

Linearization and connection problems for discrete hypergeometric polynomials

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Abstract

It is well known the importance of discrete polynomials not only for their mathematical nature but by their several applications in several branches of the actual sciences. The main aim of the present talk is to present a “relative simple” algorithm for solving the linearization problem involving certain families of discrete q -polynomials (this kind of problems can appear in some specific physical systems). Concrete examples of non-orthogonal families of Pochhammer and their q -analogues, as well as more complicated examples will be presented. Finally, comparison with other alternative approaches will be given.

1 Introduction

The expansion of any arbitrary discrete polynomial $q_m(x)$ in series of a general (albeit fixed) set of discrete hypergeometric polynomial $\{p_n(x)\}$ is a matter of great interest, solved only for some particular classical cases (for a review see [13, 18, 31] up to the middle of seventies and [9, 59, 62], since then up to now). This is particularly true for the deeper problem of linearization of a product of any two discrete polynomials. Usually, the determination of the expansion coefficients in these particular cases required a deep knowledge of special functions and, at times, ingenious induction arguments based in the three-term recurrence relation of the involved orthogonal polynomials [13, 14, 15, 16, 17, 20, 25, 27, 29, 30, 31, 38, 41, 42, 50, 54, 64, 65, 67]. Only recently, general and widely applicable strategies begin to appear [9, 10, 11, 12, 24, 33, 37, 40, 43, 45, 44, 47, 48, 49, 59, 61, 62, 63, 68].

One of the reasons for this increasing interest is the applications of such kind of problems in several branches of the Mathematics and Physics. For example, Gasper in his paper [31], write

The solution to many problems can be shown to depend on the determination of when a specific function is positive or nonnegative. ...

Sometime the problem can be reduced to a simpler one involving fewer parameters or it can be transformed into another problem that is easier to handle. For example, consider a two variable problem which consisting of proving

$$\sum_n a_n p_n(x) p_n(y) \geq 0, \quad (1.1)$$

where $p_n(x)$ is a sequence of functions and x and y satisfy appropriate restrictions. If there is an integral representation of the form

$$p_n(x) p_n(y) = \int p_n(z) d\mu_{x,y}(z), \quad d\mu_{x,y}(z) \geq 0,$$

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then the problem (1.1) can (at least formally) be reduced to the one variable problem

$$\sum_n a_n p_n(x) \geq 0,$$

... it may be possible to simplify the problems of the type

$$\int p_n(x) p_m(x) d\phi(x) \geq 0$$

by using formulas of the forms

$$p_n(x) p_m(x) = \sum_k a(k, m, n) P_k(x), \quad a(k, m, n) \geq 0, \quad (1.2)$$

$$p_k(x) = \sum_j b(j, k) q_j(x), \quad b(j, k) \geq 0, \quad (1.3)$$

...

Nine years after, one of the most famous conjecture: *The Bieberbach conjecture* ($|a_n| \leq n$) for analytic and univalent functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $|z| < 1$, has been solved by Louis de Branges using an inequality proved by Askey and Gasper in 1976 [19] (see [21] for more details)

$$\sum_{k=0}^n P_k^{\alpha,0}(t) = \frac{(\alpha+2)_n}{n!} {}_3F_2 \left(\begin{matrix} -n, n+\alpha+2, \frac{\alpha+2}{2} \\ \frac{\alpha+3}{2}, \alpha+1 \end{matrix} \middle| t \right) \geq 0, \quad 0 \leq t < 1, \quad \alpha > -2, \quad (1.4)$$

where $(a)_n$ is the Pochhammer symbol and $P_n^{\alpha,\beta}(x)$ denotes the Jacobi polynomials

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right).$$

Expansions of type (1.3) are usually called as connection or projection formulas while those of type (1.2) are referred to linearization formulas and the corresponding coefficients $b(j, k)$ and $a(k, m, n)$ are known as connection and linearization coefficients [13, 18]. Notice that, since the involved hypergeometric series in (1.4) is terminating, i.e., has a finite number of terms, the above problem can be considered as a connection problem between two families of polynomials where all the connection coefficients are positive (and equal to 1 in this example). So the Gasper's words about the importance in applications of the connection and linearization problems, and the positivity of the corresponding coefficients, become very actual and of interest.

Here in this work we will use a different notation for the connection c_{mn} and linearization c_{jmn} coefficients, i. e., the coefficients on the expansions [18]

$$q_m(x) = \sum_{n=0}^m c_{mn} p_n(x), \quad (1.5)$$

$$q_m(x) r_j(x) = \sum_{n=0}^{m+j} c_{jmn} p_n(x), \quad (1.6)$$

respectively, where $q_m(x)$ and $r_j(x)$ are any m th-degree and j th-degree polynomials, and $\{p_n\}$ denotes an arbitrary set of polynomials.

The first who considered the linearization problem for discrete polynomials (notice that in the de Branges's proof the "continuous" Jacobi polynomials have been used) was Eagleson in 1969

for Kravchuk polynomials [27]. Later on, Gasper [31] study the connection problem for the Hahn $h^{\alpha,\beta}(x, N)$ polynomials

$$h_j^{\gamma,\mu}(x, M) = \sum_{n=0}^j c_{jn} h_n^{\alpha,\beta}(x, N), \quad j \leq \min\{N-1, M-1\},$$

and completely solved it (the particular case $N = M$, of interest because $c_{jn} \geq 0$, he solved one year earlier in [30]), from where, by limiting process it is possible to obtain the connection coefficients for Jacobi polynomials as well as for other continuous and discrete families (see [30, 31] for further information on this). Some years later, Askey and Gasper [20] considered the linearization problem when the involved polynomials were the discrete polynomials of Hahn, Meixner Kravchuk and Charlier (for a review on discrete polynomials see [51, 52]) but only in the special case when all r_m , q_j and p_n belong to the same family with the same parameters (in [31] some preliminary results regarding to the positivity of such coefficients were discussed).

In all these cases, continuous and discrete, the proofs were based on very specific characteristic of the involved families, particularly their hypergeometric representation and generating functions have been exploited for finding the corresponding solution.

It is important to remark that, even in the case when it is possible to compute explicitly the connection or the linearization coefficients, not always is easy to show that they are nonnegative which were important as we already pointed out. This led to a recurrent method, i.e., to find a difference equation for the coefficients c_{mn} and c_{jmn} , respectively, and from it to deduce their non negativity. The first who did it was Hylleraas [38] in 1962 for a product of two Jacobi polynomials. In fact Hylleraas was able to solve the obtained recurrence relation for some special cases and prove the non negativity of the coefficients in some of these cases. Later, this method has been used by Askey and Gasper [13, 16, 17, 20] to prove the non negativity of the linearization coefficients for certain families of orthogonal polynomials.

More recently, Ronveaux, Zarzo, Area and Godoy [10, 33, 61], developed a recurrent method, called *NaViMa* algorithm, for solving the connection problem (1.5) for all families of classical polynomials, as well as some special kind of linearization problem and used it for solving different problems related with the associated, Sobolev-type polynomials, etc [34, 36, 60]. Although, they use it only for solving a very special linearization problem, it can be easily extended for solving the general problem (1.6) [24, 44]. Let us point out that there is a very similar algorithm for finding the recurrence relation for both, connection and linearization coefficients due to Lewanowicz [43, 45, 47]. The most important tool in the both aforesaid algorithms was the *structure relations* (or Salam-Chihara characterization) that the polynomials p_n in (1.5) and (1.6) satisfy.

Both problems, connection and linearization, are of great interest also in Physics. For example, for the 2^l -pole transitions in hydrogen-like atoms (and other related systems) the radial part of the probability is proportional to integrals of the form

$$T_l^{12} = \int_0^\infty [L_{n_1}^{2l_1+1}(\alpha_1 r) L_{n_2}^{2l_2+1}(\alpha_2 r)] r^m e^{-r} dr,$$

where L_n^l are the Laguerre polynomials. This kind of integrals also appears in the theory of Morse oscillators as well as in transitions for spherical-symmetric systems [54]. Furthermore, for spherical-symmetric the Wigner-Eckart theorem [28, 66] allows to write the matrix elements of certain irreducible operators in terms of products of two (or more) 3j symbols (Hahn and dual Hahn polynomials [52]), 6j symbols (Racah polynomials [52]), etc as well as their q -analogues.

To conclude this introduction we need to say that in the world of q -polynomials [22, 32, 39, 52, and reference contained therein] there are not so many results concerning to these problems. One

of the first who was interested on this was Rogers [57, 58] who used a q -analogue of the connection formula for Jacobi polynomials $P_n^{\gamma, \gamma}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} c_{j,n} P_{n-2j}^{\beta, \beta}(x)$, $c_{n,j} \geq 0$, for the q -ultraspherical polynomials to prove some Rogers-Ramanujan identities. Also, very recently, this problem has been considered in [7, 8, 46] for q -polynomials in the exponential lattice $x(s) = q^s$ [5, 53, 52], where the authors obtained recurrence relations for the coefficients in (1.5) and (1.6). Again, in these works the use of the structure relations plays a fundamental role. But not for any arbitrary family of q -polynomials there exist such relations. In [5] it is proven that all families of q -polynomials on the exponential lattice $x(s) = c_1 q^s + c_3$ satisfy such a relation, but for the general lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ [22, 52] the problem is still open. Then, the following question naturally arises: What to do in case when we do not have structure relations? This question was solved for the continuous case in [12, 63] and for the discrete case in [9].

2 The NAVIMA algorithm.

In this section we will describe a recurrent algorithm for finding the connection coefficients in the expansion (1.5) for classical polynomials.

This method uses the following properties of the classical polynomials:

1. A second order differential equation:

$$\sigma(x)p_n''(x) + \tau(x)p_n'(x) + \lambda_n p_n(x) = 0, \quad \deg \sigma \leq 2, \quad \deg \tau = 1, \quad (2.1)$$

2. A structure relation

$$\sigma(x)p_n'(x) = \tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x), \quad n \geq 0, \quad p_{-1} \equiv 0, \quad (2.2)$$

and a three-term recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x). \quad (2.3)$$

Also the q_m family satisfy equations of the same type

$$\bar{\sigma}(x)q_m''(x) + \bar{\tau}(x)q_m'(x) + \bar{\lambda}_m q_m(x) = 0, \quad \bar{\sigma} \leq 2, \quad \deg \bar{\tau} = 1, \quad (2.4)$$

$$\bar{\sigma}(x)q_m'(x) = \bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x), \quad m \geq 0, \quad q_{-1}(x) \equiv 0, \quad (2.5)$$

$$xq_m(x) = \bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x). \quad (2.6)$$

Let us describe the main idea of this method.

First of all, we apply the operator $\mathcal{L}_2 : \mathbb{P} \rightarrow \mathbb{P}$ defined by

$$\mathcal{L}_2[\pi(x)] = \bar{\sigma}(x) \frac{d^2 \pi(x)}{dx^2} + \bar{\tau}(x) \frac{d\pi(x)}{dx} + \bar{\lambda}_m \pi(x)$$

to both sides of (1.5). Since (2.4),

$$0 = \sum_{n=0}^m c_{mn} [\bar{\sigma}(x)p_n''(x) + \bar{\tau}(x)p_n'(x) + \bar{\lambda}_m p_n(x)].$$

Next, we multiply both sides by σ and use (2.1) and (2.2). This yields

$$0 = \sum_{n=0}^m c_{mn} \left\{ [\sigma(x)\bar{\lambda}_m - \bar{\sigma}(x)\lambda_n] p_n(x) - \bar{\sigma}(x)\tau(x)p_n'(x) + \bar{\tau}(x)[\tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x)] \right\}.$$

To eliminate the term p'_n , we again multiply by σ , and use the (2.2). Thus,

$$0 = \sum_{n=0}^m c_{mn} \left\{ \sigma(x) [\sigma(x)\bar{\lambda}_m - \bar{\sigma}(x)\lambda_n] p_n(x) + [\bar{\tau}(x)\sigma(x) - \bar{\sigma}(x)\tau(x)] [\tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x)] \right\}.$$

Now, taking into account that τ , $\bar{\tau}$, σ and $\bar{\sigma}$ are polynomials of first and second degree (at most), and using the recurrence relation (2.3) we obtain an expression of the form

$$0 = \sum_{n=0}^M F[c_{m0}, \dots, c_{mn}] p_n(x).$$

Since $\deg \sigma$ could be equal 2, then we obtain a recurrence of order 8 (at most):

$$\sum_{k=m-4}^{m+4} f[m, n, p_n, q_m] c_{mn} = 0.$$

Remark 1: Notice that to obtain the recurrence relation we have multiply two times by σ , which, obviously artificially increase the order of the recurrence.

Remark 2: Obviously, the same procedure can be applied to the discrete case, since there are the corresponding analogues of (2.1) and (2.2).

Remark 3: It is possible to get the minimal order for the recurrence relation if we use also the following relation for the classical polynomials p_n

$$p_n(x) = a_n p'_{n+1}(x) + b_n p'_n(x) + c_n p'_{n-1}(x).$$

This yields to a recurrence of order 4 instead of the above 8-th order one (see [10, 33]).

Remark 4: Notice that the algorithm remains valid if q_m is any polynomial satisfying a linear differential equation with polynomials coefficients. This implies that the above algorithm can be used for solving also the linearization problem (1.6) for classical orthogonal polynomials as it is pointed out in [33].

Before describing the q -analogue of the NAVIMA algorithm [8], we need to introduce some notations and definitions.

3 Properties of the q -polynomials.

Here we will summarize some of the properties of the q -polynomials [52] useful for the rest of the work.

Let us consider the *second order difference equation of hypergeometric type* for some lattice function $x(s)$,

$$\bar{\sigma}(x(s)) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \frac{\bar{\tau}(x(s))}{2} \left[\frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda y(s) = 0, \tag{3.1}$$

$$\nabla f(s) = f(s) - f(s - 1), \Delta f(s) = f(s + 1) - f(s),$$

where $\nabla f(s)$ and $\Delta f(s)$, denote the backward and forward finite difference derivatives, respectively, $\bar{\sigma}(x)$ and $\bar{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant.

Usually the above equation is written in the form [52, 51]:

$$\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad (3.2)$$

$$\sigma(s) = \bar{\sigma}(x(s)) - \frac{1}{2} \bar{\tau}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \bar{\tau}(x(s)).$$

Notice that $\sigma(s)$ and $\tau(s)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively. It is important to remark that the above difference equations have polynomial solutions of hypergeometric type iff $x(s)$ is a function of the form

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q), \quad (3.3)$$

where c_1, c_2, c_3 and $q^\mu = \frac{c_1}{c_2}$ are constants which, in general, depend on q [22, 52, 53].

In the especial case when $x(s) = s$, Eq. (3.1) becomes the classical second order difference equation of hypergeometric type for the uniform lattice:

$$\sigma(x) \nabla \Delta y(x) + \tau(x) \Delta y(x) + \lambda y(x) = 0, \quad (3.4)$$

Usually, the equation (3.2) is written in the compact or selfadjoint form

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[\sigma(s) \rho(s) \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda \rho(s) y(s) = 0, \quad (3.5)$$

where $\rho(s)$ is the solution of the Pearson-type difference equations

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} [\sigma(s) \rho(s)] = \tau(s) \rho(s) \quad (3.6)$$

The polynomial solutions of (3.2) is determined by the analogue of the Rodrigues Formula [52, page 66, Eq. (3.2.19)]

$$P_n(s)_q = \frac{B_n}{\rho(s)} \nabla^{(n)} [\rho_n(s)], \quad \nabla^{(n)} \equiv \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)}, \quad (3.7)$$

where the function $\rho_n(s)$ is given by

$$\rho_n(s) = \rho(s+n) \prod_{i=1}^n \sigma(s+i), \quad (3.8)$$

and

$$x_m(s) = x(s + \frac{m}{2}). \quad (3.9)$$

From the (3.7) as well as the expression [52, Eq. (3.2.28), page 68] gives

$$\nabla^{(n)} f(s) = \sum_{k=0}^n (-1)^{n-k} \frac{[n]_q!}{[k]_q! [n-k]_q!} \prod_{l=0}^n \frac{\nabla x(s+k - \frac{n-1}{2})}{\nabla x(s + \frac{k-l+1}{2})} f(s-n+k). \quad (3.10)$$

we can obtain an explicit expression for the polynomials $P_n(s)_q$

$$P_n(s)_q = B_n \sum_{m=0}^n \frac{[n]_q! (-1)^{m+n}}{[m]_q! [n-m]_q!} \frac{\nabla x(s+m - \frac{n-1}{2})}{\prod_{l=0}^n \nabla x(s + \frac{m-l+1}{2})} \frac{\rho_n(s-n+m)}{\rho(s)}, \quad (3.11)$$

which, with the help of (3.6) transforms [53]

$$\begin{aligned}
 P_n(s)_q &= B_n \sum_{m=0}^n \frac{[n]_q! (-1)^{m+n} \nabla x(s + m - \frac{n-1}{2})}{[m]_q! [n-m]_q! \prod_{l=0}^n \nabla x(s + \frac{m-l+1}{2})} \times \\
 &\times \prod_{l=0}^{n-m-1} [\sigma(s-l)] \prod_{l=0}^{m-1} [\sigma(s+l) + \tau(s+l) \Delta x(s+l - \frac{1}{2})].
 \end{aligned} \tag{3.12}$$

Here and throughout the paper $[n]_q$ denotes the so called q -numbers and $[n]_q!$ are the q -factorials

$$[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q.$$

These polynomial solutions $P_n(s)_q$ correspond to some values of λ_n [52, 53]

$$\lambda_n = -[n]_q \left\{ \frac{1}{2} (q^{n-1} + q^{-n+1}) \tilde{\tau}' + [n-1]_q \frac{\tilde{\sigma}''}{2} \right\}, \tag{3.13}$$

where (see Eq. (3.2)) $\tilde{\sigma}(s) = \frac{\tilde{\sigma}''}{2} x(s)^2 + \tilde{\sigma}'(0)x(s) + \tilde{\sigma}(0)$, and $\tilde{\tau}(s) \equiv \tau'x(s) + \tau(0)$.

