r and s in (2.3) and (2.4). Expanding the determinant in (2.5) by the first column we get

$$\tilde{\sigma}(x;n)\tilde{P}_n(x-1) + \tilde{\phi}_n(x)\tilde{P}_n(x) + \tilde{\psi}_n(x)\tilde{P}_n(x+1) = 0, \tag{2.6}$$

where

$$\tilde{\sigma}(x;n) = s(x;n)[a(x;n)d(x;n) - c(x;n)b(x;n)],$$

$$\tilde{\phi}_n(x) = q(x;n)[c(x;n)f(x;n) - e(x;n)d(x;n)], \tag{2.7}$$

$$\tilde{\psi}_n(x) = r(x;n)[e(x;n)b(x;n) - a(x;n)f(x;n)],$$

or, equivalently,

$$\tilde{\sigma}(x;n)\bigtriangleup\bigtriangledown\tilde{P}_n(x) + \tilde{\tau}(x;n)\bigtriangleup\tilde{P}_n(x) + \tilde{\lambda}(x;n)\tilde{P}_n(x) = 0, \tag{2.8}$$

where $\tilde{\tau}(x;n) = \tilde{\psi}(x;n) - \tilde{\sigma}(x;n)$ and $\tilde{\lambda}(x;n) = \tilde{\psi}(x;n) + \tilde{\sigma}(x;n) + \tilde{\phi}(x;n)$. ■

3 Second order linear difference equation for quasi-orthogonal-type polynomials.

Here we will describe an algorithm for finding the second order linear difference equation which satisfy the polynomials $\widetilde{P}_n$ defined by (1.5).

Theorem 3.1 *Suppose that the polynomials $\{\widetilde{P}_n\}$ are defined by (1.5) where the polynomial P_n satisfy the* difference-recurrence *relations (1.6) and (1.7). Then $\{\widetilde{P}_n\}$ satisfy a SODE of the form*

$$\tilde{\sigma}(x;n)\bigtriangleup^2\tilde{P}_n(x) + \tilde{\tau}(x;n)\bigtriangleup\tilde{P}_n(x) + \tilde{\lambda}(x;n)\tilde{P}_n(x) = 0, \tag{3.1}$$

where $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{\lambda}$ are given explicitly in (3.9).

<u>Proof.</u> We start from Eq. (1.5)

$$q(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P_{n-1}(x), \tag{3.2}$$

and we will use the properties (1.6)-(1.7) which the family $\{P_n\}$ satisfies. The main idea, which is a generalization to the one presented in [17], is the following. We apply the operator $\triangle$ in (3.2)

$$\triangle q(x;n)\widetilde{P}_n(x) + q(x+1;n) \triangle \widetilde{P}_n(x) =$$

$$= [\triangle a(x;n)]\, P_n(x) + a(x+1;n) \triangle P_n(x) + [\triangle b(x;n)]\, P_{n-1}(x) + b(x+1;n) \triangle P_{n-1}(x),$$

and then multiply it by $\sigma(x;n)\bar{\sigma}(x;n-1)q(x;n)$ and use (1.6) to eliminate the term $\triangle P_n(x)$, (1.7) (changing n by $n-1$) for $\triangle P_{n-1}(x)$, as well as (3.2) for the $\widetilde{P}_n$ polynomial. This allows us to rewrite the above equation as a combination of P_n and P_{n-1},

$$r(x;n) \triangle \widetilde{P}_n(x) = c(x;n)P_n(x) + d(x;n)P_{n-1}(x), \tag{3.3}$$

where

$$\begin{aligned}
r(x;n) &= \bar{\sigma}(x;n-1)\sigma(x;n)q(x;n)q(x+1;n),\\
c(x;n) &= \bar{\sigma}(x;n-1)\sigma(x;n)[a(x+1;n)q(x;n) - q(x+1;n)a(x;n)]+\\
&\quad +q(x;n)[a(x+1;n)\bar{\sigma}(x;n-1)\tau(x;n) + b(x+1;n)\sigma(x;n)\bar{\gamma}(x;n-1)],\\
d(x;n) &= \bar{\sigma}(x;n-1)\sigma(x;n)[b(x+1;n)q(x;n) - q(x+1;n)b(x;n)]+\\
&\quad +q(x;n)[a(x+1;n)\bar{\sigma}(x;n-1)\gamma(x;n) + b(x+1;n)\sigma(x;n)\bar{\tau}(x;n-1)].
\end{aligned} \tag{3.4}$$

