

# The dual Hahn q-Polynomials in the lattice $x(s) = [s]_q[s+1]_q$ and the q-Algebras $SU_q(2)$ and $SU_q(1,1)$ .<sup>1</sup>

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## Abstract

The dual q-Hahn polynomials in the non uniform lattice  $x(s) = [s]_q[s+1]_q$  are obtained. The main data for these polynomials are calculated ( the square of the norm the coefficients of the three term recurrence relation, etc), as well as its representation as a q-hypergeometric series. The connection with the Clebsch-Gordan Coefficients of the Quantum Algebras  $SU_q(2)$  and  $SU_q(1,1)$  is also given.

## §1 Introduction

It is well known that the Lie Groups Representation Theory plays a very important role in the Quantum Theory and in the Special Function Theory. The group theory is an effective tool for the investigation of the properties of different special functions, moreover, it gives the possibility to unify various special functions systematically. In a very simple and clear way, on the basis of group representation theory concepts, the Special Function Theory was developed in the classical book of N.Ya.Vilenkin [1] and in the monography of N.Ya.Vilenkin and A.U.Klimyk [2], which have an encyclopedic character.

In recent years, the development of the quantum inverse problem method [3] and the study of solutions of the Yang-Baxter equations [4] gave rise to the notion of quantum groups and algebras, which are, from the mathematical point of view, Hopf algebras [5]. They are of great importance for applications in quantum integrable systems, in quantum field theory, and statistical physics (see [6] and references contained therein). They are attracting much attention in quantum physics, especially after the introduction of the q-deformed oscillator [7]-[8]. Also they have been used for the description of the rotational and vibrational spectra of deformed nuclei [9]-[11] and diatomic molecules [12]-[14], etc. However to apply them it is necessary to have a well developed theory of their representations. In quantum physics, for instance, the knowledge of the Clebsch-Gordan coefficients (3j symbols ), Racah coefficients ( 6j symbols ) and 9j symbols [15] is crucial for applications because all

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the matrix elements of the physical quantities are proportional to them.

The present work represents the definite part of the investigations about the connection between different constructions of the Wigner-Racah algebras for the q-groups and q-algebras  $SU_q(2)$  and  $SU_q(1,1)$  and the orthogonal polynomials of discrete variables (see also [16], **Vol. III** [2], [17], as well as [18]-[21], [22] and [23], [24]-[26]). For a review of q-polynomials see [24] and [26]. In [19] the properties of the Clebsch-Gordan coefficients (CGC's) of these two quantum algebras  $SU_q(2)$  and  $SU_q(1,1)$  and the q-analog of the Hahn polynomials on the exponential lattice  $x(s) = q^{2s}$  were considered in detail. In a similar way the Racah coefficients ( $6j$  symbols) for such q-algebras have been connected with the Racah polynomials in the lattice  $x(s) = [s]_q[s+1]_q$  [20]-[21]. Recently the q-analogs of the Kravchuk and Meixner polynomials on the non-uniform lattice  $x(s) = q^{2s}$  were investigated (see [23] and reference contained therein) in order to find the connection of them with the Wigner D-functions and Bargmann D-functions for the q-algebras  $SU_q(2)$  and  $SU_q(1,1)$  respectively.

To continue this line it seems reasonable to investigate the interrelation between the CGC's for the quantum algebras  $SU_q(2)$  and  $SU_q(1,1)$  with q-analogs of the dual Hahn polynomials on the non-uniform lattice  $x(s) = [s]_q[s+1]_q$ . In order to solve this problem in sections 2 and 3 we discuss the properties of these q-polynomials, their explicit formula and the representation in terms of the generalized q-hypergeometric functions  ${}_3F_2$  [24] is obtained. In section 4, from the detailed analysis of the finite difference equations (2) for these q-polynomials, we deduce the relation between them and the CGC's for  $SU_q(2)$ , which help us to draw an analogy between the basic properties of the Clebsch-Gordan coefficients and these orthogonal q-polynomials. Since these coefficients are studied from view point of the theory of orthogonal polynomials, a group-theoretical interpretation arises for the basic properties of dual Hahn q-polynomials. In section 5 we find the relation between Clebsch-Gordan coefficients for the quantum algebra  $SU_q(1,1)$  and the dual Hahn q-polynomials by two different ways, the first one - as in the previous case, i.e., comparing the finite difference equation for the dual Hahn q-polynomials and the corresponding recurrence relation for the CGC's, and the second one; using the well-known relation between the CGC's for the q-algebra  $SU_q(1,1)$  and the CGC's for  $SU_q(2)$ .

Using the connection between the CGC's and these q-polynomials (see below the formulas (19) and (24) ) we find explicit formulas for the CGC's, as well as their representation in terms of the generalized q-hypergeometric functions  ${}_3F_2$  or the basis hypergeometric series  ${}_3\varphi_2$  [27].

In the conclusion of this Section it should be noted that some new approach to the investigation of the connection between the representation theory of algebras and the theory of orthogonal polynomials was suggested recently [28]-[32]. It allows to solve also a new class of problems (so called *quasi-exactly solvable problems*). This approach was extended on the q-difference equation in [33]. In [34] it was shown that the similar approach can be formulated also to study the classical orthogonal polynomials in the exponential lattice  $x(s) = q^{2s}$  [24]. As for the quadratic lattice  $x(s) = s(s+1)$  and the q-quadratic lattice  $x(s) = [s]_q[s+1]_q$ , the extension of this approach on such type of problems is not found yet. Therefore we apply here the standard method of [24] to the analysis of the dual Hahn q-polynomials in the q-quadratic lattice.

## §2 The dual Hahn $q$ -polynomials in the non-uniform lattice

$$x(s) = [s]_q [s+1]_q.$$

Let us to start with the study of some general properties of orthogonal polynomials of a discrete variable in non-uniform lattices. Let be

$$\tilde{\sigma}(x(s)) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla Y(s)}{\nabla x(s)} + \frac{\tilde{\tau}(x(s))}{2} \left[ \frac{\Delta Y(s)}{\Delta x(s)} + \frac{\nabla Y(s)}{\nabla x(s)} \right] + \lambda Y(s) = 0, \quad (1)$$

$$\nabla f(s) = f(s) - f(s-1), \Delta f(s) = f(s+1) - f(s),$$

the finite difference equation of hypergeometric type for some lattice function  $x(s)$ , where  $\nabla f(s) = f(s) - f(s-1)$  and  $\Delta f(s) = f(s+1) - f(s)$  denote the backward and forward finite difference quotients, respectively. Here  $\tilde{\sigma}(x)$  and  $\tilde{\tau}(x)$  are polynomials in  $x(s)$  of degree at most 2 and 1, respectively, and  $\lambda$  is a constant. It is convenient (see [24] and [25]) to rewrite (1) in the equivalent form

$$\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla Y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta Y(s)}{\Delta x(s)} + \lambda Y(s) = 0, \quad (2)$$

$$\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2} \tilde{\tau}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)).$$

It is known ([24] and [25]) that for some special kind of lattices, solutions of (2) are orthogonal polynomials of a discrete variable, in other words, they satisfy the orthogonality relation

$$\sum_{s_i=a}^{b-1} P_n(x(s_i)) P_m(x(s_i)) \rho(s_i) \Delta x(s_i - \frac{1}{2}) = \delta_{nm} d_n^2, \quad s_{i+1} = s_i + 1, \quad (3)$$

where  $\rho(s)$  is some non-negative function (weight function), i.e.,

$$\rho(s_i) \Delta x(s_i - \frac{1}{2}) > 0 \quad (a \leq s_i \leq b-1),$$

supported in a countable set of the real line  $(a, b)$  and such that

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} [\sigma(x) \rho(x)] = \tau(x) \rho(x),$$

$$\sigma(s) \rho(s) x^k (s - \frac{1}{2})|_{s=a,b} = 0, \quad \forall k \in \mathbb{N} \quad (\mathbb{N} = \{0, 1, 2, \dots\}).$$

Here  $d_n^2$  denotes the square of the norm of the corresponding orthogonal polynomials.