Also for the difference derivatives $y_{kn}(s)_q$ of the polynomial solution $P_n(s)_q$, defined by

$$y_{kn}(s)_q = \frac{\Delta}{\Delta x_{k-1}(s)} \frac{\Delta}{\Delta x_{k-2}(s)} \cdots \frac{\Delta}{\Delta x(s)} [P_n(s)_q] \equiv \Delta^{(k)} [P_n(s)_q], \tag{3.14}$$

a Rodrigues-type formula holds

$$y_{kn}(s)_q = \Delta^{(k)} P_n(s)_q = \frac{A_{nk} B_n}{\rho_k(s)} \nabla_k^{(n)} [\rho_n(s)], \tag{3.15}$$

where the operator $\nabla_k^{(n)}$ is defined by

$$\nabla_k^{(n)} f(s) = \frac{\nabla}{\nabla x_{k+1}(s)} \frac{\nabla}{\nabla x_{k+2}(s)} \cdots \frac{\nabla}{\nabla x_n(s)} [f(s)],$$

and

$$\begin{aligned}
 A_{nk} &= \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left\{ \left(\frac{q^{\frac{1}{2}(n+m-1)} + q^{-\frac{1}{2}(n+m-1)}}{2} \right) \tilde{\tau}' + [n+m-1]_q \frac{\tilde{\sigma}''}{2} \right\} = \\
 &= \frac{[n]_q!}{[n-k]_q!} \frac{a_k}{B_k},
 \end{aligned} \tag{3.16}$$

where a_n denotes the leading coefficient of the polynomial P_n .

Of special interest are the “discrete” orthogonal q -polynomials, i.e., polynomials with a discrete orthogonality

$$\sum_{s_i=a}^{b-1} P_n(x(s_i))_q P_m(x(s_i))_q \rho(s_i) \Delta x(s_i - \frac{1}{2}) = \delta_{nm} d_n^2, \quad s_{i+1} = s_i + 1, \tag{3.17}$$

where $\rho(x)$ is a solution of the Pearson-type equation (3.6), and it is a non-negative function (weight function), i.e.,

$$\rho(s_i) \Delta x(s_i - \frac{1}{2}) > 0 \quad (a \leq s_i \leq b-1),$$

supported on a countable subset of the real line $[a, b]$ (a, b can be $\pm\infty$). The orthogonality relation (3.17) can be obtained from the difference equation (3.2), providing that the following boundary conditions

$$\sigma(s)\rho(s)x^k(s - \tfrac{1}{2}) \Big|_{s=a,b} = 0, \quad k = 0, 1, 2, \dots, \quad (3.18)$$

hold [52, 53], where the weight function $\rho(s)$ is a solution of the Pearson-type equation (3.6). Notice that the above boundary condition (3.18) is valid for $k = 0$. Moreover, if we assume that a is finite, then (3.18) is fulfilled at $s = a$ providing that $\sigma(a) = 0$ [52, §3.3, page 70]. In the following we will assume that this condition holds. The squared norm in (3.17) is given by [52, Chapter 3, Section 3.7.2, pag. 104]

$$d_n^2 = (-1)^n A_{nn} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \Delta x_n(s - \tfrac{1}{2}). \quad (3.19)$$

As a consequence of the orthogonality, the q -orthogonal polynomials satisfy the following three-term recurrence relations (TTRR)

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \quad (3.20)$$

with the initial conditions

$$P_{-1}(s)_q = 0, \quad P_0(s)_q = 1.$$

In the most general case, the solution of the q -hypergeometric equation (3.2) corresponds to the case

$$\begin{aligned} \sigma(s) &= A[s - s_1]_q[s - s_2]_q[s - s_3]_q[s - s_4]_q, \quad A = \text{const} \neq 0, \\ \sigma(s) + \tau(s) \Delta x(s - \tfrac{1}{2}) &= A[s - \bar{s}_1]_q[s - \bar{s}_2]_q[s - \bar{s}_3]_q[s - \bar{s}_4]_q. \end{aligned} \quad (3.21)$$

and has the form [53]

$$\begin{aligned} P_n(s)_q &= B_n \left(\frac{A}{c_1(q)q^{-\frac{\mu}{2}}\kappa_q^2} \right)^n (s_1 + s_2 + \mu|q)_n (s_1 + s_3 + \mu|q)_n \times \\ &\times (s_1 + s_4 + \mu|q)_n {}_4F_3 \left(\begin{matrix} -n, 2\mu + n - 1 + \sum_{i=1}^4 s_i, s_1 - s, s_1 + s + \mu \\ s_1 + s_2 + \mu, s_1 + s_3 + \mu, s_1 + s_4 + \mu \end{matrix} ; q, 1 \right), \end{aligned} \quad (3.22)$$

or

$$\begin{aligned} P_n(s)_q &= B_n \left(\frac{-A}{c_1(q)q^\mu\kappa_q^5} \right)^n q^{-\frac{n}{2}(3s_1+s_2+s_3+s_4+\frac{3(n-1)}{2})} (q^{s_1+s_2+\mu}; q)_n \times \\ &\times (q^{s_1+s_3+\mu}; q)_n (q^{s_1+s_4+\mu}; q)_n {}_4\varphi_3 \left(\begin{matrix} 2\mu+n-1+\sum_{i=1}^4 s_i \\ q^{-n}, q, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{matrix} ; q, q \right). \end{aligned} \quad (3.23)$$

where $\kappa_q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, the q -hypergeometric function ${}_pF_q$ and the basic hypergeometric serie ${}_p\varphi_q$ are defined by [4]

$${}_rF_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k (a_2|q)_k \cdots (a_r|q)_k}{(b_1|q)_k (b_2|q)_k \cdots (b_p|q)_k} \frac{z^k}{(1|q)_k} \left[\kappa_q^{-k} q^{\frac{1}{4}k(k-1)} \right]^{p-r+1}, \quad (3.24)$$

and

$${}_r\varphi_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right]^{p-r+1}, \quad (3.25)$$

respectively, and

$$(a|q)_k = \prod_{m=0}^{k-1} [a+m]_q, \quad (a; q)_k = \prod_{m=0}^{k-1} (1-aq^m). \quad (3.26)$$

Here also we will deal with the q -polynomials in the exponential lattice $x(s) = c_1q^s + c_3$. In this case the above representations transform [53]

$$\sigma(s) = \bar{A}(q^{s-\bar{s}_1} - 1)(q^{s-\bar{s}_2} - 1), \quad (3.27)$$

$$\sigma(s) + \tau(s) \triangle x(s - \frac{1}{2}) = \bar{A}(q^{s-\bar{s}_1} - 1)(q^{s-\bar{s}_2} - 1).$$

$$P_n(s)_q = \left(\frac{\bar{A}\kappa q}{c_1} \right)^n B_n q^{-\frac{n}{2}(\bar{s}_1 + \bar{s}_2 - \frac{n-1}{2})} (s_1 - \bar{s}_1|q)_n (s_1 - \bar{s}_2|q)_n \times \quad (3.28)$$

$$\times {}_3F_2 \left(\begin{matrix} -n, s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1, s_1 - s \\ s_1 - \bar{s}_1, s_1 - \bar{s}_2 \end{matrix} ; q, q^{\frac{1}{2}(s-s_2)} \right) =$$

$$= \left(\frac{-\bar{A}\kappa q}{c_1} \right)^n B_n q^{-\frac{n}{2}(s_1 + s_2 + \frac{n-1}{2})} (s_1 - \bar{s}_1|q)_n (s_2 - \bar{s}_1|q)_n \times \quad (3.29)$$

$$\times {}_3F_2 \left(\begin{matrix} -n, s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1, s - \bar{s}_1 \\ s_1 - \bar{s}_1, s_2 - \bar{s}_1 \end{matrix} ; q, q^{\frac{1}{2}(s-\bar{s}_2)} \right),$$

or, in terms of the basic hypergeometric series [53]

$$P_n(s)_q = B_n \left(\frac{\bar{A}}{c_1(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \right)^n q^{-\frac{n(n-1)}{4} - ns_1} (q^{s_1 - \bar{s}_1}; q)_n (q^{s_1 - \bar{s}_2}; q)_n \times \quad (3.30)$$

$$\times {}_3\varphi_2 \left(\begin{matrix} q^{-n}, q^{s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1}, q^{s_1 - s} \\ q^{s_1 - \bar{s}_1}, q^{s_1 - \bar{s}_2} \end{matrix} ; q, q^{s - s_2 + 1} \right) =$$

$$= B_n \left(-\frac{\bar{A}}{c_1(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \right)^n q^{-\frac{3n(n-1)}{4} - n(s_1 + s_2 - \bar{s}_1)} (q^{s_1 - \bar{s}_1}; q)_n \times \quad (3.31)$$

$$\times (q^{s_2 - \bar{s}_1}; q)_n {}_3\varphi_2 \left(\begin{matrix} q^{-n}, q^{s_1 + s_2 - \bar{s}_1 - \bar{s}_2 + n - 1}, q^{s - s_1} \\ q^{s_1 - \bar{s}_1}, q^{s_2 - \bar{s}_1} \end{matrix} ; q, q^{s - s_2 + 1} \right).$$

Finally, the following relations holds for the q -polynomials in the exponential lattice $x(s) = c_1q^s + c_3$.

1. The *first structure relation*

$$\sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} = \tilde{S}_n P_{n+1}(s)_q + \tilde{T}_n P_n(s)_q + \tilde{R}_n P_{n-1}(s)_q, \quad (3.32)$$

2. The *second structure relation*

$$[\sigma(s) + \tau(s) \triangle x(s - \frac{1}{2})] \frac{\Delta P_n(s)_q}{\Delta x(s)} = S_n P_{n+1}(s)_q + T_n P_n(s)_q + R_n P_{n-1}(s)_q, \quad (3.33)$$

Table 1: Main data for the q -Charlier polynomials in the lattice $x(s) = \frac{q^s-1}{q-1}$.

$P_n(s)_q$	$c_n^{(\mu)}(s)_q, \quad x(s) = \frac{q^s-1}{q-1}$
(a, b)	$[0, \infty)$
$\rho(s)$	$\frac{\mu^s}{e_q[(1-q)\mu]\Gamma_q(s+1)}, \quad \mu > 0, \quad 0 < (1-q)\mu < 1$
$\sigma(s)$	$q^s x(s)$
$\tau(s)$	$\mu q^{\frac{3}{2}} - q^{\frac{1}{2}} x(s)$
$\tau_n(s)$	$-q^{-n+\frac{1}{2}} x_n(s) + \mu q^{\frac{n}{2}+\frac{3}{2}} + q^{-\frac{3}{4}n} [\frac{n}{2}]_q$
λ_n	$[n]_q q^{-\frac{(n-2)}{2}}$
B_n	$\frac{1}{\mu^n}$
d_n^2	$(\{1-q\}\mu; q)_{n+1} \frac{[n]_q!}{q^{\frac{n}{4}(n-\tau)+\frac{1}{2}} \mu^n} = \frac{e_q[(1-q)q^{n+1}\mu]}{e_q[(1-q)\mu]} \frac{[n]_q!}{q^{\frac{n}{4}(n-\tau)+\frac{1}{2}} \mu^n}$
$\rho_n(s)$	$\frac{\mu^{s+n} q^{\frac{n}{2}(n+2s+1)}}{e_q[(1-q)\mu]\Gamma_q(s+1)}$
a_n	$\frac{(-1)^n}{\mu^n} q^{-\frac{3n}{4}(n-1)+\frac{n}{2}}$

3. A difference-recurrence relation

$$P_n(s)_q = L_n \frac{\Delta P_{n+1}(s)_q}{\Delta x(s)} + M_n \frac{\Delta P_n(s)_q}{\Delta x(s)} + N_n \frac{\Delta P_{n-1}(s)_q}{\Delta x(s)}, \quad (3.34)$$

where $\tilde{S}_n, S_n, \tilde{T}_n, T_n, \tilde{R}_n, R_n, L_n, M_n$ and N_n are known constants [5].

The q -Charlier polynomials on the exponential lattice.

The q -analogue of the Charlier polynomials in the exponential lattice $x(s) = \frac{q^s-1}{q-1}$ defined by

$$\begin{aligned} c_n^{(\mu)}(s, q) &= q^{\frac{n}{4}(n+5)} {}_2\varphi_0 \left(\begin{matrix} q^{-n}, q^{-s} \\ - \end{matrix}; q, -\frac{q^s}{(q-1)\mu} \right) = \\ &= q^{\frac{n}{4}(n+5)} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{1}{\mu^k} (s)_q^{[k]}, \quad 1 < q < 1, \quad 0 < \mu < 1 \end{aligned} \quad (3.35)$$

Obviously, the q -Charlier polynomials $c_n^{(\mu)}(s, q)$ are polynomials of degree n on any exponential lattice $x(s) = c_1 q^s + c_3$. We have chosen $c_1 = -c_3 = 1/(q-1)$ in order to have $\lim_{q \rightarrow 1} x(s) = s$, i.e., the linear lattice [5]. Their main data can be found in [5].

3.1 The q -analog of the Pochhammer symbols.

Let us define the quantities $(s)_q$ by

$$(s)_q = \frac{q^s - 1}{q - 1} = q^{\frac{s}{2}-1} [s]_q, \quad (3.36)$$

Table 1: Main data for the q -Charlier polynomials in the lattice $x(s) = \frac{q^s-1}{q-1}$ (cont).