Analogously, applying the operator $\triangle$ in (3.3) and repeating the same procedure we obtain

$$s(x;n) \triangle^2 \widetilde{P}_n(x) = e(x;n)P_n(x) + f(x;n)P_{n-1}(x), \tag{3.5}$$

where, now,

$$\begin{aligned}
s(x;n) &= \bar{\sigma}(x;n-1)\sigma(x;n)r(x;n)r(x+1;n),\\
e(x;n) &= \bar{\sigma}(x;n-1)\sigma(x;n)[c(x+1;n)r(x;n) - r(x+1;n)c(x;n)]+\\
&\quad +r(x;n)[c(x+1;n)\bar{\sigma}(x;n-1)\tau(x;n) + d(x+1;n)\sigma(x;n)\bar{\gamma}(x;n-1)],\\
f(x;n) &= \bar{\sigma}(x;n-1)\sigma(x;n)[d(x+1;n)r(x;n) - r(x+1;n)d(x;n)]+\\
&\quad +r(x;n)[c(x+1;n)\bar{\sigma}(x;n-1)\gamma(x;n) + d(x+1;n)\sigma(x;n)\bar{\tau}(x;n-1)].
\end{aligned} \tag{3.6}$$

As before, the expressions (1.5), (3.3) and (3.5) yield

$$\begin{vmatrix} q(x;n)\widetilde{P}_n(x) & a(x;n) & b(x;n)\\ r(x;n) \triangle \widetilde{P}_n(x) & c(x;n) & d(x;n)\\ s(x;n) \triangle^2 \widetilde{P}_n(x) & e(x;n) & f(x;n) \end{vmatrix} = 0\,, \tag{3.7}$$

where the functions q, a and b are known from expression (1.1), and c, d, e, f, r and s can be found from (3.4) and (3.6). Expanding the determinant in (3.7) by the first column

$$\widetilde{\sigma}_n(x) \triangle^2 \widetilde{P}_n(x) + \widetilde{\tau}_n(x) \triangle \widetilde{P}_n(x) + \widetilde{\lambda}_n(x)\widetilde{P}_n(x) = 0, \tag{3.8}$$

where

$$\widetilde{\sigma}_n(x) = s(x;n)\left[a(x;n)d(x;n) - c(x;n)b(x;n)\right],$$

$$\widetilde{\tau}_n(x) = r(x;n)[e(x;n)b(x;n) - a(x;n)f(x;n)], \tag{3.9}$$

$$\widetilde{\lambda}_n(x) = q(x;n)[c(x;n)f(x;n) - e(x;n)d(x;n)],$$

or, equivalently,

$$\widetilde{\sigma}(x-1;n)\widetilde{P}_n(x+1) + [\widetilde{\lambda}(x-1;n) - \widetilde{\tau}(x-1;n) + \widetilde{\sigma}(x-1;n)]\widetilde{P}_n(x) +$$

$$+[\widetilde{\tau}(x-1;n) - 2\widetilde{\sigma}(x-1;n)]\widetilde{P}_n(x-1) = 0.$$

■

In the special case of quasi-orthogonal polynomials (1.9), the equations (1.5), (3.3) and (3.5) can be rewritten in a more convenient form

$$q(x;n) = 1, \qquad a(x;n) = A, \qquad b(x;n) = B, \quad r(x;n) = \bar{\sigma}(x;n-1)\sigma(x;n),$$

$$c(x;n) = A\bar{\sigma}(x;n-1)\tau(x;n) + B\sigma(x;n)\bar{\gamma}(x;n-1),$$

$$d(x;n) = A\bar{\sigma}(x;n-1)\gamma(x;n) + B\sigma(x;n)\bar{\tau}(x;n-1).$$

$$s(x;n) = \bar{\sigma}(x;n-1)\sigma(x;n)\bar{\sigma}(x+1;n-1)\sigma(x+1;n),$$

$$e(x;n) = c(x+1;n)r(x;n) - r(x+1;n)c(x;n) + c(x+1;n)\bar{\sigma}(x;n-1)\tau(x;n) +$$

