The q-analogue of the dual Hahn Polynomials: Its connection with $SU_q(2)$ and $SU_q(1,1)$ q-Algebras.

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Abstract

The dual q-Hahn polynomials in the non uniform lattice $x(s) = [s]_q[s+1]_q$ are obtained. The main data for these polynomials are calculated. Its connection with the Clebsch-Gordan Coefficients of the Quantum Algebras $SU_q(2)$ and $SU_q(1,1)$ is also given. Finally, some asymptotic formulas for the moments of the distribution of zeros are providing.

1 Introduction

It is well known that the close connection between q-polynomials and the representation theory of q-algebras exists (see [1],[2],[3]).The detail consideration of the interrelations between q-polynomials and quantum algebras seems to be valuable because of an important role that such algebras play in the modern physics (see [4] and references contained therein). This talk represents the part of such investigations. It is devoted to the construction of the dual Hahn q-polynomials in the non-uniform lattice $x(s) = [s]_q[s+1]_q$ and the study of their connection with the Clebsch-Gordan Coefficients (CGC's) of the Quantum Algebras $SU_q(2)$ and $SU_q(1,1)$ [5]. All main characteristics of such polynomials are calculated. In such a way it is demonstrated that the principal relations which they satisfy (i.e., the second order difference equation, the three term recurrence relation, the finite difference derivatives, etc) correspond to some properties of the CGC's of the $SU_q(2)$ q-algebra.The same connection can be established for the $SU_q(1,1)$ q-algebra. For more details see [5]. Finally from the three term recurrence relation we obtain some asymptotic formulas for the moments of the distribution of zeros of such polynomials by using the general method provided in [6].

2 The dual Hahn q-polynomials in the non-uniform lattice $x(s) = [s]_q[s+1]_q$.

The dual Hahn q-polynomials are the polynomial solution of the second order difference equation of hypergeometric type on the non-uniform lattice $x(s) = [s]_q[s+1]_q$

$$\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla Y(s)}{\nabla x(s)} + \tau(s)\frac{\triangle Y(s)}{\triangle x(s)} + \lambda Y(s) = 0, \tag{1}$$

$$\nabla f(s) = f(s) - f(s-1) \text{ and } \triangle f(s) = f(s+1) - f(s),$$

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where $\sigma(s)$ and $\tau(s)$ are given in Table I, $[s]_q = (q^s - 1/q^s)/(q - 1/q)$, q is a real number. They satisfy the orthogonality relation ($s_{i+1} = s_i + 1$) in the interval $[a, b-1]$:

$$\sum_{s_i=a}^{b-1} W_n^{(c)}(s_i, a, b)_q W_m^{(c)}(s_i, a, b)_q \rho(s_i) = \delta_{nm} d_n^2, \tag{2}$$

with the weight function $\rho(s)$. It is known that the polynomial solution of the second order difference equation of hypergeometric type on the non-uniform lattice $x(s) = [s]_q[s+1]_q$ are uniquely determined, up to a normalizing factor B_n, by the difference analog of the Rodrigues formula (see [2] and [3]). Follow [2] (Chapter III) we can find *the main data* for the polynomials $W_n^{(c)}(s, a, b)_q$. The results of these calculations are provided in Table I.

Table I. Main Data for the q-analog of the Hahn polynomials $W_n^c(s, a, b)_q$

$Y_n(s)$	$W_n^c(x(s), a, b)_q\,, \qquad x(s) = [s]_q[s+1]_q$
$\rho(s)$	$\dfrac{q^{-s(s+1)}[s+a]_q![s+c]_q!}{[s-a]_q![s-c]_q![s+b]_q![b-s-1]_q!}$ $-\frac{1}{2} \leq a \leq b-1, \qquad \lvert c\rvert < a+1$
$\sigma(s)$	$q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q$
$\tau(s)$	$-x(s) + q^{a-b+c+1}[a+1]_q[b-c-1]_q + q^{c-b+1}[b]_q[c]_q$
λ_n	$q^{-n+1}[n]_q$
B_n	$\dfrac{(-1)^n}{[n]_q!}$
d_n^2	$q^{-ab-bc+ac+a+c-b+1+2n(a+c-b)-n^2+5n} \dfrac{[a+c+n]_q!}{[n]_q![b-c-n-1]_q![b-a-n-1]_q!}$
a_n	$\dfrac{q^{-\frac{3}{2}n(n-1)}}{[n]_q!}$
α_n	$q^{3n}[n+1]_q$
β_n	$q^{2n-b+c+1}[b-a-n+1]_q[a+c+n+1]_q+$ $+q^{2n+2a+c-b+1}[n]_q[b-c-n]_q + [a]_q[a+1]_q$
γ_n	$q^{n+3+2(c+a-b)}[n+a+c]_q[b-a-n]_q[b-c-n]_q$

It is clear from this Table that these polynomials coincide with the usual dual Hahn polynomials , in the limit $q \to 1$ (cf. [2], Table 3.7 page 109). Here by $[x]_q!$ we denote the q-factorial which satisfy the relation $[x+1]_q! = [x+1]_q[x]_q!$.