They satisfy a three term recurrence relation (TTRR) of the form

$$x(s) P_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s), \quad P_{-1}(s) = 0, \quad P_0(s) = 1. \quad (4)$$

The polynomial solutions of equation (2), denoted by  $Y_n(x(s)) \equiv P_n(s)$ , are uniquely determined, up to a normalizing factor  $B_n$ , by the difference analog of the Rodrigues formula (see [24] page 66 Eq. (3.2.19))

$$P_n(s) = \frac{B_n}{\rho(s)} \nabla_n^{(n)} [\rho_n(s)], \quad \nabla_n^{(n)} = \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)} [\rho_n(s)], \quad (5)$$

where

$$x_m(s) = x\left(s + \frac{m}{2}\right), \quad \rho_n(s) = \rho(n+s) \prod_{k=1}^n \sigma(s+k).$$

These solutions correspond to some values of  $\lambda_n$  - the eigenvalues of equation (2).

Let us to start with the study of the dual Hahn  $q$ -polynomials in the particular non-uniform lattice  $x(s) = [s]_q [s+1]_q$  where  $[n]_q$  denotes the so called  $q$ -numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and  $q$  is, in general, a complex number  $|q| \neq 1$ .

We will use a result by Nikiforov, Suslov, Uvarov [24] (Theorem 1 page 59) who established that for the lattice functions  $x(s) = c_1 q^{2s} + c_2 q^{-2s} + c_3$ , where  $c_1$ ,  $c_2$  and  $c_3$  are some constants, equation (2) has a polynomial solution uniquely determined, up to a constant factor  $B_n$ , by (5). A simple calculation shows that our lattice  $x(s) = [s]_q [s+1]_q$  belongs to this class. In fact we have

$$x(s) = \frac{q}{(q - q^{-1})^2} q^{2s} + \frac{q^{-1}}{(q - q^{-1})^2} q^{-2s} - \frac{q + q^{-1}}{(q - q^{-1})^2}, \quad (6)$$

so for the lattice  $x(s) = [s]_q [s+1]_q$  it is possible to obtain polynomial solutions of the equation (2) (see Appendix), and these solutions are uniquely determined by the Rodrigues formula (5).

We are interested to construct the polynomials in such a way that, in the limit  $q \rightarrow 1$ , they and all their principal attributes ( $\sigma(s)$ ,  $\tau(s)$ ,  $\lambda_n$ ,  $\rho(s)$ ,  $d_n^2$ , TTRR coefficients  $\alpha_n, \beta_n, \gamma_n$ , etc. transform into the classical ones. These polynomials we will call *the  $q$ -analog of the classical dual Hahn polynomials in the non-uniform lattice*  $x(s) = [s]_q [s+1]_q$  and they will be denoted by  $W_n^{(c)}(s, a, b)_q$  (see also [25]). In order to obtain these  $q$ -polynomials let us define the  $\sigma(x(s))$  function such that in the limit  $q \rightarrow 1$  it coincides with the  $\sigma(s)$  for the classical polynomials, i.e.,

$$\lim_{q \rightarrow 1} \sigma(x(s)) = (s-a)(s+b)(s-c).$$

Therefore we will choose the function  $\sigma(s)$  as follows

$$\sigma(s) = q^{s+c+a-b+2} [s-a]_q [s+b]_q [s-c]_q. \quad (7)$$

Following Chapter III in [24] we can find *the main data* for the polynomials  $W_n^{(c)}(s, a, b)_q$ . The results of these calculations are provided in Table 2.1 (see also the Appendix). Everywhere,  $\forall x \in \mathbb{N}$ , by  $[x]_q!$  we denote the  $q$ -factorial which satisfies the relation  $[x+1]_q! = [x+1]_q [x]_q!$ , and coincides with the  $\tilde{\Gamma}_q(x)$  function introduced by Nikiforov et al. ([24]

page 67 Eq.(3.2.23)-(3.2.25)). In general  $\forall x \in \mathbb{R}$  the  $q$ -factorial is defined in terms of the standard  $\Gamma_q(x)$  (see [24] or [26]) by formula

$$\tilde{\Gamma}_q(x+1) = [x]_q! = q^{-x(x-1)/2} \Gamma_q(x+1).$$

It is clear that all characteristics of these  $q$ -polynomials coincide with the corresponding attributes for the classical dual Hahn polynomials (see [24] page 109 table 3.7.) in the limit  $q \rightarrow 1$ .

**Table 2.1.** Main Data for the  $q$ -analog of the Hahn polynomials  $W_n^c(s, a, b)_q$

$Y_n(s)$	$W_n^c(x(s), a, b)_q, \quad x(s) = [s]_q[s+1]_q$
$(a, b)$	$(a, b)$
$\rho(s)$	$\frac{q^{-s(s+1)}[s+a]_q![s+c]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-1]_q!}$ $-\frac{1}{2} \leq a \leq b-1, \quad  c  < a+1$
$\sigma(s)$	$q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q$
$\tau(s)$	$-x(s) + q^{a-b+c+1}[a+1]_q[b-c-1]_q + q^{c-b+1}[b]_q[c]_q$
$\lambda_n$	$q^{-n+1}[n]_q$
$B_n$	$\frac{(-1)^n}{[n]_q!}$
$\rho_n(s)$	$\frac{q^{-s(s+1+n) - \frac{n^2}{2} + n(a+c-b+\frac{3}{2})}[s+a+n]_q![s+c+n]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-n-1]_q!}$
$d_n^2$	$q^{-ab-bc+ac+a+c-b+1+2n(a+c-b)-n^2+5n} \frac{[a+c+n]_q!}{[n]_q![b-c-n-1]_q![b-a-n-1]_q!}$
$a_n$	$\frac{q^{-\frac{3}{2}n(n-1)}}{[n]_q!}$
$\alpha_n$	$q^{3n}[n+1]_q$
$\beta_n$	$q^{2n-b+c+1}[b-a-n+1]_q[a+c+n+1]_q +$ $+q^{2n+2a+c-b+1}[n]_q[b-c-n]_q + [a]_q[a+1]_q$
$\gamma_n$	$q^{n+3+2(c+a-b)}[n+a+c]_q[b-a-n]_q[b-c-n]_q$

### §3 The explicit formula for the dual Hahn q-polynomials in the lattice $x(s) = [s]_q[s+1]_q$ . The finite difference derivative formulas.