$P_n(s)_q$	$c_n^{(\mu)}(s)_q, \quad x(s) = \frac{q^s-1}{q-1}$
α_n	$-\mu q^{\frac{3}{2}n - \frac{1}{2}}$
β_n	$\mu q^{2n+1} + [n]_q \{1 - \mu(1-q)q^n\} q^{\frac{n}{2}-1}$
γ_n	$-q^n [n]_q \{1 - \mu(1-q)q^n\}$
\tilde{S}_n	$\mu q^{\frac{1}{2}(n+1)} (1 - q^n)$
\tilde{T}_n	$[n]_q q^{\frac{n}{2}} \{1 - \mu(1-q)q^n\} - \mu q^{n+2} (1 - q^n)$
\tilde{R}_n	$-q^{n+1} [n]_q \{1 - \mu(1-q)q^n\}$
S_n	0
T_n	$[n]_q q^{-\frac{n}{2}} \left(1 - q^{-\frac{1}{2}} - \mu q^n (1 - q)\right)$
R_n	$-q [n]_q \{1 - \mu(1-q)q^n\}$
L_n	$-\frac{\mu q^{n-\frac{1}{2}}}{[n+1]_q}$
M_n	$\frac{\mu q^{\frac{n}{2}} (q^{n+1} - 1)}{[n+1]_q}$
N_n	0

and let $[(s)_q]_n$, the q -Pochhammer symbol, be defined by

$$[(s)_q]_n = (s)_q (s+1)_q \cdots (s+n-1)_q = \prod_{k=0}^{n-1} \frac{q^{s+k} - 1}{q - 1}. \quad (3.37)$$

Notice that $[(s)_q]_n$ is a polynomial of degree exactly equal n in q^s . The polynomials $[(s)_q]_n$ satisfy the following difference equation

$$(s)_q [(s+1)_q]_n - (s+n)_q [(s)_q]_n = 0, \quad (3.38)$$

and a recurrence relation

$$(s)_q [(s)_q]_n - q^{-n} [(s)_q]_{n+1} + q^{-n} (n)_q [(s)_q]_n = 0. \quad (3.39)$$

Notice also that

$$[(s)_q]_n = \frac{(q^s; q)_n}{(1-q)^n}, \quad \text{where } (a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k). \quad (3.40)$$

The polynomials $(q^s; q)_n$ satisfy the following difference equation on the exponential lattice $x(s) = c_1 q^s + c_3$

$$\frac{\Delta(q^s; q)_n}{\Delta x(s)} = -q^{\frac{n-1}{2}} [n]_q c_1^{-1} (q^{s+n}; q)_{n-1}. \quad (3.41)$$

Notice that $[(s)_q]_n$ is a polynomial of degree exactly equal n in q^s , and that $\lim_{q \rightarrow 0} [(s)_q]_n = (s)_n$ is the classical Pochhammer symbol $(s)_n = (s)(s+1) \cdots (s+n-1)$.

Let us define the q -Stirling polynomials or q -falling factorials $(s)_q^{[n]}$, by

$$(s)_q^{[n]} = (s)_q(s-1)_q \cdots (s-n+1)_q = \prod_{k=0}^{n-1} \frac{q^{s-k} - 1}{q - 1} \quad (3.42)$$

Also we will use the notation

$$(s)_q^{[n]} = \frac{(q^s; q)_q^{[n]}}{(1-q)^n}, \quad (a; q)_q^{[n]} = (1-a)(1-aq^{-1}) \cdots (1-aq^{-n+1}). \quad (3.43)$$

These quantities $(s)_q^{[n]}$ are closely related to the q -Stirling numbers $\tilde{S}_q(n, k)$, $s_q^*(n, k)$ by formulas

$$(s)_q^n = \sum_{k=0}^n \tilde{S}_q(n, k) (s)_q^{[k]}, \quad (s)_q^{[n]} = \sum_{k=0}^n s_q^*(n, k) (s)_q^k, \quad (3.44)$$

and satisfy the following difference equation

$$(s)_q(s-1)_q^{[n]} - (s-n)_q(s)_q^{[n]} = 0, \quad (3.45)$$

as well as the recurrence relation

$$(s)_q(s)_q^{[n]} - q^n(s)_q^{[n+1]} - (n)_q[(s)_q]^{[n]} = 0. \quad (3.46)$$

Notice that $(a; q)_q^{[n]}$ satisfies the difference equation on the exponential lattice $x(s) = c_1 q^s + c_3$

$$\frac{\Delta(q^s; q)_q^{[n]}}{\Delta x(s)} = -q^{-\frac{n-1}{2}} [n]_q c_1^{-1} (q^s; q)_q^{[n-1]}. \quad (3.47)$$

3.2 The discrete case.

The most general polynomial solution of the hypergeometric difference equation (3.4) corresponds to the case

$$\sigma(x) = A(x-x_1)(x-x_2), \quad \sigma(x) + \tau(x) = A(x-\bar{x}_1)(x-\bar{x}_2).$$

Without loss of generality we will consider the case $A = -1$ and $x_1 = 0$. In this case, the monic polynomial solutions can be written as follows [4, 53]

$$P_n(x) = \frac{(-\bar{x}_1)_n(-\bar{x}_2)_n}{(x_2 - \bar{x}_1 - \bar{x}_2 + n - 1)_n} {}_3F_2 \left(\begin{matrix} -n, -x, x_2 - \bar{x}_1 - \bar{x}_2 + n - 1 \\ -\bar{x}_1, -\bar{x}_2 \end{matrix} \middle| 1 \right), \quad (3.48)$$

where the generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \quad (3.49)$$

The four referred families of discrete hypergeometric polynomials are the so-called classical discrete orthogonal polynomials: Hahn $h_n^{\alpha, \beta}(x, N)$, Meixner $M_n^{\gamma, \mu}(x)$, Kravchuk $K_n^p(x, N)$ and Charlier $C_n^\mu(x)$, polynomials [51, 52], whose main data in its monic form are shown in Tables 2-3. They can be expressed in terms of the hypergeometric functions by formulas [52, Section 2.7, p. 49]:

$$h_n^{\alpha, \beta}(x, N) = \frac{(1-N)_n(\beta+1)_n}{(\alpha+\beta+n+1)_n} {}_3F_2 \left(\begin{matrix} -x, \alpha+\beta+n+1, -n \\ 1-N, \beta+1 \end{matrix} \middle| 1 \right), \quad (3.50)$$

$$M_n^{\gamma, \mu}(x) = (\gamma)_n \frac{\mu^n}{(\mu-1)^n} {}_2F_1 \left(\begin{matrix} -n, -x \\ \gamma \end{matrix} \middle| 1 - \frac{1}{\mu} \right), \quad (3.51)$$

$$K_n^p(x, N) = \frac{(-p)_n N!}{(N-n)!} {}_2F_1 \left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| \frac{1}{p} \right), \quad (3.52)$$

$$C_n^\mu(x) = (-\mu)^n {}_2F_0 \left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{\mu} \right). \quad (3.53)$$

These expressions immediately follow from the above representation (3.48) and its different limits (more details can be found in [52, 53]).

Table 2: Main data for monic Hahn and Charlier polynomials.

$P_n(x)$	Hahn $h_n^{\alpha, \beta}(x; N)$	Charlier $C_n^\mu(x)$
(a, b)	$[0, N - 1]$	$[0, \infty)$
$\sigma(x)$	$x(N + \alpha - x)$	x
$\tau(x)$	$(\beta + 1)(N - 1) - (\alpha + \beta + 2)x$	$\mu - x$
$\sigma(x) + \tau(x)$	$(x + \beta + 1)(N - 1 - x)$	μ
λ_n	$n(n + \alpha + \beta + 1)$	n
$\rho(x)$	$\frac{\Gamma(N + \alpha - x)\Gamma(\beta + x + 1)}{\Gamma(N - x)\Gamma(x + 1)}$ $\alpha, \beta \geq -1, n \leq N - 1$	$\frac{e^{-\mu} \mu^x}{\Gamma(x + 1)}$ $\mu > 0$
$\rho_n(x)$	$\frac{\Gamma(N + \alpha - x)\Gamma(n + \beta + x + 1)}{\Gamma(N - n - x)\Gamma(x + 1)}$	$\frac{e^{-\mu} \mu^{x+n}}{\Gamma(x + 1)}$
B_n	$\frac{(-1)^n}{(\alpha + \beta + n + 1)_n}$	$(-1)^n$
b_n	$-\frac{n}{2} \left(\frac{2(\beta + 1)(N - 1) + (n - 1)(\alpha - \beta + 2N - 2)}{\alpha + \beta + 2n} \right)$	$-\frac{n}{2}(2\mu + n - 1)$
d_n^2	$\frac{n!\Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)\Gamma(\alpha + \beta + N + n + 1)}{(\alpha + \beta + 2n + 1)(N - n - 1)!\Gamma(\alpha + \beta + n + 1)(\alpha + \beta + n + 1)_n^2}$	$n! \mu^n$

Table 3: Main data for monic Meixner and Kravchuk polynomials.

$P_n(x)$	Meixner $M_n^{\gamma, \mu}(x)$	Kravchuk $K_n^p(x)$
(a, b)	$[0, \infty)$	$[0, N]$
$\sigma(x)$	x	x
$\tau(x)$	$(\mu - 1)x + \mu\gamma$	$\frac{Np - x}{1 - p}$
$\sigma(x) + \tau(x)$	$\mu x + \gamma\mu$	$-\frac{p}{1 - p}(x - N)$
λ_n	$(1 - \mu)n$	$\frac{n}{1 - p}$
$\rho(x)$	$\frac{\mu^x \Gamma(\gamma + x)}{\Gamma(\gamma)\Gamma(x + 1)}$ $\gamma > 0, 0 < \mu < 1$	$\frac{N! p^x (1 - p)^{N - x}}{\Gamma(N + 1 - x)\Gamma(x + 1)}$ $0 < p < 1, n \leq N$
$\rho_n(x)$	$\frac{\mu^{x+n} \Gamma(\gamma + x + n)}{\Gamma(\gamma)\Gamma(x + 1)}$	$\frac{N! p^{x+n} (1 - p)^{N - n - x}}{\Gamma(N + 1 - n - x)\Gamma(x + 1)}$
B_n	$\frac{1}{(\mu - 1)^n}$	$(-1)^n (1 - p)^n$
b_n	$n \left(\gamma + \frac{n - 1}{2} \frac{1 + \mu}{\mu} \right) \left(\frac{\mu}{\mu - 1} \right)$	$-n[Np + (n - 1)(\frac{1}{2} - p)]$
d_n^2	$\frac{n!(\gamma)_n \mu^n}{(1 - \mu)^{\gamma + 2n}}$	$\frac{n! N! p^n (1 - p)^n}{(N - n)!}$

4 The q -NAVIMA algorithm in the exponential lattice.

Here we will present the q -analogue of the NAVIMA algorithm. This algorithm have been obtained firstly in [8] for the lattice $x(s) = q^s$. Here we will extend it to all exponential lattices of the form $x(s) = c_1 q^s + c_3$.

Let us consider two families of q -polynomials $P_n(x)$ and $Q_n(x(s))$ belonging to the class of discrete orthogonal polynomials in the exponential lattice $x(s) = c_1 q^s + c_3$. Each polynomial $P_n(x(s))$ can be represented as a linear combination of the polynomials $Q_n(x(s))$. In particular

$$Q_m(x(s)) = \sum_{n=0}^m C_n(m) P_n(x(s)). \quad (4.1)$$

For the family $P_n(x(s))$ we will use the notation

1. $\sigma(s)$, $\tau(s)$ and λ_n for the difference equation (3.2)
2. α_n , β_n and γ_n for the TTRR (3.20) coefficients
3. S_n , R_n and T_n for the second structure relation (3.33)

and for the $Q_n(x(s))$

1. $\bar{\sigma}(s)$, $\bar{\tau}(s)$ and $\bar{\lambda}_n$ for the difference equation (3.2)
2. $\bar{\alpha}_n$, $\bar{\beta}_n$ and $\bar{\gamma}_n$ for the TTRR (3.20) coefficients
3. \bar{S}_n , \bar{R}_n and \bar{T}_n for the second structure relation (3.33)

Since the polynomials of the family $Q_m(x(s))$ are solutions of the second order difference equation (3.1) the action of the difference operator of second order $\mathcal{L}_2 : \mathbb{P} \rightarrow \mathbb{P}$, defined by

$$\mathcal{L}_2[\pi(x(s))] = \bar{\sigma}(s) \frac{\Delta[\pi(x(s))]}{\Delta x(s - \frac{1}{2})} \left[\frac{\nabla[\pi(x(s))]}{\nabla x(s)} \right] + \bar{\tau}(s) \frac{\Delta}{\Delta x(s)} + \bar{\lambda}_m[\pi(x(s))], \quad \pi(x(s)) \in \mathbb{P},$$

on Eq. (4.1) gives us

$$\sum_{m=0}^n C_n(m) \left[\bar{\sigma}(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[\frac{\nabla P_n(x(s))}{\nabla x(s)} \right] + \bar{\tau}(s) \frac{\Delta P_n(x(s))}{\Delta x(s)} + \bar{\lambda}_m P_n(x(s)) \right] = 0. \quad (4.2)$$

Multiplying by $\sigma(s)$ and using (3.2) for the P_n family, we obtain the relation

$$\sum_{m=0}^n C_n(m) \left\{ [\bar{\tau}(s)\sigma(s) - \tau(s)\bar{\sigma}(s)] \frac{\Delta P_n(x(s))}{\Delta x(s)} + [\bar{\lambda}_m\sigma(s) - \bar{\sigma}(s)\lambda_n] P_n(x(s)) \right\} = 0. \quad (4.3)$$

In order to eliminate $\frac{\Delta P_n(x(s))}{\Delta x(s)}$, we multiply (4.3) by $\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})$ and use the second structure relation (3.33) for the $Q_m(x(s))$ family, obtaining

$$\begin{aligned} & \sum_{m=0}^n C_n(m) \{ [\bar{\tau}(s)\sigma(s) - \tau(s)\bar{\sigma}(s)] [S_n P_{n+1}(x(s)) + R_n P_{n-1}(x(s)) + T_n P_n(x(s))] + \\ & + [\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})] [\bar{\lambda}_m\sigma(s) - \bar{\sigma}(s)\lambda_n] P_n(x(s)) \} = 0. \end{aligned} \quad (4.4)$$

The last step consists to expand the remaining terms of type $\sigma^2(s)P_n(x(s))$, $\bar{\sigma}(s)\sigma(s)P_n(x(s))$, $\sigma(s)\bar{\tau}(s)P_n(x(s))$ and $\bar{\sigma}(s)\tau(s)P_n(x(s))$ in linear combination of $P_n(x(s))$ by using the TTRR (3.20)

repeatedly for the $P_n(x(s))$ family.

After this process, (4.4) reduces to

$$\sum_{n=0}^N M_n [C_0(m), C_1(m), \dots, C_n(m)] P_n(x(s)). \quad (4.5)$$

Taking into account the linear independence of the family $P_n(x(s))$ we obtain the linear system

$$M_n [C_0(m), C_1(m), \dots, C_n(m)] = 0. \quad (4.6)$$

These relations contain (linearly) several connection coefficients $C_i(m)$ depending essentially on the degrees of $\sigma(s)$ and $\bar{\sigma}(s)$. In the most general situation they are polynomials of second degree in $x(s) = c_1 q^s + c_3$. In this case we obtain a relation of the following type the linear system we are looking for

$$M_n [C_{n+4}(m), \dots, C_{n-4}(m)] = 0, \quad (4.7)$$

which is valid for m greater or equal than the number of initial conditions needed to start the recursion ($m \geq 8$). Notice that for ($m < 8$) the system also gives the solution, but not in a recurrent way.

Notice that for the q-Hahn, q-Meixner, q-Charlier and q-Kravchuk polynomials, as it is show in [4, 5] and [52], table 3.3, page 95, the $\sigma(s)$ is a polynomial of second degree in $x(s) = q^s$. This implies that for such polynomials the recurrence relations for the connection coefficient all are of the form (4.7).

Remark 1: Notice that to obtain the recurrence relation we have multiply two times by σ , which, obviously artificially increase the order of the recurrence (in fact in 4th orders).

Remark 2: It is possible to get the minimal order for the recurrence relation if we use also the relation (3.34) for the q -polynomials P_n . This will yield a recurrence of order 4 instead of the above 8-th order one.

Remark 3: Notice that the algorithm remains valid if Q_m is any polynomial satisfying a linear difference equation with polynomials coefficients. This implies that the above algorithm can be used for solving also the linearization problem (1.6) for q -orthogonal polynomials in the exponential lattice.