$$+d(x+1;n)\sigma(x;n)\bar{\gamma}(x;n-1),$$

$$f(x;n) = d(x+1;n)r(x;n) - r(x+1;n)d(x;n) + c(x+1;n)\bar{\sigma}(x;n-1)\gamma(x;n) +$$

$$+d(x+1;n)\sigma(x;n)\bar{\tau}(x;n-1).$$

Notice that the quasi-orthogonal polynomials of order k given by the linear combination [13]

$$\widetilde{P}_{n+k}(x) = \sum_{i=n}^{n+k} h_i P_i(x), \quad h_n h_{n+k} \neq 0, \quad h_i \text{ constant},$$

where P_n is an orthogonal family, can be written, by using the three-term recurrence relation (1.8) for the P_n in the form (1.5) where the coefficients $a(x;n)$ and $b(x;n)$ are polynomials in x of fixed degree k independent on n.

The two above described algorithms can be implemented, in a very simple way, in any computer algebra system.

4 Applications.

In this section we will apply the above algorithms to the Meixner-Krall and Quasi-orthogonal Meixner polynomials, respectively. To find the explicit expressions for the coefficients $\widetilde{\sigma}$, $\widetilde{\tau}$ and $\widetilde{\lambda}$ we have used *Mathematica* [16].

Meixner-Krall polynomials.

These polynomials satisfy the orthogonality relation (1) with $N=1$, $(a,b)=[0,\infty)$ and $\rho(x)=\frac{\mu^x\Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(x+1)}$, $\gamma>0$ and $0<\mu<1$. Then, Eq. (1.1) is given by [2] ($A\equiv A_1$)

$$M_n^{\gamma,\mu,A}(x)=M_n^{\gamma,\mu}(x)+B_n\bigtriangledown M_n^{\gamma,\mu}(x),\qquad B_n=A\frac{\mu^n(1-\mu)^{\gamma-1}(\gamma)_n}{n!\left(1+A\sum_{m=0}^{n-1}\frac{(\gamma)_m\mu^m(1-\mu)^\gamma}{m!}\right)}.$$

In this case,

$$\sigma(x;n)=x,\qquad \tau(x;n)=(\mu-1)x+\mu\gamma,\qquad \lambda(x;n)=(1-\mu)n,$$

and

$$q(x;n)=\sigma(x;n),\qquad a(x;n)=\sigma(x;n)[1+B_n]-B_n\phi(x;n),\qquad b(x;n)=-B_n\psi(x;n).$$

Then, Theorem 2.1 gives

$$\tilde\sigma(x;n)=(1-x)\left(B_n\,\gamma\,\mu-B_n\,n-{B_n}^2\,n+B_n\,\mu\,n+{B_n}^2\,\mu\,n-x-B_n\,x+B_n\,\mu\,x\right),$$

$$\begin{aligned}\tilde\tau(x;n)={}&-(\gamma\,\mu)-2\,B_n\,\gamma\,\mu+B_n\,\gamma\,\mu^2-B_n\,\gamma^2\,\mu^2+B_n\,n+{B_n}^2\,n-B_n\,\mu\,n-{B_n}^2\,\mu\,n+\\&+B_n\,\gamma\,\mu\,n+{B_n}^2\,\gamma\,\mu\,n-B_n\,\gamma\,\mu^2\,n-{B_n}^2\,\gamma\,\mu^2\,n+x+B_n\,x-\mu\,x-2\,B_n\,\mu\,x+\gamma\,\mu\,x+\\&+2\,B_n\,\gamma\,\mu\,x+B_n\,\mu^2\,x-2\,B_n\,\gamma\,\mu^2\,x-B_n\,n\,x-{B_n}^2\,n\,x+2\,B_n\,\mu\,n\,x+2\,{B_n}^2\,\mu\,n\,x-\\&-B_n\,\mu^2\,n\,x-{B_n}^2\,\mu^2\,n\,x-x^2-B_n\,x^2+\mu\,x^2+2\,B_n\,\mu\,x^2-B_n\,\mu^2\,x^2,\end{aligned}$$

$$\begin{aligned}\tilde\lambda(x;n)={}&(-1+\mu)\,n\,\left(1+2\,B_n+{B_n}^2-B_n\,\mu-{B_n}^2\,\mu+B_n\,\gamma\,\mu-B_n\,n-{B_n}^2\,n+\right.\\&\left.+B_n\,\mu\,n+{B_n}^2\,\mu\,n-x-B_n\,x+B_n\,\mu\,x\right),\end{aligned}$$

which coincide with the result given in [1]. This method also works in the case of the generalized (Krall-type) Kravchuk Charlier and Hahn polynomials [4, 5].