**The explicit formula for the dual Hahn q-polynomials.** In order to obtain the explicit formula for the q-polynomials  $W_n^{(c)}(s, a, b)_q$  we will use the Rodrigues Formula (5). Firstly, notice that for the lattice  $x(s) = [s]_q[s+1]_q$  verifies the relation

$$x(s) - x(s-i) = [i]_q \nabla x(s - \frac{i-1}{2}) = [i]_q [2s-i+1]_q$$

holds. Then, by induction we can find the following expression for the operator  $\nabla_n^{(n)}[f(s)]$

$$\nabla_n^{(n)}[f(s)] = \sum_{m=0}^n \frac{(-1)^{n+m} [n]_q! [2s-n+2m+1]_q}{[m]_q! [n-m]_q! \prod_{k=0}^m [2s+m+1-k]_q} f(s-n+m).$$

Thus, the Rodrigues Formula for the lattice  $x(s) = [s]_q[s+1]_q$  takes the form (see also [24] page 69, Eq. (3.2.30) )

$$P_n(s) = B_n \sum_{m=0}^n \frac{(-1)^{n+m} [n]_q! [2s-n+2m+1]_q}{[m]_q! [n-m]_q! \prod_{k=0}^m [2s+m+1-k]_q} \frac{\rho_n(s-n+m)}{\rho(s)}. \quad (8)$$

Now using the main data for the  $W_n^{(c)}(s, a, b)_q$  polynomials (Table 2.1), the equation (8) can be rewritten in the form

$$\begin{aligned} W_n^{(c)}(s, a, b)_q &= \\ &= \frac{[s-a]_q! [s+b]_q! [s-c]_q! [b-s-1]_q!}{q^{\frac{n^2}{2}-sn-n(a+c-b+\frac{5}{2})} [s+a]_q! [s+c]_q!} \sum_{m=0}^n \frac{(-1)^m [2s-n+2m+1]_q}{[m]_q! [n-m]_q! [2s+m+1]_q!} \times \\ &\times \frac{q^{-m^2-2sm+nm-m} [2s+m-n]_q! [s+a+m]_q! [s+c+m]_q!}{[s-a-n+m]_q! [s+b-n+m]_q! [s-c-n+m]_q! [b-s-m-1]_q!}. \end{aligned} \quad (9)$$

As a consequence of this representation we obtain the values of  $W_n^{(c)}(s = a, a, b)_q$  and  $W_n^{(c)}(s = b-1, a, b)_q$  at the ends of the interval of orthogonality  $(a, b)$

$$W_n^{(c)}(s = a, a, b)_q = \frac{(-1)^n q^{-\frac{n^2}{2}+n(c-b+\frac{3}{2})} [b-a-1]_q! [a+c+n]_q!}{[n]_q! [a+c]_q! [b-a-n-1]_q!}, \quad (10)$$

$$W_n^{(c)}(s = b-1, a, b)_q = \frac{q^{-\frac{n^2}{2}+n(c+a+\frac{3}{2})} [b-a-1]_q! [b-c-1]_q!}{[n]_q! [b-c-n-1]_q! [b-a-n-1]_q!}. \quad (11)$$

In order to find the representation of these polynomials in terms of q-Hypergeometric Functions we can follow [24] (Chapter 3, section 3.11.2, page 135). Using the corresponding

constants  $c_1$ ,  $c_2$  and  $c_3$  (6) for the non-uniform lattice  $x(s) = [s]_q[s+1]_q$  we obtain (see [24] Eq. 3.11.36 page 146) the following

$$W_n^{(c)}(x(s), a, b)_q = \frac{(a-b+1|q)_n(a+c+1|q)_n}{q^{n(s+\frac{1}{2}(n-1))-(c+a-b+1)}[n]_q!} \times \times {}_3F_2 \left( \begin{matrix} -n, a-s, a+s+1 \\ a-b+1, a+c+1 \end{matrix}; q, q^{(b-c-n)} \right), \quad (12)$$

where by definition

$${}_3F_2 \left( \begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k(a_2|q)_k(a_3|q)_k}{(b_1|q)_k(b_2|q)_k(q|q)_k} z^k,$$

and

$$(\alpha|q)_n = \prod_{k=0}^{n-1} [\alpha+k]_q = [\alpha]_q[\alpha+1]_q \dots [\alpha+n-1]_q = \frac{\tilde{\Gamma}_q(\alpha+n)}{\tilde{\Gamma}_q(\alpha)}.$$

**The finite difference derivative formulas for the dual Hahn q-polynomials.** To obtain the finite difference derivative formulas for these q-polynomials we will follow [24] (page 24, Eq.(2.2.9)). Firstly, notice the relation:

$$\frac{\Delta P_n(s - \frac{1}{2})}{\Delta x(s - \frac{1}{2})} = \frac{\tilde{B}_{n-1}}{\tilde{\rho}(s)} \nabla_{n-1}^{(n-1)} [\rho_{n-1}(s)] = \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \dots \frac{\nabla}{\nabla x_{n-1}(s)} [\tilde{\rho}_{n-1}(s)] = \tilde{P}_{n-1}(s),$$

where  $\tilde{B}_{n-1} = -\lambda_n B_n$ ,  $\tilde{\rho}(s) = \rho_1(s - \frac{1}{2})$ ,  $\tilde{\rho}_{n-1}(s) = \rho_n(s - \frac{1}{2})$ . In general, the polynomials  $\tilde{P}_{n-1}(s)$  in the right hand side of this equation, are not the same that  $P_n(s)$  (because they can have a different weight function). Since for the q-analog of dual Hahn polynomials in the lattice  $x(s) = [s]_q[s+1]_q$  the following connection between weight functions holds

$$\tilde{\rho}_{n-1}(s, a', b', c') = \rho_n(s - \frac{1}{2}, a, b, c) = \rho_{n-1}(s, a + \frac{1}{2}, b - \frac{1}{2}, c + \frac{1}{2}), \quad (13)$$

we conclude that  $\tilde{P}_{n-1}(s)$  coincides with the dual Hahn q-polynomial characterized by new parameters  $a' = a + \frac{1}{2}$ ,  $b' = b - \frac{1}{2}$  and  $c' = c + \frac{1}{2}$ . Then, we obtain the following formula for the *finite difference derivative*:

$$W_n^{(c)}(s + \frac{1}{2}, a, b)_q - W_n^{(c)}(s - \frac{1}{2}, a, b)_q = q^{-3n+3} [2s+1]_q W_{n-1}^{(c+\frac{1}{2})}(s, a + \frac{1}{2}, b - \frac{1}{2})_q. \quad (14)$$

The formula (14) will be called *the first differentiation formula* for the polynomials  $W_n^{(c)}(s, a, b)_q$ .

Now, if we change the parameters  $a, b$  and  $c$  and the variable  $s$  in  $\rho(s)$  by  $a' = a - \frac{1}{2}$ ,  $b' = b + \frac{1}{2}$ ,  $c' = c - \frac{1}{2}$ ,  $s' = s - \frac{1}{2}$ , we find

$$\tilde{\rho}_{n+1}(s - \frac{1}{2}, a', b', c') = q^{-2n+a+c-b+\frac{1}{4}} \rho_n(s, a, b, c). \quad (15)$$

Then, from the Rodrigues Formula (5)

$$\tilde{P}_{n+1}(s - \frac{1}{2}, a', b', c') = \frac{\tilde{B}_{n+1}}{\tilde{\rho}(s - \frac{1}{2}, a', b', c')} \nabla_{n+1}^{(n+1)} [\tilde{\rho}_{n+1}(s - \frac{1}{2}, a', b', c')].$$

and using Eq. (15) we obtain

$$\tilde{P}_{n+1}(s - \frac{1}{2}, a', b', c') = \frac{\tilde{B}_{n+1} q^{-2n+a+c-b+\frac{1}{4}} \nabla \rho(s, a, b, c) P_n(s)}{B_n \tilde{\rho}(s - \frac{1}{2}, a', b', c') \nabla x(s)}.$$

As in the previous case, we notice that in the left hand side of this equation the dual Hahn  $q$ -polynomials with  $a', b', c'$  parameters appear which are different from the corresponding parameters in the right hand side. Namely,  $a' = a - \frac{1}{2}, b' = b + \frac{1}{2}$  and  $c' = c - \frac{1}{2}$ . As a result the following formula for the finite difference derivative holds

$$q^{2n-a-c+b} [n+1]_q [2s]_q W_{n+1}^{(c-\frac{1}{2})}(s - \frac{1}{2}, a - \frac{1}{2}, b + \frac{1}{2})_q = \tag{16}$$

$$q^s [s-a]_q [s-c]_q [s+b]_q W_n^{(c)}(s-1, a, b)_q - q^{-s} [s+a]_q [s+c]_q [b-s]_q W_n^{(c)}(s, a, b)_q.$$

The formula (16) will be called *the second differentiation formula* for the polynomials  $W_n^{(c)}(s, a, b)_q$ .