5 A general algorithm for solving the linearization problem in the exponential lattice.

In this section we will present a general algorithm [7] to find a recurrence relation for the linearization coefficients L_{mjn} in the expansion

$$Q_m(x(s))R_j(x(s)) = \sum_{n=0}^{m+j} L_{mjn} P_n(x(s)), \quad x(s) = c_1 q^s + c_3, \quad (5.1)$$

where c_1, c_3 and q are constants, $Q_m(x(s)) \equiv Q_m(s)_q$ and $R_j(x(s)) \equiv R_j(s)_q$ are polynomials which satisfy a second order difference equation of the form

$$a(s)Q_m(s+1)_q + b(s)Q_m(s)_q + c(s)Q_m(s-1)_q = 0, \quad (5.2)$$

and

$$\alpha(s)R_j(s+1)_q + \beta(s)R_j(s)_q + \gamma(s)R_j(s-1)_q = 0, \quad (5.3)$$

respectively. A special case of such polynomials are the q -polynomials of hypergeometric type [4, 52, 53], which satisfy the difference equation (3.2). Obviously, the Eq. (3.2) is of the type (5.2) ($y \equiv Q_m$), with

$$a(s) = \sigma(s) + \tau(s) \Delta x(s - \frac{1}{2}), \quad c(s) = \frac{\sigma(s)}{\nabla x(s)}, \quad b(s) = \lambda \Delta x(s - \frac{1}{2}) - \frac{a(s)}{\Delta x(s)} - c(s).$$

In the following, we will use the operators \mathcal{T} and \mathcal{I} defined as follows

$$\begin{aligned} \mathcal{T} : \mathbb{P} &\rightarrow \mathbb{P} & \mathcal{I} : \mathbb{P} &\rightarrow \mathbb{P} \\ \mathcal{T}p(s) &= p(s+1) & \mathcal{I}p(s) &= p(s) \end{aligned}$$

Using the above operators, we can rewrite the Eqs. (5.2)-(5.3) in the form

$$a(s+1)\mathcal{T}^2 Q_m(s)_q + b(s+1)\mathcal{T} Q_m(s)_q + c(s+1)\mathcal{I} Q_m(s)_q = 0, \quad (5.4)$$

and

$$\alpha(s+1)\mathcal{T}^2 R_j(s)_q + \beta(s+1)\mathcal{T} R_j(s)_q + \gamma(s+1)\mathcal{I} R_j(s)_q = 0. \quad (5.5)$$

It is known [7], that, if the polynomials $Q_m(s)_q$ and $R_j(s)_q$ satisfy the linear difference equations (5.4) and (5.5), respectively, then the product $u(s)_q \equiv Q_m(s)_q R_j(s)_q$, satisfy a four order difference equation of the form

$$\mathcal{L}_4 u(s) \equiv p_4(s)\mathcal{T}^4 u(s)_q + p_3(s)\mathcal{T}^3 u(s)_q + p_2(s)\mathcal{T}^2 u(s)_q + p_1(s)\mathcal{T} u(s)_q + p_0(s)\mathcal{I} u(s)_q. \quad (5.6)$$

The idea of the proof is the following [7, 23, 24].

Since (5.4)-(5.5),

$$\begin{aligned} a(s+1)\alpha(s+1)\mathcal{T}^2 u(s) &= \\ &= [b(s+1)\mathcal{T} Q_m(s)_q + c(s+1)\mathcal{I} Q_m(s)_q] [\beta(s+1)\mathcal{T} R_j(s)_q + \gamma(s+1)\mathcal{I} R_j(s)_q], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mathcal{L}_2 u(s) &\equiv a(s+1)\alpha(s+1)\mathcal{T}^2 u(s) - b(s+1)\beta(s+1)\mathcal{T} u(s) - c(s+1)\gamma(s+1)\mathcal{I} u(s) = \\ &= b(s+1)\gamma(s+1) [\mathcal{T} Q_m(s)_q \mathcal{I} R_j(s)_q] + c(s+1)\beta(s+1) [\mathcal{I} Q_m(s)_q \mathcal{T} R_j(s)_q] = \\ &= l_1(s) [\mathcal{T} Q_m(s)_q \mathcal{I} R_j(s)_q] + l_2(s) [\mathcal{I} Q_m(s)_q \mathcal{T} R_j(s)_q]. \end{aligned}$$

Next, we change in the last expression $s \rightarrow s+1$, and substitute in the right-hand side the expression $\mathcal{T}^2 Q_m(s)_q$ and $\mathcal{T}^2 R_j(s)_q$, using the Eqs. (5.4)-(5.5), respectively. This allows us to rewrite the resulting expression in the form

$$\mathcal{M}_3 u(s) = m_1(s) [\mathcal{T} Q_m(s)_q \mathcal{I} R_j(s)_q] + m_2(s) [\mathcal{I} Q_m(s)_q \mathcal{T} R_j(s)_q],$$

where \mathcal{M}_3 is a difference operator of third order (there is one term proportional to \mathcal{T}^3), m_1 and m_2 are known functions of s . Repeating the same procedure, but now starting from the above equation we obtain

$$\mathcal{N}_4 u(s) = n_1(s) [\mathcal{T} Q_m(s)_q \mathcal{I} R_j(s)_q] + n_2(s) [\mathcal{I} Q_m(s)_q \mathcal{T} R_j(s)_q].$$

Then

$$\begin{vmatrix} \mathcal{L}_2 u(s) & l_1(s) & l_2(s) \\ \mathcal{M}_3 u(s) & m_1(s) & m_2(s) \\ \mathcal{N}_4 u(s) & n_1(s) & n_2(s) \end{vmatrix} = 0. \quad (5.7)$$

Expanding the determinant from the first column, the Eq. (5.6) holds.

Remark: The above equation (5.7), and its proof, remains true for any lattice function $x(s)$ and not only for the exponential lattice $x(s) = c_1 q^s + c_3$.

5.1 The generalized linearization algorithm.

As before, we will suppose that $Q_m(s)_q$ and $R_j(s)_q$ satisfy the equations (5.4) and (5.5), respectively, and that $P_n(s)_q$ satisfy a the three-term recurrence relation (3.20) and a structure relation in the exponential lattice $x(s) = c_1 q^s + c_3$ (3.33). Notice that the latest can be written in the equivalent form [7]

$$\Sigma(s)\mathcal{T}P_n(s)_q = \sum_{k=n-2}^{n+2} A_k(n)P_k(s)_q, \quad \Sigma(s) = \sigma(s) + \tau(s) \Delta x(s - \frac{1}{2}), \quad (5.8)$$

To obtain (5.8) from (3.33) we need to use that $\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})$ is a polynomial of degree two in $x(s)$ and that $\Delta x(s)$ is a polynomial of first degree in $x(s)$ (which is not valid in general for any lattice $x(s)$), as well as the TTRR (3.20).

From the above expression (5.8), one easily obtains that

$$\begin{aligned} \Sigma(s)\Sigma(s+1)\mathcal{T}^2 P_n(s)_q &= \sum_{k=n-4}^{n+4} \tilde{A}_k(n)P_k(s)_q, \\ \Sigma(s)\Sigma(s+1)\Sigma(s+2)\mathcal{T}^3 P_n(s)_q &= \sum_{k=n-6}^{n+6} \hat{A}_k(n)P_k(s)_q, \\ \Sigma(s)\Sigma(s+1)\Sigma(s+2)\Sigma(s+3)\mathcal{T}^4 P_n(s)_q &= \sum_{k=n-8}^{n+8} \bar{A}_k(n)P_k(s)_q. \end{aligned} \quad (5.9)$$

To obtain a recurrence relation for the linearization coefficients we can do we can follow an idea similar to the one exposed in the previous section for the connection problem:

Since (5.6), $\mathcal{L}_4 Q_m(s)_q R_j(s)_q = 0$, then applying \mathcal{L}_4 to both sides of (5.1), we find

$$0 = \sum_{n=0}^{m+j} L_{mjn} \Sigma(s)\Sigma(s+1)\Sigma(s+2)\Sigma(s+3)\mathcal{L}_4 P_n(x(s)).$$

Taking into account that \mathcal{L}_4 is a four degree operator, and using the structure relation (5.8) as well as (5.9) we find

$$\begin{aligned} 0 = \sum_{n=0}^{m+j} L_{mjn} \left\{ p_4(s) \sum_{k=n-8}^{n+8} \bar{A}_k(n)P_k(s)_q + p_3(s)\Sigma(s+3) \sum_{k=n-6}^{n+6} \hat{A}_k(n)P_k(s)_q + \right. \\ \left. + p_2(s)\Sigma(s+2)\Sigma(s+3) \sum_{k=n-4}^{n+4} \tilde{A}_k(n)P_k(s)_q + \right. \\ \left. + p_1(s)\Sigma(s+1)\Sigma(s+2)\Sigma(s+3) \sum_{k=n-2}^{n+2} A_k(n)P_k(s)_q + \right. \\ \left. + \Sigma(s)\Sigma(s+1)\Sigma(s+2)\Sigma(s+3)p_0(s)P_n(s)_q \right\}, \end{aligned}$$

from where, and by taking into account that $\Sigma(s+k)$, $k = 0, 1, 2, 3$, is a polynomial of degree two in $x(s) = c_1 q^s + c_3$, as well as the TTRR (3.20) we obtain that the coefficients L_{mjn} satisfy a

recurrence relation of the form

$$\sum_{k=0}^r c_k(i, j, n) L_{mj} L_{n+k} = 0. \quad (5.10)$$

In general, the present algorithm may not give the minimal order recurrence for the linearization coefficients. To get the order r minimal it is necessary to use more specific properties of the families of polynomials involved in (5.1).

Remark: Notice that the present algorithm also works for the case when the product $Q_m(x(s))R_j(x(s))$ satisfy any k th-linear difference equation with polynomial coefficients (not necessary of order 4 as in (5.7)), so it can be used for solving more general linearization problems involving the product of three or more q -polynomials. Notice also that will be possible to reduce the order of the recurrence relation if we use the relation (3.34) for the q -polynomials.

Obviously the following question arises: And what happens if there is not structure relations (3.33)? For example, for the q -polynomials in the general lattice this question is still open. In the next section we will describe an alternative algorithm which will allow us avoid this problem.

5.2 An example.

Since the product $[(s)_q]_i [(s)_q]_j$ is a polynomial in q^s , it can be represented as a linear combination of the single q -Pochhammer symbols $[(s)_q]_n$. In particular,

$$[(s)_q]_i [(s)_q]_j = \sum_{n=0}^{i+j} L_{ijn}(q) [(s)_q]_n. \quad (5.11)$$

In order to obtain the recurrence relation for the linearization coefficients L_{ijn} in (5.11) we apply the operator

$$(s)_q^2 \mathcal{T} - (s+i)_q (s+j)_q \mathcal{I} \quad (5.12)$$

to both sides of (5.11). Using formula (3.38) we obtain the following expression

$$0 = \sum_{n=0}^{i+j} L_{ijn} \left[\left(\frac{q^s - 1}{q - 1} \right)^2 \mathcal{T} [(s)_q]_n - \left(\frac{q^{s+i} - 1}{q - 1} \right) \left(\frac{q^{s+j} - 1}{q - 1} \right) [(s)_q]_n \right]. \quad (5.13)$$

Taking into account the Eq. (3.38) for the q -Pochhammer symbol, we find

$$\begin{aligned} 0 &= \sum_{n=0}^{i+j} L_{ijn} [(s)_q]_n \left[\left(\frac{q^s - 1}{q - 1} \right) \left(\frac{q^{s+n} - 1}{q - 1} \right) - \left(\frac{q^{s+i} - 1}{q - 1} \right) \left(\frac{q^{s+j} - 1}{q - 1} \right) \right] = \\ &= \sum_{n=0}^{i+j} L_{ijn} [(s)_q]_n [(s)_q (s+n)_q - (s+i)_q (s+j)_q]. \end{aligned}$$

Using the identity

$$(s+n)_q = q^n (s)_q + (n)_q,$$

the last expression transforms

$$0 = \sum_{n=0}^{i+j} L_{ijn} [(s)_q]_n \{ (s)_q^2 [q^n - q^{i+j}] + (s)_q [(n)_q - q^i (j)_q - q^j (i)_q] - (i)_q (j)_q \},$$

from where, using Eq. (3.39), we arrive to the expression

$$\begin{aligned}
 & \sum_{n=0}^{i+j} L_{ijn} \left\{ q^{-2n-1} [q^n - q^{i+j}] [(s_q)_{n+2}] + \right. \\
 & + [((n)_q - q^i(j)_q - q^j(i)_q) q^{-n} - (q^n - q^{i+j}) (q^{-2n-1}(n+1)_q + q^{-2n}(n)_q)] [(s_q)_{n+1}] + \\
 & \left. + [(q^n - q^{i+j}) q^{-2n} (n)_q^2 - ((n)_q - q^i(j)_q - q^j(i)_q) q^{-n} - (i)_q(j)_q] [(s_q)_n] \right\} = \\
 & = \sum_{n=0}^{i+j} L_{ijn} \left\{ q^{-2n-1} [q^n - q^{i+j}] [(s_q)_{n+2}] - \right. \\
 & - \left\{ q^{-n-1}(n+1)_q + q^{i+j-1-2n} [1 + q^{n+j+1}(j)_q + q^{n+i+1}(i)_q - 2(n+1)_q] \right\} [(s_q)_{n+1}] - \\
 & \left. - q^{i+j-2n} [(n)_q - q^{n+j}(j)_q] [(n)_q - q^{n+i}(i)_q] [(s_q)_n] \right\} = 0.
 \end{aligned}$$

Then, the following three-term recurrence relation for the linearization coefficients L_{ijn} holds

$$A_n L_{ijn-2} + B_n L_{ijn-1} + C_n L_{ijn} = 0, \quad (5.14)$$

where

$$\begin{aligned}
 A_n &= q^{-2n+3} [q^{n-2} - q^{i+j}], \\
 B_n &= -q^{-n}(n)_q - q^{i+j+1-n} [q^{-j}(j)_q + q^{-i}(i)_q - q^{-n}(n)_q - q^{-n+1}(n-1)_q], \\
 C_n &= -q^{i+j} [q^{-n}(n)_q - q^{-j}(j)_q] [q^{-n}(n)_q - q^{-i}(i)_q],
 \end{aligned} \quad (5.15)$$

with the initial conditions $L_{ij i+j+1} = 0$ and $L_{ij i+j} = q^{-ij}$.

To solve the above recurrence we apply the algorithm qHyper [1, 2, 56] which allows us to find an equivalent two-term recurrence relation for the linearization coefficients. Namely,

$$L_{ijn+1} = -\frac{q^{-k-1}(i+j-n)_q}{(i-n-1)_q(j-n-1)_q} L_{ijn}, \quad (5.16)$$

so that,

$$L_{ijn} = (-1)^{i+j-n} q^{\frac{i(i+1)+j(j+1)-n(n+1)}{2}} \frac{[(-j)_q]_{i+j-n} [(-i)_q]_{i+j-n}}{(i+j-n)_q!}, \quad (5.17)$$

for $n \geq \max(i, j)$ and vanishes otherwise.

Notice that, in the limit $q \rightarrow 1$, the above recurrence relations (5.14)-(5.16) transform into a two-term recurrence relations for the standard Pochhammer symbols $(s)_n$ of the form

$$(k-i-j-1)L_{ijn-1} - (k^2 - (i+j)k + ij)L_{ijn} = 0, \quad L_{ij i+j+1} = 0, \quad L_{ij i+j} = 1,$$

which solution

$$L_{ijn} = \begin{cases} \frac{(-1)^{i+j+n} (-j)_{i+j-n} (-i)_{i+j-n}}{(i+j-n)!} & n \geq \max(i, j) \\ 0 & \text{otherwise} \end{cases},$$

corresponds to (5.17) in the limit $q \rightarrow 1$.

The same can be done in the case of q -Stirling polynomials [7].

6 An alternative algorithm for the connection and linearization problem.

In this section we will describe the q -analogue [6] of the method presented in [9, 11, 12, 63] for finding explicit expression of the coefficients c_{mn} and l_{jmn} of (1.5) and (1.6) in terms of the coefficients of the second order difference equation of hypergeometric-type in the general non-uniform lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$. The resulting expansion coefficients will be given in a compact, analytic, closed and formally simple form in terms of the polynomial coefficients of the corresponding second-order difference equation(s). Notice that the above lattice contains, as a particular case, the exponential lattice $x(s) = c_1q^s + c_3$ considered in the previous Sections [7, 8, 46]. The advantage of the present approach is that it only requires the knowledge of the second order difference equation satisfied by the involved hypergeometric q -polynomials as well as their hypergeometricity, i.e., the Rodrigues-type formula, and it does not require neither information about any kind of recurrence relation of the involved discrete hypergeometric q -polynomials nor to solve any “high” order recurrence relation for the connection coefficients themselves.