3 The distribution of zeros of q-Dual Hahn polynomials.

In this section we provide the asymptotic behavior of the moments of zeros of the q-Dual Hahn polynomials $W_n^{(c)}(x(s),a,b)_q$ using a general method presented in [6]. To calculate the discrete density of zeros $\rho_N(x)$ of the polynomial $W_N^{(c)}(x(s),a,b)_q$, the knowledge of all its moments

$$\mu_m'^{(N)} = \int_a^b x^m \rho_N(x)\, dx, \qquad m = 0,1,2,...,N$$

is necessary. This method allows us to compute, asymptotically, the moments of the distribution of zeros around the origin $\mu_m = \frac{1}{n}\sum_{i=1}^{n} x_{n,i}^m$. A general formula to calculate these quantities was found in the Ref.6. We will apply this approach to obtain the general expression for the moments μ_m. The main idea is to use the three term recurrence relation for the coefficients of the monic orthogonal polynomials (i.e., $a_n = 1$). Follow to Ref.6 we find that for the moments of the q-Dual Hahn polynomials the following formula holds

$$\mu_m'^{(N)} = \sum_{(m)} m \prod_{i=1}^{j+1} \frac{(r_{i-1} + r_i' + r_i - 1)!}{(r_{i-1}-1)!\, r_i!\, r_i'!} \frac{q^{5m-3R}}{[q^{-2c}-1]^{m-2R}} \frac{q^{2(c+a-b)m}}{(q-q^{-1})^{2m}} \frac{q^{8m(N-t)}}{q^{4m}-1}, \qquad (3)$$

where the sumation $\sum_{(m)}$ runs over all partitions $(r_1', r_1, ..., r_{j+1}')$ of the number m such that: 1) $R' + 2R = m$, R and R' denote the sums $R = \sum_{i=1}^{j} r_i$ and by $R' = \sum_{i=1}^{j-1} r_i'$, 2) if $r_s = 0,\ 1 < s < j$, then $r_k = r_k' = 0$ for each $k > s$ and 3) $j = \frac{m}{2}$ or $j = \frac{m-1}{2}$ for m even or odd respectively. For the factorial coefficient F we take the convention $r_0 = r_p = 1$ $r_0 = r_p = 1$ (see [6])

Thus $\mu_m'^{(N)}$ depends on N. To have a *normalized density of zeros* we need to define a scaled density (see [6], Section 5). In our case we must use the scaled function

$$\rho_1^{++}(x) = \lim_{N\to\infty} \frac{q^m - 1}{q^{mN}-1} \rho_N\left(xq^{-7N}\right),$$

and then the corresponding moments are given by the expression ($m \geq 1$, $\mu_0^{++}(1) = 1$):

$$\mu_m^{++}(1) = \sum_{(m)} m \prod_{i=1}^{j+1} \frac{(r_{i-1} + r_i' + r_i - 1)!}{(r_{i-1}-1)!\, r_i!\, r_i'!} \frac{q^{5m-3R}}{[q^{-2c}-1]^{m-2R}} \frac{q^{2(c+a-b)m}}{(q-q^{-1})^{2m}} \frac{q^m-1}{q^{4m}-1}.$$

4 The connection with the Clebsch-Gordan coefficients for the q-algebras $SU_q(2)$ and $SU_q(1,1)$.

In this section we discuss the connection between the dual Hahn q-polynomials and the Clebsch-Gordan coefficients for the q-algebra $SU_q(2)$. For a review of the q-algebras $SU_q(2)$ see [1], [4], [5]. We suggest the following relation between dual q-Hahn polynomials and CGCs:

$$(-1)^{J_1+J_2-J} < J_1 M_1 J_2 M_2 | JM >_q = \frac{\sqrt{\rho(s) \bigtriangledown x(s-\frac{1}{2})}}{d_n} W_n^{(c)}(x(s),a,b)_{q^{-1}}. \qquad (4)$$

where in analogy with the Ref.2 we assume
$|J_1 - J_2| < M, n = J_2 - M_2, s = J, a = M, c = J_1 - J_2, b = J_1 + J_2 + 1$. If we substitute the expression (4) into the finite difference equation (1) the well known recurrent relation, connecting the CGCs with the total angular momenta $J, J+1$ and $J-1$, will be obtained. Similarly, from the properties of the dual q-Hahn polynomials it is possible to obtain a number of other properties of the CGC's.
The CGC's $< j_1 m_1 j_2 m_2 | jm >_q$ for the $SU_q(1,1)$ algebra, can be expressed in terms of the q-dual Hahn as follows

$$(-1)^{m-j-1} < j_1 m_1 j_2 m_2 | jm >_q = \frac{\sqrt{\rho(s) \nabla x(s-\frac{1}{2})}}{d_n} W_n^{(c)}(x(s), a, b)_{q^{-1}}, \tag{5}$$

where

$$n = m_1 - j_1, s = j, a = j_1 + j_2 + 1, c = j_1 - j_2, b = m$$

They are connected with the $SU_q(2)$ CGCs by the relation:

$$< J_1 M_1 J_2 M_2 | JM >_{su_q(2)} = < j_1 m_1 j_2 m_2 | jm >_{su_q(1,1)}, \tag{6}$$

where

$$\begin{array}{lll} J_1 = (m + j_1 - j_2 - 1)/2, & M_1 = (m_1 - m_2 + j_1 + j_2 + 1)/2, & J = j, \\ J_2 = (m - j_1 + j_2 - 1)/2, & M_2 = (m_2 - m_1 + j_1 + j_2 + 1)/2, & M = j_1 + j_2 + 1. \end{array}$$

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