#### §4 Clebsch-Gordan coefficients for the $q$ -algebra $SU_q(2)$ and the dual Hahn $q$ -polynomials.

The quantum algebra  $SU_q(2)$  is defined by three generators  $J_0, J_+$  and  $J_-$  with the following properties (see [35]-[36] and references therein)

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = [2J_0]_q,$$

$$J_0^\dagger = J_0, \quad J_{\pm}^\dagger = J_{\mp}.$$

Here we use the standard notation  $[A, B] = AB - BA$  for the commutators,  $[n]_q$  for  $q$ -numbers and  $[2J_0]_q$  means the corresponding infinite formal series. Let  $D^{J_1}$  and  $D^{J_2}$  be two irreducible representations (IR) of the algebra  $SU_q(2)$ . The tensor product of two irreducible representations  $D^{J_1} \otimes D^{J_2}$  can be decomposed into the direct sum of IR  $D^J$  components

$$D^{J_1} \otimes D^{J_2} = \sum_{J=|J_1-J_2|}^{J_1+J_2} \oplus D^J.$$

For the basis vectors of the IR  $D^J$  we have

$$|J_1 J_2, JM \rangle_q = \sum_{M_1, M_2} \langle J_1 M_1 J_2 M_2 | JM \rangle_q |J_1 M_1 \rangle_q |J_2 M_2 \rangle_q, \tag{17}$$

where a symbol  $\langle J_1 M_1 J_2 M_2 | JM \rangle_q$  denotes the Clebsch-Gordan coefficients (CGC) for the quantum algebra  $SU_q(2)$ . In [35]-[38] have been proved that these CGC's satisfy the following recurrence relation



$$\begin{aligned}
& \sqrt{\frac{[J-M]_q[J+M]_q[J_1+J_2+J+1]_q[J_2-J_1+J]_q[J-J_2+J_1]_q[J_1+J_2-J+1]_q}{[2J+1]_q[2J-1]_q[2J]_q^2}} \times \\
& \quad \times \langle J_1 M_1 J_2 M_2 | J - 1 M \rangle_q + \\
& - \frac{(q^J[J+M+1]_q - q^{-J}[J-M+1]_q)([2J]_q[2J_2+2]_q - [2]_q[J_2+J_1-J+1]_q[J+J_1-J_2]_q)}{[2J+2]_q[2J]_q[2]_q} \times \\
& \quad \times \langle J_1 M_1 J_2 M_2 | JM \rangle_q + \tag{18} \\
& + \sqrt{\frac{[J-M+1]_q[J+M+1]_q[J_1+J_2+J+2]_q[J_2-J_1+J+1]_q[J-J_2+J_1+1]_q[J_1+J_2-J]_q}{[2J+3]_q[2J]_q[2J+2]_q^2}} \times \\
& \quad \times \langle J_1 M_1 J_2 M_2 | J + 1 M \rangle_q + \\
& + \frac{(q^{J_2+M_1}[J_2+M_2+1]_q - q^{M_1-J_2}[J_2-M_2+1]_q)}{[2]_q} \langle J_1 M_1 J_2 M_2 | JM \rangle_q = 0.
\end{aligned}$$

Comparing the difference equation (2) for the q-analog of the dual Hahn polynomials  $W_n^{(c)}(s, a, b)$  in the non-uniform lattice  $x(s) = [s]_q[s+1]_q$  with the recurrence relation for CGC's, we conclude that CGC's  $\langle J_1 M_1 J_2 M_2 | JM \rangle_q$  can be expressed in terms of the dual Hahn q-polynomials by the formula

$$(-1)^{J_1+J_2-J} \langle J_1 M_1 J_2 M_2 | JM \rangle_q = \frac{\sqrt{\rho(s) \Delta x(s - \frac{1}{2})}}{d_n} W_n^{(c)}(x(s), a, b)_{q^{-1}}. \tag{19}$$

$$|J_1 - J_2| < M, n = J_2 - M_2, s = J, a = M, c = J_1 - J_2, b = J_1 + J_2 + 1.$$

Here  $\rho(x)$  and  $d_n$  denote the weight function and the normalization factor for the polynomials  $W_n^{(c)}(x(s), a, b)_{q^{-1}}$ , respectively. It should be noted that in (19) the parameter  $q$  is prescribed to the CGC  $\langle J_1 M_1 J_2 M_2 | JM \rangle_q$ , meanwhile the inverse parameter  $q^{-1}$  corresponds to the dual Hahn q-polynomial.

From the last expression and the orthogonality (3) of the  $W_n^{(c)}(s, a, b)$  polynomials the orthogonality of the CGC's follows, i.e.,

$$\sum_{J, M} \langle J_1 M_1 J_2 M_2 | JM \rangle_q \langle J_1 M'_1 J_2 M'_2 | JM \rangle_q = \delta_{M_1 M'_1} \delta_{M_2 M'_2}. \tag{20}$$

In the same way, we can show using (19) that the recursive relation (4) for the dual Hahn q-polynomials  $W_n^{(c)}(s, a, b)$  is equivalent to the the *recursive relation* in  $M_1$  and  $M_2$  for the CGC's [35]-[38]

$$\begin{aligned}
& q^{-2} \sqrt{[J_2 - M_2 + 1]_q [J_2 + M_2]_q [J_1 + M_1 + 1]_q [J_1 - M_1]_q} \langle J_1 M_1 + 1 J_2 M_2 - 1 | JM \rangle_q + \\
& \sqrt{[J_2 + M_2 + 1]_q [J_2 - M_2]_q [J_1 + M_1]_q [J_1 - M_1 + 1]_q} \langle J_1 M_1 - 1 J_2 M_2 + 1 | JM \rangle_q + \\
& \left( q^{-2M_1} [J_2 + M_2 + 1]_q [J_2 - M_2]_q + q^{2M_2} [J_1 + M_1 + 1]_q [J_1 - M_1]_q + [M + \frac{1}{2}]_q^2 - \right. \\
& \quad \left. - [J + \frac{1}{2}]_q^2 \right) q^{-M_2+M_1-1} \langle J_1 M_1 J_2 M_2 | JM \rangle_q = 0.
\end{aligned} \tag{21}$$

The phase factor  $(-1)^{J_1+J_2-J}$  in (19) was obtained by the comparison of the values of the  $W_n^{(c)}(s, a, b)$  polynomials at the ends of the interval of orthogonality (see (10) and (11)) with the corresponding values of the CGC's at  $J = M$  and  $J = J_1 + J_2 + 1$ . Using the relation (19) and the finite difference derivative formulas (14) and (16) we find the two recurrence relations for the CGC's:

$$\begin{aligned}
q^{-J-1} \sqrt{\frac{[J-M+1]_q [J_1+J_2+J+2]_q [J_2-J_1+J+1]_q [2J+2]_q}{[2J+3]_q [J_2-M_2]_q}} &< J_1 M_1 J_2 M_2 | J+1 M \rangle_q + \\
+ \sqrt{\frac{[J+M+1]_q [J_1+J_2-J]_q [J-J_2+J_1+1]_q [2J+2]_q}{[2J+1]_q [J_2-M_2]_q}} &< J_1 M_1 J_2 M_2 | J M \rangle_q = \\
= q^{(-J-J_2-M_2+M-1/2)} [2J+2]_q &< J_1 M_1 J_2 - \frac{1}{2} M_2 + \frac{1}{2} | J + \frac{1}{2} M + \frac{1}{2} \rangle_q
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
q^{-J} \sqrt{\frac{[J-M]_q [J_1+J_2+J+1]_q [J_2-J_1+J]_q [2J]_q}{[2J-1]_q [J_2-M_2+1]_q}} &< J_1 M_1 J_2 M_2 | J-1 M \rangle_q + \\
+ \sqrt{\frac{[J+M]_q [J_1+J_2-J+1]_q [J-J_2+J_1]_q [2J]_q}{[2J+1]_q [J_2-M_2+1]_q}} &< J_1 M_1 J_2 M_2 | J M \rangle_q = \\
= q^{(-J-J_2-M_2+M-1)/2} [2J]_q &< J_1 M_1 J_2 + \frac{1}{2} M_2 - \frac{1}{2} | J - \frac{1}{2} M - \frac{1}{2} \rangle_q
\end{aligned} \tag{23}$$