6.1 Main theorems.

Here we find the explicit expression of the coefficients c_{mn} in the expansion of an arbitrary q -polynomial $Q_m(x(s)) \equiv Q_m(s)_q$ on $x(s)$ in series of the orthogonal discrete hypergeometric set of q -polynomials $\{P_n\}$ in the same non-uniform lattice $x(s)$, i.e.

$$Q_m(s)_q = \sum_{n=0}^m c_{mn} P_n(s)_q. \quad (6.1)$$

Theorem 6.1 [6] *The explicit expression of the coefficients c_{mn} in the expansion (6.1) is*

$$\begin{aligned} c_{mn} &= \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \Delta^{(n)} [Q_m(s)_q] \rho_n(s) \Delta x_n(s - \frac{1}{2}) \\ &= (-1)^n \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \frac{\nabla}{\nabla x(s - \frac{n-1}{2})} \cdots \frac{\nabla}{\nabla x(s)} [Q_m(s)_q] \rho_n(s - n) \Delta x(s - \frac{n+1}{2}). \end{aligned} \quad (6.2)$$

Proof: Multiply both sides of Eq. (6.1) by $P_k(s)_q \rho(x) \Delta x(s - \frac{1}{2})$, and summing between a and $b - 1$, the orthogonality relation (3.17) immediately gives

$$c_{mn} = \frac{1}{d_n^2} \sum_{s=a}^{b-1} Q_m(s)_q P_n(s)_q \rho(s) \Delta x(s - \frac{1}{2}). \quad (6.3)$$

Using the Rodrigues formula (3.7) for $P_n(s)_q$ we find

$$c_{mn} = \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s)_q \nabla^{(n)} [\rho_n(s)] \Delta x(s - \frac{1}{2}) = \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s)_q \nabla \left[\nabla_1^{(n)} [\rho_n(s)] \right]. \quad (6.4)$$

Then, by using the formula of summation by parts

$$\sum_{x_i=a}^{b-1} f(x_i) \nabla g(x_i) = f(x_i)g(x_i) \Big|_{a-1}^{b-1} - \sum_{x_i=a}^{b-1} g(x_i - 1) \nabla f(x_i),$$

we have

$$c_{mn} = \frac{B_n}{d_n^2} Q_m(s)_q \nabla_1^{(n)} [\rho_n(s)] \Big|_{a-1}^{b-1} - \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \nabla Q_m(s)_q \nabla_1^{(n)} [\rho_n(t)] \Big|_{t=s-1}. \quad (6.5)$$

Notice that the first term is proportional to $\rho_1(s) = \sigma(s+1)\rho(s+1)$, so, since the condition (3.18), it vanishes. Now, making the change $s \rightarrow s-1$ in the second term, we find

$$c_{mn} = -\frac{B_n}{d_n^2} \sum_{s=a-1}^{b-2} \Delta Q_m(s)_q \nabla_1^{(n)} [\rho_n(s)].$$

But

$$\nabla_1^{(n)}[\rho_n(s)] = \frac{\nabla}{\nabla x_2(s)} \nabla_2^{(n)} [\rho_n(s)], \quad \nabla x_2(s) = \nabla x(s+1) = \Delta x(s),$$

then, the last equation transforms

$$c_{mn} = -\frac{B_n}{d_n^2} \sum_{s=a-1}^{b-2} \frac{\Delta}{\Delta x(s)} [Q_m(s)_q] \nabla \left[\nabla_2^{(n)} [\rho_n(s)] \right].$$

Repeating this process n times, and using $\nabla_n^{(n)}[\rho_n(s)] = \rho_n(s)$ as well as that $\rho_n(a-k) = 0$ for $k = 1, 2, \dots, n$ (see Eq. (3.8)), we obtain the desired expression (6.2) for c_{mn} .

The second expression can be obtained analogously [6]. ■

If we now assume that Q_m is an hypergeometric polynomials which satisfy an equation of the form (3.1) but with coefficients $\tilde{\sigma}$, $\tilde{\tau}$, and $\tilde{\lambda}_m$, then, by using the Rodrigues-type formula (3.14) for the polynomials Q_m we obtain the following result.

Corollary 6.1 *The explicit expression of the coefficients c_{mn} in the expansion (6.1) when both polynomials are of hypergeometric type is*

$$c_{mn} = \frac{(-1)^n B_n \tilde{B}_m \tilde{A}_{mn}}{d_n^2} \sum_{l=0}^{m-n} (-1)^l \frac{[m-n]_q!}{[l]_q! [m-n-l]_q!} \times \sum_{s=a}^{b-n-1} \frac{\tilde{\rho}_m(s-l) \rho_n(s)}{\tilde{\rho}_n(s)} \frac{\Delta x_m(s-l-\frac{1}{2}) \Delta x_n(s-\frac{1}{2})}{\prod_{k=0}^{m-n} \Delta x_m(s-\frac{k+l+1}{2})}. \quad (6.6)$$

A simple consequence of theorem 6.1 is the following result for the linearization problem:

$$R_j(s)_q Q_m(s)_q = \sum_{n=0}^{m+j} l_{jmn} P_n(s)_q, \quad (6.7)$$

where $\{P_n\}$ is a discrete orthogonal set of hypergeometric q -polynomials which satisfy the difference equation (3.1) and Q_m and R_j are arbitrary q -polynomials on the same lattice $x(s)$.

Corollary 6.2 *The explicit expression of the coefficients l_{jmn} in the expansion (6.7) is*

$$l_{jmn} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \Delta^{(n)} [Q_m(s)_q R_j(s)_q] \rho_n(s) \Delta x_n(s-\frac{1}{2}). \quad (6.8)$$

In the special case when R_j is the j -degree q -hypergeometric polynomial satisfying the following second order difference equation on the non-uniform lattice $x(s)$

$$\tilde{\sigma}(s) \frac{\Delta}{\Delta x(s-\frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tilde{\tau}(s) \frac{\Delta y(s)}{\Delta x(s)} + \tilde{\lambda}_j y(s) = 0, \quad (6.9)$$

the following theorem holds

Theorem 6.2 *The explicit expression of the coefficients l_{jmn} in the expansion (6.7) is given by*

$$l_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \tilde{A}_{jk} \times \sum_{s=a}^{b-n-1} \frac{\rho_n(s) \Delta x_n(s - \frac{1}{2})}{\tilde{\rho}_k(s+n-k)} [\Delta^{(n-k)} Q_m(s)_q] [\nabla_k^{(j)} \tilde{\rho}_j(s+n-k)], \quad (6.10)$$

or, equivalently,

$$l_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \tilde{A}_{jn-k} \times \sum_{s=a}^{b-n-1} \frac{\rho_n(s) \Delta x_n(s - \frac{1}{2})}{\tilde{\rho}_{n-k}(s)} [\Delta^{(k)} Q_m(s+n-k)_q] [\nabla_{n-k}^{(j)} \tilde{\rho}_j(s)]. \quad (6.11)$$

Proof: We will start from Eq. (6.8) and we will use the analog of the Leibnitz formula in the non-uniform lattice (3.3) [4]

$$\Delta^{(n)} [f(s)g(s)] = \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \Delta^{(k)} f(s+n-k) \Delta^{(n-k)} g(s), \quad (6.12)$$

to $\Delta^{(n)} [Q_m(s)_q R_j(s)_q]$. Then, using the Rodrigues-type formula (3.14) for $\Delta^{(k)} R_j(s+n-k)_q$

$$\Delta^{(k)} R_j(s+n-k)_q = \frac{\tilde{A}_{jk} \tilde{B}_j}{\tilde{\rho}_k(s+n-k)} \nabla_k^{(j)} [\tilde{\rho}_j(s+n-k)],$$

the desired result holds. The second formula can be obtained analogously. \blacksquare

A simple corollary of the above theorem is the following

Corollary 6.3 *The explicit expression of the coefficients l_{jmn} in the expansion (6.7) is given by*

$$l_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \tilde{A}_{jk} \sum_{l=0}^{j-k} (-1)^l \frac{[j-k]_q!}{[l]_q! [j-k-l]_q!} \times \sum_{s=a}^{b-n-1} \frac{\rho_n(s) \Delta x_n(s - \frac{1}{2}) \tilde{\rho}_j(s+n-k-l)}{\tilde{\rho}_k(s+n-k)} \frac{\Delta x_j(s+n-k-l - \frac{1}{2})}{\prod_{m=0}^{j-k} \Delta x_j(s+n-k - \frac{m+l+1}{2})} [\Delta^{(n-k)} Q_m(s)_q], \quad (6.13)$$

Notice that the corollary 6.1 also follows from the above formula if we put $m = 0$ since $Q_0 \equiv 1$.

6.1.1 The classical discrete case.

In the especial case when $x(s)$ is the linear lattice, i.e, $x(s) = s$, from Theorem 6.1 and 6.3 we recover the main results in [9] for the connection and linearization problems, respectively

Theorem 6.3 *Let be $x(s)$ the linear lattice $x(s) = s$. Then, the explicit expression of the coefficients c_{mn} in the expansion (6.1) is*

$$c_{mn} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \Delta^n Q_m(s) \rho(s+n) \prod_{k=1}^n \sigma(s+k) = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-1} \nabla^n Q_m(s) \rho(s) \prod_{k=0}^{n-1} \sigma(s-k). \quad (6.14)$$

If Q_m is also an hypergeometric polynomial, then

$$\begin{aligned} c_{mn} &= \frac{(-1)^n B_n \tilde{B}_m \tilde{A}_{mn}}{d_n^2} \sum_{s=a}^{b-n-1} \sum_{k=0}^{m-n} \frac{\rho_n(s)}{\tilde{\rho}_n(s)} \binom{m-n}{k} (-1)^k \tilde{\rho}_m(s-k) = \\ &= \frac{(-1)^n B_n \tilde{B}_m \tilde{A}_{mn}}{d_n^2} \sum_{s=a}^{b-1} \sum_{k=0}^{m-n} \frac{\rho_n(s-n)}{\tilde{\rho}_n(s-n)} \binom{m-n}{k} (-1)^k \tilde{\rho}_m(s-n-k). \end{aligned} \quad (6.15)$$

Corollary 6.4

$$(x)_m = \sum_{n=0}^m a_{mn} p_n(x), \quad a_{mn} = \frac{(-1)^n m! B_n}{(m-n)! d_n^2} \sum_{x=a}^{b-1} (x)_{m-n} \rho_n(x-n), \quad (6.16)$$

$$x^{[m]} = \sum_{n=0}^m d_{mn} p_n(x), \quad d_{mn} = \frac{(-1)^n m! B_n}{(m-n)! d_n^2} \sum_{x=a}^{b-1} (x-n)^{[m-n]} \rho_n(x-n). \quad (6.17)$$

Theorem 6.4 Let be $x(s)$ the linear lattice $x(s) = s$. Then, the explicit explicit expression of the coefficients c_{jmn} in the expansion (6.7) is given by

$$\begin{aligned} c_{jmn} &= \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} \tilde{A}_{jk} \times \\ &\times \sum_{s=a}^{b-n-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l} \frac{\rho_n(s)}{\tilde{\rho}_k(s+n-k)} \tilde{\rho}_j(s+n-k-l) [\nabla^{n-k} Q_m(s+n-k)] = \\ &= \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} \tilde{A}_{jk} \sum_{s=a}^{b-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l} \frac{\rho_n(s-n)}{\tilde{\rho}_k(s-k)} \tilde{\rho}_j(s-k-l) [\nabla^{n-k} Q_m(s-k)], \end{aligned}$$

where $k_- = \max(0, n-m)$ and $k_+ = \min(n, j)$.

For completeness, let us point out that there is another equivalent expression for the connection coefficients $c_{j0n} \equiv c_{jn}$ which sometimes is very useful. The general polynomial solution of the equation (3.4) is given by (3.48). Then, the solution for the *direct* connection problem

$$q_j(x) = \sum_{k=0}^j a_{jk} x^{[k]}, \quad (6.18)$$

is given by

$$a_{jk} = \frac{(-1)^k (-\bar{x}_1)_j (-\bar{x}_2)_j (x_2 - \bar{x}_1 - \bar{x}_2 + j - 1)_k (-n)_k}{(-\bar{x}_1)_k (-\bar{x}_2)_k (x_2 - \bar{x}_1 - \bar{x}_2 + j - 1)_j k!}. \quad (6.19)$$

This formula immediately follows from the identity $x^{[k]} = (-1)^k (x)_k$ and the definition of the generalized hypergeometric function (3.49). Let us also remark that sometimes it is better to use the combination of the above formula with formula (6.17) wo obtain the searched expansion coefficients. Notice that

$$q_j(x) = \sum_{k=0}^j a_{jk} x^{[k]} = \sum_{k=0}^j a_{jk} \sum_{n=0}^k d_{kn} p_n(x) = \sum_{n=0}^j \underbrace{\left(\sum_{k=0}^{j-n} a_{j k+n} d_{k+n n} \right)}_{c_{j0n}} p_n(x), \quad (6.20)$$

where $a_{j k+n}$ and $d_{k+n n}$ are given by (6.19) and (6.17), respectively. Again here, the coefficients $c_{j0n} \equiv c_{jn}$ depend only on the coefficients of the second order difference equation of hypergeometric type (3.4).

6.1.2 The classical continuous case.

Finally, we will show how from Theorem 6.1 we can recover (formally) the general results for the continuous case [11, 63]. In order to do this we notice that, formally, if we make the change $x(s) = sh \rightarrow x$, then [52],

$$\frac{P_n(x(s+1)) - P_n(x(s))}{x_k(s+1) - x_k(s)} = \frac{P_n(sh+h) - P_n(sh)}{h} = \frac{P_n(x+h) - P_n(x)}{h}.$$

Thus, $\lim_{h \rightarrow 0} \frac{\Delta P_n(x(s))}{\Delta x_k(s)} = P_n'(x)$ and $\lim_{h \rightarrow 0} \Delta^{(k)} P_n(s)_q = \frac{d^k P_n(x)}{dx^k}$. Then, by similar limiting processes Eq. (3.1) transforms into the classical hypergeometric differential equation [51, 52]

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0.$$

where $\sigma(x) = \lim_{h \rightarrow 0} \bar{\sigma}(x(s))$, $\tau(x) = \lim_{h \rightarrow 0} \bar{\tau}(x(s))$ being $x = sh$. Furthermore, the Pearson-type equation (3.6) becomes $[\sigma(x)\rho(x)]' = \tau(x)\rho(x)$ and also [52] $\rho_n(s; h) \rightarrow \rho(x)\sigma^n(x)$. Finally, the Rodrigues-type formula (3.7) transforms into

$$\Delta^{(k)} P_n(s)_q = \frac{A_{nk} B_n}{\rho_k(s)} \nabla_k^{(n)} [\rho_n(s)] \rightarrow \frac{d^k P_n(x)}{dx^k} = \frac{A_{nk} B_n}{\rho_k(x)} \frac{d^{n-k}}{dx^{n-k}} [\rho(x)\sigma^n(x)].$$

Now we put $x(s) = sh$ in (6.2)

$$\begin{aligned} c_{mn}(h) &= \frac{(-1)^n B_n(h)}{d_n^2(h)} \sum_{x_i=ah}^{(b-1)h-nh} \Delta^{(n)} [Q_m(x_i)_q] \rho_n(x_i/h; h) h = \\ &= \frac{(-1)^n B_n(h)}{d_n^2(h)} \sum_{x=A}^{B-nh} \Delta^{(n)} [Q_m(x_i)_q] \rho_n(x_i/h; h) h, \quad x_{i+1} = x_i + h. \end{aligned}$$

Let us prove that the above sum transforms in the limit in an integral from which the main result in [63, Theorem 3.1, page 163] easily follows. More concretely,

$$\lim_{h \rightarrow 0} c_{mn}(h) = \frac{(-1)^n B_n}{d_n^2} \int_A^B \frac{d^k Q_m(x)}{dx^k} \rho(x) \sigma^n(x) dx,$$

where d_n^2 is the squared norm for the polynomials orthogonal with respect to $\rho(x)$ [52].