Quasi-orthogonal Meixner polynomials.

The Quasi-orthogonal Meixner polynomials are defined by

$$Q_n^{\gamma,\mu}(x)=AM_n^{\gamma,\mu}(x)+BM_{n-1}^{\gamma,\mu}(x),$$

and the classical Meixner polynomials satisfy the Eqs. [12]

$$(\mu x+\gamma\mu)\bigtriangleup M_n^{\gamma,\mu}(x)=n\mu M_n^{\gamma,\mu}(x)+\frac{n\mu(n-1+\gamma)}{1-\mu}M_{n-1}^{\gamma,\mu}(x),$$

and

$$(\mu x+\gamma\mu)\bigtriangleup M_n^{\gamma,\mu}(x)=\Big[(1-\mu)x-n-\mu\gamma\Big]M_n^{\gamma,\mu}(x)-(1-\mu)M_{n+1}^{\gamma,\mu}(x),$$

which are of type (1.6) and (1.7) where

$$\sigma(x;n)=\mu(x+\gamma),\quad \tau(x;n)=n\mu,\quad \gamma(x;n)=\frac{n\mu(n-1+\gamma)}{1-\mu},$$

and

$$\bar\sigma_n(x)=\mu(x+\gamma),\quad \bar\tau_n(x)=(1-\mu)x-n\gamma\mu,\quad \bar\gamma_n(x)=\mu-1.$$

Then, Theorem 3.1 gives

$$\begin{aligned}\tilde{\sigma}(x;n) = \; & A\mu\big(-A + A\mu + B\mu n\big)\big(\gamma + x\big) - \\ & -B\Big(B\mu^2 n\big(-1+\gamma+n\big)\big(\gamma+x\big)(1-\mu)^{-1} + A\mu\big(\gamma+x\big)\big(1-\gamma\mu-n+x-\mu x\big)\Big),\end{aligned}$$

$$\begin{aligned}\tilde{\tau}(x;n) = \; & \tfrac{1}{1-\mu}\big(\gamma+x\big)\big(1-\mu-\gamma\mu-n+\mu n+x-\mu x\big)\big(A^2+AB-2A^2\mu-AB\mu-AB\gamma\mu+A^2\mu^2+ \\ & +AB\gamma\mu^2-ABn-B^2\mu n+B^2\gamma\mu n+AB\mu^2 n+B^2\mu n^2+ABx-2AB\mu x+AB\mu^2 x\big),\end{aligned}$$

$$\begin{aligned}\tilde{\lambda}(x;n) = \; & n\big(\gamma+x\big)\big(-A^2-AB+2A^2\mu+AB\mu+AB\gamma\mu-A^2\mu^2-AB\gamma\mu^2+ABn+ \\ & +B^2\mu n-B^2\gamma\mu n-AB\mu^2 n-B^2\mu n^2-ABx+2AB\mu x-AB\mu^2 x\big).\end{aligned}$$

When $A = 1$ and $B = 0$, a straightforward calculation shows that the above equation transforms in the SODE

$$\mu(x+\gamma+1)\,\triangle^2 M_n^{\gamma,\mu}(x) + [(\mu-1)(x-n+1)+\mu\gamma]\,\triangle M_n^{\gamma,\mu}(x) + n(1-\mu)M_n^{\gamma,\mu}(x) = 0,$$

which is the SODE for the classical Meixner polynomials in the form (1.4). Obviously, the same can be done for the quasi-orthogonal Kravchuk [2], Charlier [2] and Hahn [3] polynomials, as well as, for the quasi-orthogonal polynomials of any order k.

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