The formula (22)-(23) can be obtained independently using the  $q$ -analog of the Quantum Theory of Angular Momentum ([35] - [38]). Let  $T_{\mu}^{\frac{1}{2}}(2)$  be a tensor operator of rank  $\frac{1}{2}$  acting on the variables  $J_2, M_2$ . If we calculate the matrix element  $\langle J_1 M_1 J_2 M_2 | T_{\mu}^{\frac{1}{2}}(2) | J'_1 J'_2; J' M' \rangle_q$ , on the one hand, using the Wigner-Eckart theorem for  $SU_q(2)$  [35] we find that

$$\begin{aligned}
&< J_1 M_1 J_2 M_2 | T_{\mu}^{\frac{1}{2}}(2) | J'_1 J'_2; J' M' \rangle_q = \\
= -\delta_{J_1, J'_1} \sum_{M'_1 M'_2} &< J'_1 M'_1 J'_2 M'_2 | J' M' \rangle_q \frac{\langle J'_2 M'_2 \frac{1}{2} \mu | J_2 M_2 \rangle_q}{\sqrt{[2J_2+1]_q}} < J_2 || T_{\mu}^{\frac{1}{2}} || J'_2 \rangle_q .
\end{aligned}$$

On the other hand, the application of the algebra of tensor operators [38] gives

$$\begin{aligned}
&< J_1 M_1 J_2 M_2 | T_{\mu}^{\frac{1}{2}}(2) | J'_1 J'_2; J' M' \rangle_q = \\
= \sum_{J'' M''} &< J_1 M_1 J_2 M_2 | J'' M'' \rangle_q < J_1 J_2; J'' M'' | T_{\mu}^{\frac{1}{2}}(2) | J'_1 J'_2; J' M' \rangle_q = \\
= \sum_{J'' M''} &< J_1 M_1 J_2 M_2 | J'' M'' \rangle_q \frac{\langle J' M' \frac{1}{2} \mu | J'' M'' \rangle_q}{\sqrt{[2J''+1]_q}} (-1)^{J_1+J_2+J''+\frac{1}{2}} \times \\
&\times \sqrt{[2J''+1]_q [2J'+1]_q} \left\{ \begin{array}{ccc} J_1 & J'' & J_2 \\ \frac{1}{2} & J'_2 & J' \end{array} \right\}_q < J_2 || T_{\mu}^{\frac{1}{2}} || J'_2 \rangle_q .
\end{aligned}$$

Putting in both of these equations  $J' = J + \frac{1}{2}, M' = M + \frac{1}{2}, J'_2 = J_2 - \frac{1}{2}, J'_1 = J_1, M'_1 = M_1, M'_2 = M_2 + \frac{1}{2}, \mu = -\frac{1}{2}$  and taking into account that at such a choice of the angular

momenta and their projections we obtain that only the values  $M'' = M, J'' = J, J + 1$  are possible. From this fact the relation (22) follows.

To obtain the equation (23) we put  $J' = J - \frac{1}{2}, M' = M - \frac{1}{2}, J_2' = J_2 + \frac{1}{2}, J_1' = J_1, M_1' = M_1, M_2' = M_2 - \frac{1}{2}, \mu = \frac{1}{2}$ . All necessary quantities  $\left\{ \begin{array}{ccc} J_1 & J'' & J_2 \\ \frac{1}{2} & J_2' & J' \end{array} \right\}_q$  and  $\langle J_1 M_1 \frac{1}{2} \mu | JM \rangle_q$  are tabulated in [35] and [36], respectively.

From relation (19) we also see that the dual Hahn q-polynomial with  $n = 0$  corresponds to the CGC with the maximal value of the projection of the angular momentum  $J_2$ , i.e.,  $M_2 = J_2$ . For this reason it will be called *the backward* (we start from  $n = 0$  and obtain CGC at  $M_2 = J_2$ , for  $n = 1$  CGC at  $M_2 = J_2 - 1$ , and so on ). There exists another possibility corresponding to the inverse case, i.e., when the polynomial with  $n = 0$  is proportional to the CGC with the minimal value of  $M_2 = -J_2$ , this relation will be called *the forward way* (we start from  $n = 0$  and obtain CGC at  $M_2 = -J_2$ , when  $n = 1$  we find CGC at  $M_2 = -J_2 + 1$ , and so on ). In fact, comparing the difference equation for the q-analog of the dual Hahn polynomials  $W_n^{(c)}(s, a, b)$  (2) with the recurrence relation for CGC's, we conclude that CGC's  $\langle J_1 M_1 J_2 M_2 | JM \rangle_q$  can be also expressed in terms of the q-dual Hahn polynomials as follows

$$(-1)^{J_1+J_2-J} \langle J_1 M_1 J_2 M_2 | JM \rangle_q = \frac{\sqrt{\rho(s) \Delta x(s - \frac{1}{2})}}{d_n} W_n^{(c)}(x(s), a, b)_q, \quad (24)$$

$$|J_1 - J_2| \langle -M, n = J_2 + M_2, s = J, a = -M, c = J_1 - J_2, b = J_1 + J_2 + 1.$$

Here, as earlier,  $\rho(x)$  and  $d_n$  denote the weight function and the normalization factor for the polynomials  $W_n^{(c)}(x(s), a, b)_q$ , respectively.

Notice that if in the previous relation we provide the change of parameters  $M_1 = -M_1, M_2 = -M_2, M = -M$  and  $q = q^{-1}$  then, the right hand side of (24) coincides with the right hand side of (19). Then, we can conclude that for the CGC's the following *symmetry property* holds

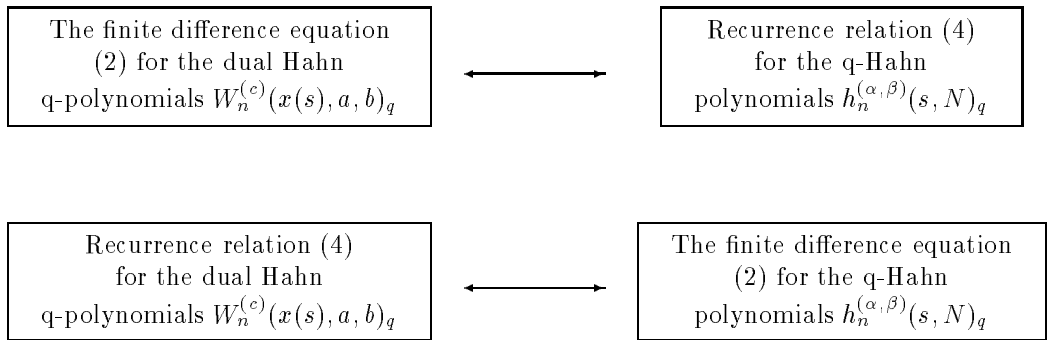
$$(-1)^{J_1+J_2-J} \langle J_1 - M_1 J_2 - M_2 | J - M \rangle_{q^{-1}} = \langle J_1 M_1 J_2 M_2 | JM \rangle_q. \quad (25)$$

To conclude this Section we provide a table in which the corresponding properties of the Hahn q-polynomials  $h_n^{(\alpha, \beta)}(s, N)_q$  defined on the exponential lattice  $q^{2s}$  [19] (see also [24] and [26]) and the dual Hahn q-polynomials  $W_n^{(c)}(x(s), a, b)_q$  defined on the lattice  $x(s) = [s]_q [s + 1]_q$  are compared with the corresponding properties for the CGC's of the q-algebra  $SU_q(2)$ . The last, help us to establish the interrelation between these two types of orthogonal q-polynomials.