In order to do that, let us show that the quantity

$$I_n(Q_m, \rho_n) \equiv \left| \sum_{x=A}^{B-nh} \Delta^{(n)} [Q_m(sh)_q] \rho_n(x_i/h; h) h - \int_A^B Q_m^{(n)}(x) \rho(x) \sigma^n(x) dx \right|$$

can be small enough for h sufficiently small.

$$\begin{aligned} |I_n(Q_m, \rho_n)| &\leq \sum_{x_i=A}^{B-nh} \left| \Delta^{(n)} [Q_m(sh)_q] - Q_m^{(n)}(x_i) \right| \rho_n(x_i/h; h) h + \\ &+ \sum_{x_i=A}^{B-nh} \left| Q_m^{(n)}(x_i) \{ \rho_n(x_i/h; h) - \rho_n(x_i) \} \right| h + \left| \sum_{x_i=A}^{B-nh} Q_m^{(n)}(x_i) \rho_n(x_i) h - \int_A^B Q_m^{(n)}(x) \rho_n(x) dx \right|, \end{aligned}$$

where $Q_m^{(n)}$ denotes the n -th derivative of Q_m and $\rho_n(x) = \rho(x)\sigma^n(x)$. Let consider first the case when B is bounded. In this case the first integral can be small enough (less than $\epsilon/3$) for h sufficient

small providing that $\rho_n(x_i/h; h)$ is bounded. In the following we will suppose that the limit function $\rho_n(x)$, $n \geq 1$ is a continuous function in $[A, B]$. For the second sum we can do the same since Q_m is a polynomial and then it is bounded in any closed interval. Finally we will consider the last sum which can be rewritten in the form

$$\left| \sum_{x_i=A}^B Q_m^{(n)}(x_i) \rho_n(x_i) h - \int_A^B Q_m^{(n)}(x) \rho_n(x) dx \right| + \left| \sum_{x_i=B-hn}^B Q_m^{(n)}(x_i) \rho_n(x_i) h \right|.$$

Notice that the first sum can be less $\epsilon/6$ since it is a Riemann sum corresponding to the integral $\int_A^B Q_m^{(n)}(x) \rho_n(x) dx$, and the last sum obviously tends to zero so, for sufficiently small h , it is less than $\epsilon/6$. So, for any given $\epsilon > 0$, one can chose a sufficiently small h so that $|I_n(Q_m, \rho_n)| \leq \epsilon$.

Finally, to prove the result for the unbounded B we use the fact that, in this case, the functions $\rho_n(x_i/h; h)$ as well as $\rho_n(x_i)$ tend to zero faster than any polynomial tends to infinity when $x_i \rightarrow \infty$ (see the boundary conditions (3.18) for the polynomials on the lattice $x(s)$ as well as for the continuous case [52, Eq. (1.3.1) page 7]. Then,

$$\begin{aligned} & |I_n(Q_m, \rho_n)| \leq \\ & \sum_{x_i=A}^{\infty} \left| \Delta^{(n)} [Q_m(sh)_q] - Q_m^{(n)}(x_i) \right| \rho_n(x_i/h; h) h + \sum_{x_i=A}^{\infty} \left| Q_m^{(n)}(x_i) \{ \rho_n(x_i/h; h) - \rho_n(x_i) \} \right| h + \\ & + \left| \sum_{x_i=A}^{\infty} Q_m^{(n)}(x_i) \rho_n(x_i) h - \int_A^{\infty} Q_m^{(n)}(x) \rho_n(x) dx \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

To conclude this Section let us point out that here we have taken the limit formally and have proved that our main result, i.e., formula (6.2), transforms into the corresponding one for the continuous case [63], but solving concrete examples one must to be very careful since, for instance, in the limit Hahn \rightarrow Jacobi, the parameter $h = 1/N$ where N is the total number of points in the lattice and the Hahn polynomials explicitly depend on it. More information on how to take limits for concrete families can be found in [32, 39, 52, 53].

7 Examples.

7.1 q -polynomials.

7.1.1 Connection between $(q^s; q)^{[m]}$ and $c_n^\mu(x, q)$.

Now will apply theorem 6.1 for finding the connection coefficients c_{mn}^q in the expansion

$$(q^s; q)^{[m]} = \sum_{n=0}^m d_{mn}^q c_n^\mu(s, q), \tag{7.21}$$

where $(a; q)^{[k]}$ is defined in (3.42), and $c_n^\mu(s, q)$ is, as before, the q -Charlier polynomials on the lattice $x(s) = \frac{q^s-1}{q-1}$ (3.35). In this case, since (3.47)

$$\begin{aligned} \Delta^{(n)} \left[(q^s; q)^{[m]} \right] &= q^{-\frac{n}{4}(n-1)} \left[\frac{\Delta}{\Delta x(s)} \right]^n (q^s; q)^{[m]} = \frac{(1-q)^n [m]_q! q^{-\frac{n}{2}(m-1)}}{[m-n]_q!} (q^s; q)^{[m-n]} \\ &= q^{\frac{n}{4}(n-1) - n(m-1)} \frac{(1-q)^n \Gamma_q[m+1]}{\Gamma_q[m-n+1]} (q^{s+n}; q)_{m-n}. \end{aligned}$$

In this case, using formula (6.2), the expression $\frac{(q^s; q)^{[m-n]}}{(q; q)_s} = \frac{1}{(q; q)_{s-m+n}}$, as well as

$$\sum_{s=0}^{\infty} \frac{(q^s; q)^{[m-n]} z^s}{(q; q)_s} = \sum_{s=m-n}^{\infty} \frac{(q^s; q)^{[m-n]} z^s}{(q; q)_s} = z^{m-n} \sum_{s=0}^{\infty} \frac{z^s}{(q; q)_s} = z^{m-n} e_q(z),$$

we obtain

$$d_{mn}^q = q^{m+\frac{n}{4}(n-7)} \binom{m}{n}_q (1-q)^m (-1)^n \mu^m. \quad (7.22)$$

The above formula is the q -analogue of the so-called *inversion formula* for hypergeometric polynomials (compare with the explicit expression of the q -Charlier polynomials (3.35).

Remark. If we rewrite (7.21) in the form

$$(s)_q^{[m]} = \sum_{n=0}^m \tilde{d}_{mn}^q c_n^\mu(s, q), \quad \tilde{d}_{mn}^q = q^{m+\frac{n}{4}(n-7)} \binom{m}{n}_q (-1)^n \mu^m, \quad (7.23)$$

taking into account that

$$\lim_{q \rightarrow 1} \frac{(q^s; q)^{[m]}}{(1-q)^m} = (s)^{[m]},$$

we obtain in the limit $q \rightarrow 1$

$$(s)^{[m]} = \sum_{n=0}^m d_{mn} c_n^\mu(s), \quad d_{mn} = \binom{m}{n} (-1)^n (\mu)^m.$$

Using again the fact that for the polynomials $c_n^\mu(s)$, the leading coefficients are given by $a_n = (-\mu)^{-n}$, the above result coincides with well know classical result (see e. g. [9])

7.1.2 Connection between $(q^s; q)_m$ and $c_n^\mu(x, q)$.

First of all we will apply theorem 6.1 for finding the connection coefficients c_{mn}^q in the expansion

$$(q^s; q)_m = \sum_{n=0}^m c_{mn}^q c_n^\mu(s, q), \quad (7.24)$$

where $(a; q)_k$ is defined in (3.26), and $c_n^\mu(s, q)$ is the afore mentioned q -Charlier polynomials.

Since we are working in the lattice $x(s) = \frac{q^s-1}{q-1}$, we have

$$\begin{aligned} \Delta^{(n)} [(q^s; q)_m] &= q^{-\frac{n}{4}(n-1)} \left[\frac{\Delta}{\Delta x(s)} \right]^n (q^s; q)_m = \frac{(1-q)^n [m]_q! q^{\frac{n}{2}(m-1)}}{[m-n]_q!} (q^{s+n}; q)_{m-n} \\ &= q^{\frac{n}{4}(n-1)} \frac{(1-q)^n \Gamma_q[m+1]}{\Gamma_q[m-n+1]} (q^{s+n}; q)_{m-n}. \end{aligned}$$

Then (6.2) gives

$$c_{mn}^q = \frac{q^{\frac{n}{4}(5n-7)} (q-1)^n \mu^n}{e_q[(1-q)\mu q^{n+1}]} \binom{m}{n}_q \sum_{s=0}^{\infty} \frac{(q^{s+n}; q)_{m-n} [(1-q)\mu q^{n+1}]^s}{(q; q)_s},$$

where the q -binomial coefficients are defined by

$$\binom{m}{n}_q = \frac{(q; q)_m}{(q; q)_n (q; q)_{m-n}}.$$

In order to take the sum in the above expression we will use the identity [32, Eq. (1.2.34) page 6]

$$(aq^s; q)_k = \frac{(a; q)_k (aq^k; q)_s}{(a; q)_s}, \quad (7.25)$$

as well as the expression [32, Eq. (1.5.2) page 11]

$$\frac{(q^m; q)_s}{(q^n; q)_s} = \sum_{k=0}^s \frac{(q^{-s}; q)_k (q^{n-m}; q)_k}{(q^n; q)_k} \frac{q^{m+s}}{(q; q)_k}. \quad (7.26)$$

Then, denoting by $z = (1 - q)\mu q^{n+1}$, we have

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{(q^{s+n}; q)_{m-n} z^s}{(q; q)_s} &= \sum_{s=0}^{\infty} \frac{(q^n; q)_{m-n} (q^m; q)_s}{(q^n; q)_s (q; q)_s} z^s = \\ &= (q^n; q)_{m-n} \sum_{k=0}^{\infty} \frac{(q^{n-m}; q)_k q^{mk}}{(q^n; q)_k (q; q)_k} \sum_{s=0}^{\infty} \frac{(q^{-s}; q)_k q^{sk}}{(q; q)_s} z^s = \\ &= (q^n; q)_{m-n} \sum_{k=0}^{\infty} \frac{(q^{n-m}; q)_k q^{mk} z^k}{(q^n; q)_k (q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right] \sum_{s=k}^{\infty} \frac{z^{s-k}}{(q; q)_{s-k}} = \\ &= (q^n; q)_{m-n} e_q[(1 - q)\mu q^{n+1}]_1 \varphi_1 \left(\begin{matrix} q^{n-m} \\ q^n \end{matrix}; q, \mu q^{n+m+1}(1 - q) \right). \end{aligned}$$

For the third equality we have used the identity [32, Eq. (1.2.32) page 6]

$$\frac{(q^{-s}; q)_k}{(q; q)_s} = \frac{(-1)^k q^{\frac{k}{2}(k-1) - ks}}{(q; q)_{s-k}}. \quad (7.27)$$

Then, for the coefficients $c_{m n}^q$ we finally obtain

$$c_{m n}^q = (q^n; q)_{m-n} \mu^n (q - 1)^n q^{\frac{n}{4}(5n-7)} \binom{m}{n}_q {}_1\varphi_1 \left(\begin{matrix} q^{n-m} \\ q^n \end{matrix}; q, \mu q^{n+m+1}(1 - q) \right). \quad (7.28)$$

Remark. Notice that, since

$$\frac{(q^s; q)_m}{(1 - q)^m} = \sum_{n=0}^m \frac{c_{m n}^q}{(1 - q)^m} c_n^\mu(x, q),$$

and taking into account that

$$\lim_{q \rightarrow 1} \frac{(q^s; q)_m}{(1 - q)^m} = (s)_m, \quad \lim_{q \rightarrow 1} c_n^\mu(x, q) = c_n^\mu(s),$$

we obtain taking the limit $q \rightarrow 1$

$$(s)_m = \sum_{n=0}^m c_{m n} c_n^\mu(s), \quad c_{m n} = \binom{m}{n} \frac{(m-1)!}{(n-1)!} (-\mu)^n {}_1F_1 \left(\begin{matrix} n-m \\ n \end{matrix} \middle| -\mu \right),$$

where $c_n^\mu(s)$ denotes the classical (non monic) Charlier polynomials [52, 53]. Since for these polynomials the leading coefficients are given by $a_n = (-\mu)^{-n}$, the above result coincides with the classical result given in [9].

7.1.3 The q -Charlier polynomials in the exponential lattice.

We will solve now the connection problem

$$c_m^\gamma(s, q) = \sum_{n=0}^m c_{mn}^q c_n^\mu(s, q). \quad (7.29)$$

Then, by using the expression (6.6) of the corollary (6.1) where $Q_m(s)_q = c_m^\gamma(s, q)$ and $P_n(s)_q = c_n^\mu(s, q)$, respectively, we obtain

$$\begin{aligned} c_{mn}^q &= \left(\frac{\mu}{\gamma}\right)^n \binom{m}{n}_q \frac{q^{\frac{1}{4}(m-n)(m-n+5)}}{e_q[(1-q)q^{n+1}\mu]} \sum_{l=0}^{m-n} \frac{(-1)^l q^{\frac{l}{2}(l-2m-1)}}{(1-q)^l \gamma^l} \binom{m-n}{l}_q \sum_{s=l}^{\infty} \frac{[(1-q)\mu q^{n+1}]_{s-l}}{(q; q)_{s-l}} = \\ &= \left(\frac{\mu}{\gamma}\right)^n \binom{m}{n}_q q^{\frac{1}{4}(m-n)(m-n+5)} \sum_{l=0}^{m-n} (-1)^l \left(\frac{\mu}{\gamma} q^{n-m+1}\right)^l \binom{m-n}{l}_q q^{\frac{l(l-1)}{2}}, \end{aligned}$$

where we also use the fact that $\sum_{s=0}^{\infty} \frac{z^k}{\Gamma_q(s-k)} = \sum_{s=k}^{\infty} \frac{z^k}{\Gamma_q(s-k)} = \sum_{s=k}^{\infty} \frac{z^k (1-q)^{s-k}}{(q; q)_{s-k}}$.

Now, applying the identity (7.27) to $(q; q)_{m-n-l}$ ($k = l$), and using the q -binomial theorem [32, §1.3, Eq. (1.3.14) page 9],

$$\sum_{l=0}^k \frac{(q^{-k}; q)_l}{(q; q)_l} z^l = {}_1\varphi_0 \left(\begin{matrix} q^{-k} \\ - \end{matrix}; q, z \right) = (zq^{-k}; q)_k,$$

we obtain the following expression for the coefficient c_{mn}^q

$$c_{mn}^q = \left(\frac{\mu}{\gamma}\right)^n \binom{m}{n}_q q^{\frac{1}{4}(m-n)(m-n+5)} (q^{n-m+1} \mu \gamma^{-1}; q)_{m-n}. \quad (7.30)$$

Remark. A simple calculation shows that the equation (7.29) transforms in the limit $q \rightarrow 1$ into

$$c_m^\gamma(s) = \sum_{n=0}^m \binom{m}{n} \left(\frac{\mu}{\gamma}\right)^n \left(1 - \frac{\mu}{\gamma}\right)^{m-n} c_n^\mu(s),$$

for the (non monic) Charlier polynomials and this coincides with the classical results for monic polynomials (see e.g. [9]) since the leading coefficients for the Charlier polynomials $c_n^\mu(s)$ is equal to $(-\mu)^{-n}$.

7.1.4 Examples of linearizations.

Let us show here how we can combine the results of the previous section to solve some linearization problems. We start finding the coefficient \tilde{L}_{ijn} in the expansion

$$(s)_q^{[i]} (s)_q^{[j]} = \sum_{n=0}^{i+j} \tilde{L}_{ijn}(q) (s)_q^{[n]}, \quad (7.31)$$

and use the identity $(s)_q^{[n]} = (-1)^n q^{-n} [(-s)_{q^{-1}}]_n$, then, from (5.11) and (5.17) we find

$$\tilde{L}_{ijn}(q) = (-1)^{i+j-n} q^{n-i-j} L_{ijn}(q^{-1}), \quad (7.32)$$

thus

$$\tilde{L}_{ijn} = q^{i+j+i-j-n} \frac{[(-j)_q]_{i+j-n} [(-i)_q]_{i+j-n}}{(i+j-n)_q!}, \quad \text{for } n \geq \max(i, j), \quad (7.33)$$

and vanishes otherwise. Notice that in the limit $q \rightarrow 1$

$$\tilde{L}_{ijn} = \begin{cases} \frac{(-j)_{i+j-n}(-i)_{i+j-n}}{(i+j-n)!} & n \geq \max(i, j) \\ 0 & \text{otherwise} \end{cases}, \quad (7.34)$$

The above problem can be used to solve the linearization problem

$$(s)_q^{[m]}(s)_q^{[j]} = \sum_{n=0}^{m+j} c_{m,j,n}^q c_n^\mu(s, q), \quad (7.35)$$

since

$$c_{m,j,n}^q = \sum_{k=0}^{m+j-n} \tilde{L}_{mj \ k+n}(q) c_{k+n \ n}^q,$$

where $\tilde{L}_{mj \ k+n}(q)$ are given in (7.33) and $c_{k+n \ n}^q$ by (7.23). Doing, some straightforward calculations (in which we use some identities involving the $(a; q)_n$ and $(a; q)^{[n]}$ symbols [32, 39]) lead us to the expression

$$c_{m,j,n}^q = \frac{q^{m+j+mj-\frac{n}{2}(n+1)} \mu^{m+j-n}}{(q-1)^n} \binom{j+m}{n}_q {}_3\hat{\varphi}_1 \left(\begin{matrix} q^{-m}, q^{-j} q^{n-m-j} \\ q^{-m-j} \end{matrix}; q, \frac{1}{(1-q)q^{n+1}\mu} \right).$$

Here, we use the function ${}_r\hat{\varphi}_p$ defined by

$${}_r\hat{\varphi}_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k}. \quad (7.36)$$

7.2 Examples for discrete classical polynomials.

Inversion problems of classical polynomials.

Here we will give the explicit closed expressions for the coefficients of the inversion formulas which follows from Theorem (6.3) of the classical discrete polynomials associated to the polynomials $(x)_m$ and $x^{[m]}$, respectively. From then, the corresponding inversion formulas associated to the polynomials x^m follow in a straightforward manner.

7.2.1 Charlier Polynomials $C_n^\mu(x)$.

The use of the inversion formula (6.16) related to $(x)_m$ and the main data of the monic Charlier polynomials (see Table 2), as well as formula (7.43), allows us to find the corresponding expansion coefficients

$$a_{mn} = \begin{cases} 1 & m = n = 0 \\ \mu m! {}_1F_1 \left(\begin{matrix} 1-m \\ 2 \end{matrix} \middle| -\mu \right) & m \neq 0, n = 0 \\ \binom{m}{n} \frac{\Gamma(m)}{\Gamma(n)} {}_1F_1 \left(\begin{matrix} n-m \\ n \end{matrix} \middle| -\mu \right) & m \neq 0, n \neq 0 \end{cases}.$$

For the expansion of $x^{[m]}$, we use Eq. (6.17) and Eq.

$${}_1F_1 \left(\begin{matrix} a \\ a \end{matrix} \middle| x \right) = e^x, \quad \forall a \in \mathbb{R}. \quad (7.37)$$

to obtain that

$$d_{mn} = \binom{m}{n} \mu^{m-n}.$$

7.2.2 Meixner polynomials $M_n^{\gamma, \mu}(x)$.

Analogously, for the monic Meixner polynomials we find

$$a_{mn} = \begin{cases} 1 & m = n = 0 \\ \frac{\mu\gamma m!}{1-\mu} {}_2F_1\left(\begin{matrix} 1-m, 1+\gamma \\ 2 \end{matrix} \middle| \frac{\mu}{\mu-1}\right) & m \neq 0, n = 0 \\ \binom{m}{n} \frac{\Gamma(m)}{\Gamma(n)} {}_2F_1\left(\begin{matrix} n-m, n+\gamma \\ n \end{matrix} \middle| \frac{\mu}{\mu-1}\right) & m \neq 0, n \neq 0 \end{cases},$$

and

$$d_{mn} = \binom{m}{n} (\gamma + n)_{m-n} \left(\frac{\mu}{1-\mu}\right)^{m-n}.$$

7.2.3 Kravchuk polynomials $K_n^p(x, N)$.

For the monic Kravchuk polynomials, we obtain

$$a_{mn} = \begin{cases} 1 & m = n = 0 \\ Npm! {}_2F_1\left(\begin{matrix} 1-m, 1-N \\ 2 \end{matrix} \middle| p\right) & m \neq 0, n = 0 \\ \binom{m}{n} \frac{\Gamma(m)}{\Gamma(n)} {}_2F_1\left(\begin{matrix} n-m, n-N \\ n \end{matrix} \middle| p\right) & m \neq 0, n \neq 0 \end{cases},$$

and

$$d_{mn} = \binom{m}{n} p^{m-n} (N - m + 1)_{m-n}.$$

7.2.4 Hahn polynomials $h_n^{\alpha, \beta}(x, N)$.

Finally, for the monic Hahn polynomials, one has

$$a_{mn} = \begin{cases} 1 & m = n = 0 \\ \frac{m!(\beta+1)(N-1)}{\alpha+\beta+2} {}_3F_2\left(\begin{matrix} m+1, 2-N, 2+\beta \\ 2, 2-N-\alpha \end{matrix} \middle| 1\right) & m \neq 0, n = 0 \\ \binom{m}{n} \frac{\Gamma(m)}{\Gamma(n)} {}_3F_2\left(\begin{matrix} n-m, 1+n-N, n+\beta+1 \\ n, 2n+\alpha+\beta+2 \end{matrix} \middle| 1\right) & m \neq 0, n \neq 0 \end{cases},$$

and

$$d_{mn} = \binom{m}{n} \frac{(N-m)_{m-n} (n+\beta+1)_{m-n}}{(2n+\alpha+\beta+2)_{m-n}}.$$

Some of the above formulas have been found by different authors using different approaches. This is so for the Stirling inversion problems of the Charlier [24, 40, 62, 68], Meixner [40, 62, 68], Kravchuk [40, 62, 68] and Hahn [30] polynomials.

Connection problem between discrete hypergeometric polynomials.

In this section we will provide the formulas connecting the different families of classical hypergeometric discrete polynomials, which generalize results already obtained by different authors using different approaches, e.g. [10, 30, 40, 43, 62], in particular, the most general case involving two Hahn polynomials is given (see formula (7.51) from below).

The first eight cases can be computed by using (6.15) and the other ones with the help of (6.20). Notice that if we equate both expressions (6.17) and (6.20) one can obtain different summation formulas involving terminating hypergeometric series of the type given in the Appendix.

7.2.5 Charlier-Charlier

From formula (6.15) and using the main data of the Charlier polynomials (see Table 2) we find for the connection coefficients between the families

$$C_j^\mu(x) = \sum_{n=0}^j c_{jn} C_n^\gamma(x),$$

the expression

$$c_{jn} = \binom{j}{n} (\gamma - \mu)^{j-n}. \quad (7.38)$$

7.2.6 Meixner-Meixner

For the Meixner-Meixner problem we have

$$M_j^{\gamma, \mu}(x) = \sum_{n=0}^j c_{jn} M_n^{\alpha, \beta}(x),$$

where

$$c_{jn} = \binom{j}{n} \frac{(1 - \beta)^{n+\alpha} \mu^{j-n} \Gamma(j + \gamma)}{\Gamma(\alpha + n) (\mu - 1)^{j-n}} \times \\ \times \sum_{k=0}^{j-n} (-1)^k \binom{j-n}{k} \left(\frac{\beta}{\mu}\right)^k \frac{\Gamma(n+k+\alpha)}{\Gamma(n+k+\gamma)} {}_2F_1 \left(\begin{matrix} n+k+\alpha, j+\gamma \\ n+k+\gamma \end{matrix} \middle| \beta \right).$$

Using the transformation formula [35, p. 425]

$${}_2F_1 \left(\begin{matrix} a & b \\ c \end{matrix} \middle| x \right) = (1-x)^{-a} {}_2F_1 \left(\begin{matrix} a & c-b \\ c \end{matrix} \middle| \frac{x}{x-1} \right) = \\ = (1-x)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a & c-b \\ c \end{matrix} \middle| x \right), \quad (7.39)$$

the identity $\binom{j-n}{k} = (-1)^k \frac{(n-j)_k}{k!}$ as well as formula [35, Eq. 65.2.2, p. 426]

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} y^k {}_2F_1 \left(\begin{matrix} c-a & c-b \\ c+k \end{matrix} \middle| x \right) = (1-x)^{a+b-c} {}_2F_1 \left(\begin{matrix} a & b \\ c \end{matrix} \middle| x+y-xy \right). \quad (7.40)$$

we finally obtain

$$c_{jn} = \binom{j}{n} \left(\frac{\mu}{\mu-1} \right)^{j-n} (\gamma+n)_{j-n} {}_2F_1 \left(\begin{matrix} n-j, n+\alpha \\ n+\gamma \end{matrix} \middle| \frac{\beta(1-\mu)}{\mu(1-\beta)} \right). \quad (7.41)$$

In particular, for the special case $\alpha = \gamma$, Eq. (7.41) becomes

$$c_{jn} = \binom{j}{n} (\gamma+n)_{j-n} \left(\frac{\beta-\mu}{(\beta-1)(\mu-1)} \right)^{j-n},$$

The second case corresponds to $\beta = \mu$, then (7.41) becomes

$$c_{jn} = \binom{j}{n} \left(\frac{\mu}{\mu-1} \right)^{j-n} (\gamma-\alpha)_{j-n}.$$

7.2.7 Kravchuk-Kravchuk.

For the Kravchuk-Kravchuk expansion,

$$K_j^p(x, N) = \sum_{n=0}^j c_{jn} K_n^q(x, M), \quad j \leq \min\{N, M\},$$

the same procedure used in the Meixner-Meixner case gives us

$$c_{jn} = \binom{j}{n} (M-j+1)_{j-n} (-p)^{j-n} {}_2F_1 \left(\begin{matrix} n-j, n-N \\ n-M \end{matrix} \middle| \frac{q}{p} \right). \quad (7.42)$$

In the particular case $p = q$ its reduces to

$$c_{jn} = \binom{j}{n} p^{j-n} (N-M)_{j-n},$$

and for the case $M = N$

$$c_{jn} = \binom{j}{n} \left(\frac{p}{q} \right)^{j-n} (q-p)^{j-n} (N-j+1)_{j-n}.$$

7.2.8 Meixner-Charlier.

In this case we have the expansion

$$M_j^{\gamma, \mu}(x) = \sum_{n=0}^j c_{jn} C_n^\alpha(x),$$

with

$$c_{jn} = \binom{j}{n} \frac{e^{-\alpha} \mu^{j-n} \Gamma(j+\gamma)}{(\mu-1)^{j-n}} \sum_{k=0}^{j-n} \frac{(-1)^k}{\Gamma(\gamma+n+k)} \binom{j-n}{k} \left(\frac{\alpha}{\mu} \right)^k {}_1F_1 \left(\begin{matrix} j+\gamma \\ m+k+\gamma \end{matrix} \middle| \alpha \right).$$

If we use the transformation formula [35, p. 431]

$${}_1F_1 \left(\begin{matrix} a \\ c \end{matrix} \middle| x \right) = e^x {}_1F_1 \left(\begin{matrix} c-a \\ c \end{matrix} \middle| -x \right). \quad (7.43)$$

and the summation formula [35, Eq. (66.2.5), p. 431]

$$\sum_{k=0}^{\infty} \frac{(c-a)_k}{k!(c)_k} y^k {}_1F_1 \left(\begin{matrix} a \\ c+k \end{matrix} \middle| x \right) = e^y {}_1F_1 \left(\begin{matrix} a \\ c \end{matrix} \middle| x-y \right). \quad (7.44)$$

we find

$$c_{jn} = \binom{j}{n} \left(\frac{\mu}{\mu-1} \right)^{j-n} (\gamma+n)_{j-n} {}_1F_1 \left(\begin{matrix} n-j \\ n+\gamma \end{matrix} \middle| \frac{\alpha(1-\mu)}{\mu} \right). \quad (7.45)$$

7.2.9 Charlier-Meixner.

For the Charlier-Meixner expansion

$$C_j^\alpha(x) = \sum_{n=0}^j c_{jn} M_n^{\gamma, \mu}(x),$$

one finds from Eq. (6.15) that

$$c_{jn} = \binom{j}{n} (-\alpha)^{j-n} {}_2F_0 \left(\begin{matrix} n-j, \gamma+n \\ - \end{matrix} \middle| \frac{\mu}{\alpha(1-\mu)} \right). \quad (7.46)$$

7.2.10 Meixner-Kravchuk.

In the Meixner-Kravchuk case,

$$M_j^{\gamma, \mu}(x) = \sum_{n=0}^j c_{jn} K_n^p(x, N), \quad j \leq N$$

we find

$$c_{jn} = \binom{j}{n} (n+\gamma)_{j-n} \left(\frac{\mu}{\mu-1} \right)^{j-n} {}_2F_1 \left(\begin{matrix} n-j, n-N \\ n+\gamma \end{matrix} \middle| \frac{p(\mu-1)}{\mu} \right). \quad (7.47)$$

7.2.11 Kravchuk-Meixner.

For the Kravchuk-Meixner connection problem,

$$K_j^p(x, N) = \sum_{n=0}^j c_{jn} M_n^{\alpha, \beta}(x), \quad j \leq N$$

we have

$$c_{jn} = \binom{j}{n} (N+1-j)_{j-n} (-p)^{j-n} {}_2F_1 \left(\begin{matrix} n-j, n+\alpha \\ n-N \end{matrix} \middle| \frac{\beta}{(\beta-1)p} \right). \quad (7.48)$$

7.2.12 Kravchuk-Charlier.

For the Kravchuk-Charlier connection problem,

$$K_j^p(x, N) = \sum_{n=0}^j c_{jn} C_n^\mu(x), \quad j \leq N,$$

we have

$$c_{jn} = \binom{j}{n} (N+1-j)_{j-n} (-p)^{j-n} {}_1F_1 \left(\begin{matrix} n-j \\ n-N \end{matrix} \middle| -\frac{\mu}{p} \right). \quad (7.49)$$

7.2.13 Charlier-Kravchuk.

For the Charlier-Kravchuk problem,

$$C_j^\mu(x) = \sum_{n=0}^j c_{jn} K_n^p(x, N), \quad j \leq N,$$

we have

$$c_{jn} = \binom{j}{n} (-\mu)^{j-n} {}_2F_0 \left(\begin{matrix} n-j, n-N \\ - \end{matrix} \middle| -\frac{p}{\mu} \right). \quad (7.50)$$

7.2.14 Hahn-Hahn

For the Hahn-Hahn problem, we use Eq. (6.20). A straightforward study of the problem

$$h_j^{\gamma,\mu}(x, M) = \sum_{n=0}^j c_{jn} h_n^{\alpha,\beta}(x, N), \quad j \leq \min\{N-1, M-1\},$$

allows us to find

$$c_{jn} = \binom{j}{n} \frac{(1+n-M)_{j-n}(1+n+\mu)_{j-n}}{(1+n+j+\gamma+\mu)_{j-n}} \times \\ \times {}_4F_3 \left(\begin{matrix} n-j, 1+n-N, n+\beta+1, 1+j+n+\gamma+\mu \\ 1+n-M, n+\mu+1, 2n+\alpha+\beta+2 \end{matrix} \middle| 1 \right). \quad (7.51)$$

In the particular case $N = M$ (7.51) reduces to

$$c_{jn} = \binom{j}{n} \frac{(1+n-N)_{j-n}(1+n+\mu)_{j-n}}{(1+n+j+\gamma+\mu)_{j-n}} {}_3F_2 \left(\begin{matrix} n-j, n+\beta+1, 1+j+n+\gamma+\mu \\ n+\mu+1, 2n+\alpha+\beta+2 \end{matrix} \middle| 1 \right).$$

7.2.15 Hahn-Charlier

For the Hahn-Charlier problem,

$$h_j^{\alpha+\beta}(x, N) = \sum_{n=0}^j c_{jn} C_n^\alpha(x), \quad j \leq M-1,$$

we find that

$$c_{jn} = \binom{j}{n} \frac{(1+n-N)_{j-n}(1+n+\beta)_{j-n}}{(1+n+j+\alpha+\beta)_{j-n}} {}_2F_2 \left(\begin{matrix} n-j, 1+j+n+\alpha+\beta \\ 1+n-N, n+\beta+1 \end{matrix} \middle| -\alpha \right). \quad (7.52)$$

7.2.16 Charlier-Hahn

For the Charlier-Hahn problem,

$$C_j^\mu(x) = \sum_{n=0}^j c_{jn} h_n^{\alpha,\beta}(x, N), \quad j \leq N-1,$$

we find that

$$c_{jn} = \binom{j}{n} (-\mu)^{j-n} {}_3F_1 \left(\begin{matrix} n-j, 1+n-N, n+\beta+1 \\ 2n+\alpha+\beta+2 \end{matrix} \middle| -\frac{1}{\mu} \right). \quad (7.53)$$

7.2.17 Hahn-Meixner

For the Hahn-Meixner problem,

$$h_j^{\alpha,\beta}(x, N) = \sum_{n=0}^j c_{jn} M_n^{\gamma,\mu}(x), \quad j \leq N-1,$$

we find that

$$c_{jn} = \binom{j}{n} \frac{(1+n-N)_{j-n}(1+n+\mu)_{j-n}}{(1+n+j+\gamma+\mu)_{j-n}} \times \\ \times {}_3F_2 \left(\begin{matrix} n-j, \alpha+\beta+j+n+1, \gamma+n \\ 1+n-N, n+\beta+1 \end{matrix} \middle| \frac{\mu}{\mu-1} \right). \quad (7.54)$$