**Table 4.1** CGC's and the q-analog of Hahn polynomials.

$P_n(s)_q$	$\langle J_1 M_1 J_2 M_2   J M \rangle_q$
Finite difference equation (2) for the $W_n^{(c)}(x(s), a, b)_q$ and TTRR (4) for $h_n^{(\alpha, \beta)}(s, N)_q$	Recurrence relation (18) for the CGC's
Finite difference equation (2) for the $h_n^{\alpha, \beta}(s, N)_q$ and TTRR (4) for $W_n^{(c)}(x(s), a, b)_q$	Recurrence relation (21) for the CGC's
$\frac{\rho(s)}{d_n^2}$ in (19)	$\langle J_1 M_1 J_2 J_2   J M \rangle_q^2$
$\frac{\rho(s)}{d_n^2}$ in (24)	$\langle J_1 M_1 J_2 - J_2   J M \rangle_q^2$
Differentiation formulas (14) and (16) for $W_n^{(c)}(x(s), a, b)_q$	Recurrence relation (22) and (23) for the CGC's
Equivalence of relation (19) and (24)	Symmetry property (25) for the CGC's
Orthogonality Relation (3)	Orthogonality Relations (20)

Comparing the finite difference equation and the TTRR which the polynomials  $h_n^{\alpha, \beta}(s, N)_q$  and  $W_n^{(c)}(x(s), a, b)_q$  satisfy we conclude



Moreover, since for the Hahn q-polynomials  $h_n^{(\alpha, \beta)}(s, N)_q$  in the exponential lattice  $x(s) = q^{2s}$  [19] the following relation holds (for the classical case see [24] and for q-case see [19])

$$(j_1 m_1 j_2 m_2 | j m)_{q^{-1}} = (-1)^s \sqrt{\frac{\rho(s) \Delta x(s - \frac{1}{2})}{d_n^2}} h_n^{\alpha, \beta}(s, N)_q$$

where  $s = j_2 - m_2$ ,  $N = j_1 + j_2 - m + 1$ ,  $\alpha = m - j_1 + j_2$ ,  $\beta = m + j_1 - j_2$ ,  $n = j - m$ , where  $\rho(x)$  and  $d_n$  denote the weight function and the normalization factor for the polynomials  $h_n^{(\alpha, \beta)}(s, N)_q$  given by formulas

$$\rho(x) = q^{\frac{1}{2}\alpha(\alpha+2N+2s-3)+\frac{1}{2}\beta(\beta+2s-1)} \frac{[\alpha + N - 2 - 1]![\beta + s]!}{[N - s - 1]![s]!}$$

$$d_n^2 = (q - q^{-1})^{2n} B_n^2 \frac{[n]![\alpha + n]![\beta + n]![\alpha + \beta + N + n]!}{[N - n - 1]![\alpha + \beta + n]![\alpha + \beta + 2n + 1]!} \\ \times q^{2\alpha+2N+N(N-1)+(N-1)(2\alpha+\beta+N)+\frac{1}{2}\beta(\beta+1)+n(\alpha+\beta+2)}$$

where  $B_n = (-1)^n \frac{1}{[n]!q^{2n}(q - q^{-1})^n}$ , we obtain the following relation between Hahn q-polynomials  $h_n^{(\alpha, \beta)}(s, N)_q$  and the dual Hahn q-polynomials  $W_n^{(c)}(x(s), a, b)_q$

$$(-1)^{s+n} q^{+(s-3)(\frac{\beta}{2}+\alpha)+3\alpha-2n\alpha-\frac{3}{2}(n+s)+(n-s)(N-\frac{\alpha+\beta}{2})} h_n^{(\alpha, \beta)}(s, N)_q = \\ = \frac{q^{2n} [s]_q! [N - s - 1]_q! [n + \beta]_q!}{[n]_q! [N - n - 1]_q! [s + \beta]_q!} W_s^{(\frac{\beta-\alpha}{2})} \left( t_n, \frac{\beta + \alpha}{2}, \frac{\beta + \alpha}{2} + N \right)_q \quad (26)$$

$$\left( t_n = s_n(s_n + 1) \quad s_n = \frac{\beta + \alpha}{2} + n \quad s, n = 0, 1, 2, \dots, N - 1 \right).$$

Observe that in the limit  $q \rightarrow 1$  this relation take the form of the classical relation between the classical Hahn and dual Hahn polynomials [24] (page 76, Eq.(3.5.14)).

## §5 The explicit formula for the CGC's. Its representation in terms of a basic hypergeometric function.

**The explicit formula for the Clebsch-Gordan coefficients of the  $SU_q(2)$  quantum algebra.** In order to obtain the explicit formula for the CGC  $\langle J_1 M_1 J_2 M_2 | JM \rangle_q$  we will use the explicit expression for the dual Hahn q-polynomials (9) and the Eq.(19), connecting them with CGC's. Providing some straightforward calculations we obtain the following general analytical formula to calculate the CGC's for the algebra  $SU_q(2)$

$$\langle J_1 M_1 J_2 M_2 | JM \rangle_q q^{-\frac{1}{2}(J(J+1)-J_1(J_1+1)+J_2(J_2+1)+(M+1)J_2+J(J_2-M_2))} = \\ = (-1)^{J_1+J_2-J} \sqrt{\frac{[J_2 - M_2]_q! [J_1 - M_1]_q! [J - M]_q! [J_2 + M_2]_q!}{[J + M]_q!}} \times \\ \times \sqrt{\frac{[J + J_1 + J_2 + 1]_q! [J_2 - J_1 + J]_q! [J_2 + J_1 - J]_q! [2J + 1]_q!}{[J_1 + M_1]_q! [J_1 - J_2 + J]_q!}} \times \\ \times \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2+2Jk-(J_2-M_2-1)k} [J + J_1 - J_2 + k]_q! [J + M + k]_q!}{[k]_q! [2J + 1 + k]_q! [J - M_1 - J_2 + k]_q! [J - J_1 + M_2 + k]_q!} \times \\ \times \frac{[2J - J_2 + M_2 + k]_q! [2J - J_2 + M_2 + 2k + 1]_q}{[J_2 - M - 2 - k]_q! [J + J_1 + M_2 + k + 1]_q! [J_1 + J_2 - J - k]_q!} \quad (27)$$

**Representation in terms of the basic hypergeometric function.** In order to find the representation of the CGC's in terms of q-Hypergeometric Functions we can use the representation of the dual Hahn q-polynomials (12). Then, from formula (19) we obtain

$$(-1)^{J_1+J_2-J} \langle J_1 M_1 J_2 M_2 | JM \rangle_{q^{-1}} = \frac{\sqrt{\rho(s)[2s+1]_q}}{d_n} \times \quad (28)$$

$$\times \frac{(a-b+1|q)_n (a+c+1|q)_n}{q^{n(s+\frac{1}{2}(n-1))-(c+a-b+1)} [n]_q!} {}_3F_2 \left( \begin{matrix} -n, a-s, a+s+1 \\ a-b+1, a+c+1 \end{matrix}; q, q^{(b-c-n)} \right)$$

where  $|J_1 - J_2| < M$ ,  $n = J_2 - M_2$ ,  $s = J$ ,  $a = M$ ,  $c = J_1 - J_2$ ,  $b = J_1 + J_2 + 1$  and  $\rho(x)$  and  $d_n$  denote, as usually, the weight function and the normalization factor for the polynomials  $W_n^{(c)}(x(s), a, b)_q$ .