7.2.18 Meixner-Hahn

In the Meixner-Hahn case,

$$M_j^{\gamma, \mu}(x) = \sum_{n=0}^j c_{jn} h_n^{\alpha, \beta}(x, N), \quad j \leq N - 1,$$

we find that

$$c_{jn} = \binom{j}{n} \left(\frac{\mu}{\mu - 1} \right)^{j-n} (\gamma + n)_{j-n} {}_3F_2 \left(\begin{matrix} n - j, 1 + n - N, n + \beta + 1 \\ \alpha + \beta + 2n + 2, \gamma + n \end{matrix} \middle| \frac{\mu - 1}{\mu} \right). \quad (7.55)$$

7.2.19 Hahn-Kravchuk

For the Hahn-Kravchuk problem,

$$h_j^{\alpha, \beta}(x, N) = \sum_{n=0}^j c_{jn} K_n^p(x, M), \quad j \leq \min\{M - 1, N\},$$

we find that

$$\begin{aligned} c_{jn} &= \binom{j}{n} \frac{(1 + n - N)_{j-n} (1 + n + \mu)_{j-n}}{(1 + n + j + \gamma + \mu)_{j-n}} \times \\ &\times {}_3F_2 \left(\begin{matrix} n - j, \alpha + \beta + j + n + 1, n - M \\ 1 + n - N, n + \beta + 1 \end{matrix} \middle| p \right). \end{aligned} \quad (7.56)$$

7.2.20 Kravchuk-Hahn

In the Kravchuk-Hahn case,

$$K_j^p(x, M) = \sum_{n=0}^j c_{jn} h_n^{\alpha, \beta}(x, N), \quad j \leq \min\{N - 1, M\},$$

we find that

$$c_{jn} = \binom{j}{n} p^{j-n} (n - M)_{j-n} {}_3F_2 \left(\begin{matrix} n - j, 1 + n - N, n + \beta + 1 \\ \alpha + \beta + 2n + 2, n - M \end{matrix} \middle| \frac{1}{p} \right). \quad (7.57)$$

7.2.21 Some linearization formulas.

Here we apply Theorem 6.3 when r_m is the product of two Stirling polynomials. More concretely, we will solve the linearization of a product of two Stirling polynomials $x^{[m]}x^{[j]}$ in terms of the Charlier polynomials

$$x^{[m]}x^{[j]} = \sum_{n=0}^{m+j} c_{m,j,n} C_n^\mu(x), \quad (7.58)$$

In fact, the aforesaid theorem gives

$$\begin{aligned} c_{m,j,n} &= \binom{m}{p-j} \binom{p}{n} \frac{j!}{(p-m)!} \mu^{p-n} \times \\ &{}_3F_3 \left(\begin{matrix} p - m - j, p + 1, 1 \\ p - j + 1, p - m + 1, p - n + 1 \end{matrix} \middle| -\mu \right), \end{aligned} \quad (7.59)$$

where $p = \max(n, m, j)$.

Next, we apply Th. 6.4 to find the solution of the following linearization problem

$$x^{[m]}C_j^\gamma(x) = \sum_{n=0}^{m+j} c_{m,j,n}C_n^\mu(x), \quad (7.60)$$

obtaining

$$c_{m,j,n} = \sum_{k=\max(0,n-m)}^j \binom{j}{k} \binom{m}{p-k} \binom{p}{n} \frac{k! (-\gamma)^{j-k} \mu^{p-n}}{(p-m)!} \times {}_3F_3 \left(\begin{matrix} p-m-k, p+1, 1 \\ p-k+1, p-m+1, p-n+1 \end{matrix} \middle| -\mu \right), \quad p = \max(n, m, k). \quad (7.61)$$

This result can be alternatively found by means of Eqs. (7.58) and (7.59) together with Eqs. (3.53) and the definition $x^{[n]} = x(x-1)\cdots(x-n+1) \equiv (-1)^n(-x)_n$. Notice the finiteness of the k -summation and the terminating character of the involved hypergeometric function ${}_3F_3$.

Expressions similar to Eq. (7.60) referred to the rest of classical discrete hypergeometric polynomials with the non-orthogonal polynomials $x^{[m]}$ and $(x)_m$ may be equally found.

Conclusions

To conclude this work let us said that all the results here are valid for classical discrete polynomials since they are polynomials of hypergeometric type in the linear lattice $x(s)$. In such away, we can recover the results given in [9, 24, 33].

References

- [1] S. A. Abramov, P. Paule, and M. Petkovšek, q -Hypergeometric Solutions of q -Difference Equations. *Discrete Mathematics*. **180** (1998), 3-22
- [2] S. A. Abramov and M. Petkovšek, Finding All q -Hypergeometric Solutions of q -Difference Equations. *Proc. FPSAC '95, Universite de Marne-la-Valle*. Noisy-le-Grand, B. Leclerc, J.-Y. Thibon, eds. 1995, 1-10.
- [3] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover Publ., New York, 1964.
- [4] R. Álvarez-Nodarse, *Polinomios hipergeométricos y q -polinomios*. Monografías de la Academia de Ciencias de Zaragoza, Spain, (1999) (to appear).
- [5] R. Álvarez-Nodarse and J. Arvesú, On the q -polynomials in the exponential lattice $x(s) = c_1q^s + c_3$. *Integral Transform. Special Funct.* **8** (1999) 1-21.
- [6] R. Álvarez-Nodarse, J. Arvesú and R. J. Yáñez, Linearization formulas between orthogonal q -polynomials. *Preprint* (1999).
- [7] R. Álvarez-Nodarse, N. R. Quintero and A. Ronveaux, On the linearization problems involving Pochhammer symbols and their q -analogues. *J. Comput. Appl. Math.* **107** (1999) 133-146.
- [8] R. Álvarez-Nodarse and A. Ronveaux, Recurrence relation for connection coefficients between q -orthogonal polynomials of discrete variables in the non-uniform lattice $x(s) = q^{2s}$. *J. Phys. A: Math. and Gen.* **29** (1996), 7165-7175.
- [9] R. Álvarez-Nodarse, R. J. Yáñez and J. S. Dehesa, Modified Clebsch-Gordan-type expansions for products of discrete hypergeometric polynomials. *J. Comput. Appl. Math.* **89** (1998) 171-197.

- [10] I. Area, E. Godoy, A. Ronveaux and A. Zarzo, Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Discrete case. *J. Comp. Appl. Math.* **89** (1998) 309-325.
- [11] P.L. Artes, J. Sánchez-Ruiz, A. Martínez-Finkelshtein and J.S. Dehesa, Linealization and connection coefficients for hypergeometric-type polynomials. *J. Comput. Appl. Math.* **99** (1998) 15-26.
- [12] P.L. Artes, J. Sánchez-Ruiz, A. Martínez-Finkelshtein and J.S. Dehesa, General linearization formulas for products of continuous hypergeometric polynomials. *J. Phys. A: Math. Gen.* **32** (1999) 7345-7366.
- [13] R. Askey, *Orthogonal polynomials and positivity*. In *Studies in Applied Mathematics 6* Special Functions and Wave Propagation (D. Ludwig and F. W. J. Olver Eds.) SIAM, Philadelphia, Pennsylvania, 1970, 64-85.
- [14] R. Askey, Orthogonal expansions with positive coefficients. *Proc. Amer. Math. Soc.* **26**, (1965), 1191-1194.
- [15] R. Askey, Orthogonal expansions with positive coefficients II. *SIAM. J. Math. Anal.* **26**, (1971), 340-346.
- [16] R. Askey, Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients. *Proc. Camb. Phil. Soc.* **70**, (1971), 243-255.
- [17] R. Askey and G. Gasper, Linearization of the product of Jacobi polynomials III. *Can. J. Math.* **23**, (1971), 332-338.
- [18] R. Askey, *Orthogonal Polynomials and Special Functions*. Regional Conferences in Applied Mathematics. **21**, SIAM, Philadelphia, Pennsylvania, 1975.
- [19] R. Askey and G. Gasper, Positive Jacobi polynomials sum. II *Amer. J. Math.* **98**, (1976), 709-737.
- [20] R. Askey and G. Gasper, Convolution structures for Laguerre polynomials. *J. d'Analyse Math.* **31**, (1977), 48-68.
- [21] R. Askey and G. Gasper, Inequalities for polynomials. In *The Bieberbach Conjecture*. Proceedings of the Symposium on the occasion of the Proof. A. Baernstein II, D. D Drazin, P. Duren, and A. Marden Eds. *Mathematical Surveys and Monographs 21* American Mathematical Society, Providence, Rhode Island 1986. 7-32.
- [22] N. M. Atakishiyev, M. Rahman, and S. K. Suslov, On Classical Orthogonal Polynomials. *Constructive Approx.* **11** (1995), 181-226.
- [23] N. M. Atakishiyev, A. Ronveaux and K.B. Wolf: Difference Equation for the Associated Polynomials on the linear lattice. *Journal of Theoretical and Mathematical Physics* **106**(1) (1996), 76-83.
- [24] S. Belmehdi, S. Lewanowicz and A. Ronveaux, Linearization of products of orthogonal polynomials of a discrete variable. *Applicationes Mathematicae.* **24** (1997) 445-455.
- [25] C. Dunkl and D.E. Ramirez, Krawtchouk polynomials and the symmetrization of hypergroups. *SIAM J. Math. Anal.* **179** (1974), 351-366.
- [26] C.F. Dunkl, Spherical functions on compact groups and applications to special functions. *Symposia Math.* **22** (1977), 145-161.
- [27] G. K. Eagleson, A characterization theorem for positive definite sequence of Krawtchouk polynomials. *Austral. J. Statist.* **11** (1969) 28-38.
- [28] A.R.M. Edmonds, *Angular Momentum in Quantum Mechanics* Princeton Univ. Press, Princeton, 1957.
- [29] G. Gasper, Linearization of the product of Jacobi polynomials II. *Can. J. Math.* **22**, (1970), 582-593.
- [30] G. Gasper, Projection formulas for orthogonal polynomials of a discrete variable, *J. Math. Anal. Appl.* **45** (1974), 176-198.
- [31] G. Gasper, Positivity and Special Functions. In *Theory and Applications of Special Functions*. R. Askey Ed., (Academic Press, N.Y. 1975), 375-433.

- [32] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Cambridge University Press, 1990.
- [33] E. Godoy, A. Ronveaux, A. Zarzo and I. Area, Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case, *J. Comput. Appl. Math* **84** (1997) 257-275.
- [34] E. Godoy, A. Ronveaux, A. Zarzo and I. Area, On the limit relations between classical continuous and discrete orthogonal polynomials. *J. Comput. Appl. Math* **91** (1998) 97-105
- [35] E.R. Hansen, *A Table of Series and Products*. Prentice Hall, New Jersey, 1975.
- [36] E. Godoy, A. Ronveaux, A. Zarzo and I. Area, Connection problems for polynomial solutions of non-homogeneous differential and difference equations. *J. Comput. Appl. Math.* **99** (1998), 177-187.
- [37] M. N. Hounkonnou, S. Belmedi and A. Ronveaux, Linearization of arbitrary products of classical orthogonal polynomials. *Preprint* (1997).
- [38] E. Hylleraas, Linealization of products of Jacobi polynomials. *Math. Scand.* **10** (1962) 189-200.
- [39] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue. *Reports of the Faculty of Technical Mathematics and Informatics* No. 98-17. Delft University of Technology. Delft 1998.
- [40] W. Koepf and D. Schmersau, Representation of orthogonal polynomials. *J. Comput. Appl. Math* **90** (1998) 59-96.
- [41] P.A. Lee, An integral representation and some summation formulas for the Hahn polynomials. *SIAM J. Appl. Math.* **19** (1970), 266-272.
- [42] P.A. Lee, Explicit formula for the multiple generating function of product of generalized Laguerre polynomials. *J. Phys. A: Math. Gen.* **30** (1997), L183-L186.
- [43] S. Lewanowicz, Recurrence relations for the connection coefficients of orthogonal polynomials of a discrete variable. *J. Comp. Appl. Math.* **76** (1996), 213-229.
- [44] S. Lewanowicz and A. Ronveaux, Linearization of powers of an orthogonal polynomial of a discrete variable. *Preprint* (1996).
- [45] S. Lewanowicz, Second-order recurrence relations for the linearization coefficients of the classical orthogonal polynomials. *J. Comput. Appl. Math.* **69** (1996), 159-160.
- [46] S. Lewanowicz, Recurrence relations for the connection coefficients of orthogonal polynomials of a discrete variable on the lattice $x(s) = \exp(2ws)$. *J. Comput. Appl. Math.* **99** (1998) 275-286.
- [47] S. Lewanowicz, A general approach to the connection and linearization problems for the classical orthogonal polynomials. *Preprint* (1997).
- [48] S. Lewanowicz, The hypergeometric functions approach to the connection problem for the classical orthogonal polynomials. *Preprint* (1997) and Private communication.
- [49] C. Market, Linearization of the product of symmetric orthogonal polynomials. *Constr. Approx.* **10**, (1994), 317-338.
- [50] A. Messina and E. Paladino, An operator approach to the construction of generating function for products of associated Laguerre polynomials. *J. Phys. A: Math. Gen.* **29** (1996), L263-L270.
- [51] A.F. Nikiforov and V. B. Uvarov, *Special Functions in Mathematical Physics*, (Birkhauser Verlag, Basel, 1988).
- [52] A.F. Nikiforov, S. K. Suslov and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, (Springer Verlag, Berlin, 1991).
- [53] A. F. Nikiforov and V. B. Uvarov, Polynomial solutions of hypergeometric type difference equations and their classification. *Integral Transforms and Special Functions.* **1** (1993), 223-249.

- [54] A.W. Niukkanen, Clebsch-Gordan-type linearization relations for the products of Laguerre and hydrogen-like functions. *J. Phys. A: Math. Gen.* **18** (1985), 1399-1437.
- [55] M. Petkovšek, Hypergeometric solutions of difference equations with polynomial coefficients. *J. Symb. Comput.* **14** (1992), 243–264.
- [56] M. Petkovšek, H. S. Wilf and D. Zeilberger, *A=B*. A. K. Peters, Wellesley, Massachusetts, 1996.
- [57] L. J. Rogers, Second memoir on the expansion of certain infinite products. *Proc. London Math. Soc.* **25** (1894) 318-343.
- [58] L. J. Rogers, Third memoir on the expansion of certain infinite products. *Proc. London Math. Soc.* **26** (1895) 15-32.
- [59] A. Ronveaux, I. Area, E. Godoy and A. Zarzo, Lectures on recursive approach to connection and linearization coefficients between polynomials. In *Proceedings Int. Workshop on Special Functions and Differential Equations* (Madras, India, 13-24 January 1997).
- [60] A. Ronveaux E. Godoy and A. Zarzo, First associated and co-recursive of classical discrete polynomials. Difference equation and connection problems. *Preprint* (1995)
- [61] A. Ronveaux, A. Zarzo and E. Godoy, Recurrence relation for connection coefficients between two families of orthogonal polynomials. *J. Comput. Appl. Math.*, **62**, (1995), 67-73.
- [62] A. Ronveaux, S. Belmehdi, E. Godoy and A. Zarzo, Recurrence relation for connection coefficients. Applications to classical discrete orthogonal polynomials. In *Proceedings of the Workshop on Symmetries and Integrability of Difference Equations* D. Levi, L. Vinet and P. Witerwitz Eds., C.R.M. Proceed. Lecture Notes Series **9** (AMS Providence, R.I. 1996), 321-337.
- [63] J. Sánchez-Ruiz and J.S. Dehesa, Expansions in series of orthogonal hypergeometric polynomials. *J. Comput. Appl. Math.* **89** (1997) 155-170.
- [64] R. Szwarz, Linearization and connection coefficients of orthogonal polynomials. *Mh. Math.*, **113**, (1992), 319-329.
- [65] R. Szwarz, Connection coefficients of orthogonal polynomials. *Canad. Math. Bull.*, **35** (4), (1992) 548-556.
- [66] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonsky QUANTUM THEORY OF ANGULAR MOMENTUM. (World Scientific, Singapore, 1989).
- [67] M. W. Wilson, Non-negative expansion of polynomials. *Proc. Amer. math. Soc.* **24** (1970) 100-102.
- [68] A. Zarzo, I. Area, E. Godoy and A. Ronveaux, Results on some inversion problems for classical continuous and discrete orthogonal polynomials. *J. Phys. A: Math. Gen.* **30** (1997), L35-L40.