## §6 Clebsch-Gordan coefficients for the q-algebra $SU_q(1, 1)$ and the dual Hahn q-polynomials.

In the previous sections we have studied the connection between dual Hahn q-polynomials and the CGC's of the  $SU_q(2)$  quantum algebra. Let us now to study the connection between the dual Hahn polynomials and the Clebsch-Gordan coefficients of the quantum algebra  $SU_q(1, 1)$  (for a survey see [2] and [39]). The quantum algebra  $SU_q(1, 1)$  is defined by three generators  $K_0$ ,  $K_+$  and  $K_-$  with the following properties [35]

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -[2K_0]_q,$$

$$K_0^{\dagger} = K_0, \quad K_{\pm}^{\dagger} = K_{\mp}.$$

Since this algebra is non-compact the Irreducible Representations (IR) can be classified in two series, the continuous and the discrete series of IR. In this work we will study the discrete case only, more concretely the *positive* discrete series  $D^{j+}$ . The basis vectors  $|jm \rangle_q$  of the IR  $D^{j+}$  can be found from the *lowest weight vector*  $|jj+1 \rangle$  ( $K_-|jj+1 \rangle = 0$ ) by the formula

$$|jm \rangle = \sqrt{\frac{[2j+1]_q!}{[j+m]_q! [m-j-1]_q!}} K_+^{m-j-1} |jj+1 \rangle.$$

Let  $D^{j_1+}$  and  $D^{j_2+}$  be two irreducible representations (IR) from the positive discrete series of the algebra  $SU_q(1, 1)$ . The tensor product of this two IRs,  $D^{j_1+} \otimes D^{j_2+}$ , can be decomposed into the direct sum of IRs  $D^{j+}$  components

$$D^{j_1+} \otimes D^{j_2+} = \sum_{j=j_1+j_2+1}^{\infty} \oplus D^{j+}.$$

For the basis vectors of the IR  $D^{j=}$  we have

$$|j_1 j_2, jm \rangle_q = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | jm \rangle_q |j_1 m_1 \rangle_q |j_2 m_2 \rangle_q, \quad (29)$$

where the symbols  $\langle j_1 m_1 j_2 m_2 | j m \rangle_q$  denote the Clebsch-Gordan coefficients (CGC) for the quantum algebra  $SU_q(1, 1)$ . In [19] it was proved that these CGC's satisfy the following recurrence relation

$$\begin{aligned} & \sqrt{[m_2 - j_2 - 1]_q [j_2 + m_2]_q [m_1 - j_1]_q [j_1 + m_1 + 1]_q} \langle j_1 m_1 + 1 j_2 m_2 - 1 | j m \rangle_q + \\ & q^2 \sqrt{[m_2 - j_2]_q [j_2 + m_2 + 1]_q [j_1 + m_1]_q [m_1 - j_1 - 1]_q} \langle j_1 m_1 - 1 j_2 m_2 + 1 | j m \rangle_q + \\ & \left( q^{-2m_1} [j_2 + m_2 + 1]_q [m_2 - j_2]_q + q^{2m_2} [j_1 + m_1 + 1]_q [m_1 - j_1]_q + \right. \\ & \left. + [j + \frac{1}{2}]_q^2 - [m + \frac{1}{2}]_q^2 \right) q^{-m_2 + m_1 + 1} \langle j_1 m_1 j_2 m_2 | j m \rangle_q = 0. \end{aligned} \quad (30)$$

Comparing the recurrence relation for the q-analog of the dual Hahn polynomials  $W_n^{(c)}(s, a, b)$  (4) with (30) for CGC's, we conclude that CGC's  $\langle j_1 m_1 j_2 m_2 | j m \rangle_q$  can be expressed in terms of the dual Hahn q- polynomials by the formula

$$(-1)^{m-j-1} \langle j_1 m_1 j_2 m_2 | j m \rangle_q = \frac{\sqrt{\rho(s) \Delta x(s - \frac{1}{2})}}{d_n} W_n^{(c)}(x(s), a, b)_{q^{-1}}, \quad (31)$$

$$n = m_1 - j_1 - 1, s = j, a = j_1 + j_2 + 1, c = j_1 - j_2, b = m.$$

We obtain the phase factor  $(-1)^{m-j-1}$  comparing the values of the  $W_n^{(c)}(s, a, b)$  polynomials at the ends of the interval (10) with the corresponding values of the CGC's. Now we can observe that if we provide the following substitution:

$$\begin{aligned} J_1 &= \frac{m+j_1-j_2-1}{2}, & M_1 &= \frac{m_1-m_2+j_1+j_2+1}{2}, & J &= j, \\ J_2 &= \frac{m-j_1+j_2-1}{2}, & M_2 &= \frac{m_2-m_1+j_1+j_2+1}{2}, & M &= j_1 + j_2 + 1, \end{aligned}$$

the right hand sides of the equations (19) and (31) become to be identical (see also [19] ). This imply that for the CGC's for these two quantum algebras the following relation holds

$$\langle J_1 M_1 J_2 M_2 | J M \rangle_{su_q(2)} = \langle j_1 m_1 j_2 m_2 | j m \rangle_{su_q(1,1)}. \quad (32)$$

Now we can obtain a general formula to calculate the CGC's for the  $SU_q(1, 1)$  algebra. Using the explicit expression for the q-analog of the dual Hahn polynomials (9) and the Eq.(31) we obtain

$$\begin{aligned}
& \langle j_1 m_1 j_2 m_2 | j m \rangle_q q^{-\frac{1}{2}(j(j+1)+j_1(j_1+1)-j_2(j_2+1))+(m-1)(j_1+1)+j(m_1-j_1-1)} = \\
& = (-1)^{m-j-1} \sqrt{\frac{[j+m]_q! [m-j-1]_q! [m_2+j_2]_q!}{[j_1+m_1]_q!}} \times \\
& \times \sqrt{\frac{[j-j_1-j_2-1]_q! [j_2-j_1+j]_q! [m_1-j_1-1]_q! [m_2-j_2-1]_q! [2j+1]_q!}{[j+j_1+j_2+1]_q! [j_1-j_2+j]_q!}} \times \quad (33) \\
& \times \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2+2jk-(m_1-j_1-2)k} [2j+j_1+1-m+k]_q! [j-1+j_2+j+k+1]_q!}{[k]_q! [2j+1+k]_q! [m_1-j_1-1-k]_q! [j-m_1-j_2+k]_q! [m-j-1-k]_q!} \times \\
& \times \frac{[j+j_1-j_2+k]_q! [2j+j_1-m_1+2k+2]_q}{[j-m_1+j_2+k]_q! [j+j_1+m_2+k+1]_q!}.
\end{aligned}$$

Using the formula (31) we find the following representation for the CGC of the  $SU_q(1, 1)$  quantum algebra in terms of the q-hypergeometric function:

$$(-1)^{m-j-1} \langle j_1 m_1 j_2 m_2 | j m \rangle_{q^{-1}} = \frac{\sqrt{\rho(s)[2s+1]_q}}{d_n} \times \quad (34)$$

$$\times \frac{(a-b+1|q)_n (a+c+1|q)_n}{q^{n(s+\frac{1}{2}(n-1))-(c+a-b+1)} [n]_q!} {}_3F_2 \left( \begin{matrix} -n, a-s, a+s+1 \\ a-b+1, a+c+1 \end{matrix}; q, q^{(b-c-n)} \right),$$

where  $n = m_1 - j_1 - 1$ ,  $s = j$ ,  $a = j_1 + j_2 + 1$ ,  $b = m$ ,  $c = j_1 - j_2$ . We remind that  $\rho(x)$  and  $d_n$  denote the weight function and the normalization factor for the polynomials  $W_n^{(c)}(x(s), a, b)_q$ , respectively.

To conclude this Section we want to remark that the same procedure can be applied to the negative discrete series of IR. Moreover, from the finite difference equation and the differentiation formulas (2), (14) and (16) we can obtain some new recurrence relations for the CGC's of the  $SU_q(1, 1)$  quantum algebra.

## Appendix. Calculation of the main data of the dual Hahn q-polynomials in the non-uniform lattice $x(s) = [s]_q [s+1]_q$

The dual Hahn q-polynomials are the polynomial solution of the second order finite difference equation of the hypergeometric type on the non-uniform lattice  $x(s) = [s]_q [s+1]_q$

$$\begin{aligned}
& \frac{q^{s+c+a-b+2} [s-a]_q [s+b]_q [s-c]_q}{[2s+1]_q} \Delta \left[ \frac{\nabla W_n^{(c)}(s, a, b)_q}{[2s-2]_q} \right] + \\
& \left\{ -[s]_q [s+1]_q + q^{c-b+1} [c]_q [b]_q + q^{a+c-b+1} [a+1]_q [b-c-1]_q \right\} \frac{\Delta W_n^{(c)}(s, a, b)_q}{[2s]_q} + \quad (35) \\
& + q^{-n+1} [n]_q W_n^{(c)}(s, a, b)_q = 0.
\end{aligned}$$



Since  $x(-s-1) = x(s)$  and  $\Delta x(s - \frac{1}{2}) = -\Delta x(t - \frac{1}{2})|_{t=-s-1}$  the coefficient  $\tau(s)$  in (35) is completely determined by the formula ([24], Equation (3.5.3), page 75))

$$\tau(s) = \frac{\sigma(-s-1) - \sigma(s)}{\Delta x(s - \frac{1}{2})}$$

The  $k$ -order finite difference derivative of the polynomials  $W_n^{(c)}(s, a, b)_q$  is defined as follows

$$v_{kn}(s) = \frac{\Delta}{\Delta x_{k-1}(s)} \frac{\Delta}{\Delta x_{k-2}(s)} \cdots \frac{\Delta}{\Delta x(s)} [W_n^{(c)}(s, a, b)_q] \equiv \Delta^{(k)} [W_n^{(c)}(s, a, b)_q],$$

where  $x_m(s) = x(s + \frac{m}{2})$ , satisfies the following equation of the same type

$$\frac{q^{s+c+a-b+2} [s-a]_q [s+b]_q [s-c]_q}{[2s+1-k]_q} \Delta \left[ \frac{\nabla v_{kn}(s)}{[2s-k-2]_q} \right] + \tau_n(s) \frac{\Delta v_{kn}(s)}{[2s-k]_q} + \mu_k v_{kn}(s) = 0, \quad (36)$$

where (see [24], page 62, Equation (3.1.19)). Furthermore, we have

$$\tau_k(s) = \frac{\sigma(s+k) - \sigma(s) + \tau(s+k) \Delta x(s + m - \frac{1}{2})}{\Delta x_{k-1}(s)}, \quad \mu_k = q^{-n+1} [n]_q + \sum_{m=0}^{k-1} \frac{\Delta \tau_m(x)}{\Delta x_m(x)}.$$

Thus, as a result, we obtain

$$\begin{aligned} \tau_k(s) = & -q^{2k} [s + \frac{k}{2}]_q [s + \frac{k}{2} + 1]_q + q^{c-b+k+1} [c + \frac{k}{2}]_q [b - \frac{k}{2}]_q + \\ & + q^{a+c-b+1-\frac{k}{2}} [a + \frac{k}{2} + 1]_q [b - c - k - 1]_q. \end{aligned} \quad (37)$$

The solution of the Pearson-type finite difference equation

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} [\sigma(x)\rho(x)] = \tau(x)\rho(x),$$

gives the weight function  $\rho(s)$

$$\rho(s) = \frac{q^{-s(s+1)} [s+a]_q! [s+c]_q!}{[s-a]_q! [s-c]_q! [s+b]_q! [b-s-1]_q!}.$$

Using the definition  $\rho_n(s) = \rho(n+s) \prod_{k=1}^n \sigma(s+k)$  (see (5)) we obtain

$$\rho_n(s) = \frac{q^{-s(s+1+n) - \frac{n^2}{2} + n(a+c-b+\frac{3}{2})} [s+a+n]_q! [s+c+n]_q!}{[s-a]_q! [s-c]_q! [s+b]_q! [b-s-n-1]_q!}. \quad (38)$$

Let us to find the squared normalization factor for the dual Hahn  $q$ -polynomials. Firstly, we use the formula ([24], section 3.2.2 page 64, Equation (3.7.15))

$$d_n^2 = q^{-\frac{3}{2}n^2 + \frac{3}{2}n} [n]_q! B_n^2 S_n, \quad (39)$$

where  $B_n = \frac{(-1)^n}{[n]_q!}$  and  $S_n$  is a sum

$$\sum_{s_i=a}^{b-n-1} \rho_n(s_i) \Delta x_n(s_i - \frac{1}{2}). \quad (40)$$

To calculate it we will use the identity ( $N = b - a - 1 \in \mathbb{N}$ )

$$S_n = \frac{S_n}{S_{n+1}} \frac{S_{n+1}}{S_{n+2}} \cdots \frac{S_{N-2}}{S_{N-1}} S_{N-1}. \quad (41)$$

From (38) and (40) we find

$$S_{N-1} = \frac{[a + c + N - 1]_q!}{[a - c]_q!} q^{-a^2 - aN - \frac{(N-1)^2}{2} + (N-1)(a+c-b+\frac{3}{2})}.$$

To obtain  $S_n$  we will follow [24] (page 105-106). Using the formulas, given at these pages, we find that  $\frac{S_n}{S_{n+1}} = \frac{1}{\sigma(x_{n-1}^*)}$ , where  $x_{n-1}^*$  is the solution of the equation  $\tau_{n-1}(x_{n-1}^*) = 0$ . Some straightforward but tedious algebra gives the following expression

$$\sigma(x_{n-1}^*) = q^{-2a+2c-2b+n+2} [a + c + n]_q [b - a - n]_q [b - c - n]_q.$$

Now, collecting the expressions (39), (40) and (41), we obtain that the squared normalization factor for the dual Hahn  $q$ -polynomials is equal to

$$d_n^2 = q^{-ab-bc+ac+a+c-b+1+2n(a+c-b)-n^2+5n} \frac{[a + c + n]_q!}{[n]_q! [b - c - n - 1]_q! [b - a - n - 1]_q!}$$

To obtain the leading coefficient  $a_n$  of the polynomial and the coefficients  $\alpha_n, \beta_n$  and  $\gamma_n$  of the three term recurrence relation (4) we use [24] (Equation (3.7.2), page 100) and formulas

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2},$$

$$\beta_n = -\frac{\alpha_n W_{n+1}^{(c)}(a, a, b)_q + \gamma_n W_{n-1}^{(c)}(a, a, b)_q}{W_n^{(c)}(a, a, b)_q} - x(a).$$

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