LIMIT RELATIONS BETWEEN GENERALIZED ORTHOGONAL POLYNOMIALS .[1]

R. Álvarez-Nodarse[2] and F. Marcellán[3]
Departamento de Matemáticas. Escuela Politécnica Superior.
Universidad Carlos III de Madrid. Butarque 15, 28911,Leganés,Madrid.

Key words and phrases: Classical Orthogonal polynomials, Limit relations.
AMS (MOS) subject classification: **33A65**

Abstract

We consider the different limit transition for modifications of the classical polynomials via the addition of one or two point masses at the ends of the interval of orthogonality. The connections between Jacobi, Laguerre, Charlier, Meixner, Kravchuk and Hahn generalized polynomials are established.

1 Introduction.

Polynomials orthogonal with respect to measures which are more general than weight functions appear as eigenfunctions of a fourth order linear differential operator with polynomial coefficients. This spectral approach leads to Laguerre-type, Legendre-type and Jacobi-type polynomials introduced by H.L.Krall [20].

A general analysis when a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one delta Dirac measure was started by Chihara [9] in the positive definite case and Marcellán and Maroni [22] for quasi-definite linear functionals. For two point masses there exist very few examples in the literature. (see [19], [11], [17] and [21])

A special emphasis was given to the modifications of classical linear functionals (Hermite, Laguerre, Jacobi and Bessel) in the framework of the so-called semiclassical orthogonal polynomials.

For discrete orthogonal polynomials, Bavinck and van Haeringen [7] obtained an infinite order difference equation for generalized Meixner polynomials, i.e., polynomials orthogonal with respect to the modification of the Meixner weight with a point mass at $x = 0$. The same was found for generalized Charlier polynomials by Bavinck and Koekoek [8].

In a series of papers [2]-[4] we obtained the representation as hypergeometric functions for generalized Meixner, Charlier, Kravchuk and Hahn polynomials as well as the corresponding second order difference equation that such polynomials satisfy. Notice that the coefficients of those difference equations are polynomials of fixed degree and they depend on n as a

[1] October 29, 1996
[2] E-mail: renato@dulcinea.uc3m.es
[3] E-mail: pacomarc@elrond.uc3m.es

parameter.

The aim of the present contribution is to obtain an analogue of the Askey tableau for such a kind of generalized polynomials with the description of the continuous generalized orthogonal polynomials as limit case of the discrete generalized orthogonal polynomials. Furthermore, we deduce the explicit second order linear differential equations for two examples which attracted the interest of the researchers: the Laguerre [13] and the Jacobi [19] case.

In Section 2 we present a summary of the more useful properties of classical polynomials both in the discrete and continuous case.

Section 3 is devoted to an explicit representation of generalized polynomials in terms of the classical ones when we add one point mass at zero (Laguerre, Meixner, Charlier, Kravchuk) or two mass points at the ends of the convex hull of the support of the measure (Jacobi and Hahn). Further, we obtain the explicit expression for second order differential equations (SODE) in the cases of Laguerre and Jacobi. Notice that this SODE was found in [13] for Laguerre case while for the Jacobi case [19] the coefficients were not deduced explicitly. Moreover, an infinite order equation for the Laguerre case was found in [13] as well as for the Gegenbauer case in [16].

In Section 4 the continuous case is obtained as a limit of the discrete one, as well as the different transitions between the discrete families.

2 Some Preliminary Results.

In this Section we have summarized some formulas of the classical orthogonal monic polynomials ($P_n(x) = x^n + \ldots$) which we will use later on. These polynomials are orthogonal with respect to a linear functional $\mathcal{C}$ on the linear space of polynomials with real coefficients defined as ($\mathbb{N} = \{0, 1, 2, ...\}$)

$$< \mathcal{C}, P > \; = \begin{cases} \displaystyle\sum_{x \in \mathbb{N}} \rho(x) P(x), & \text{Hahn, Meixner, Kravchuk and Charlier} \\ \displaystyle\int_a^b \rho(x) P(x)\, dx, & \text{Jacobi and Laguerre} \end{cases} \tag{1}$$

where $\rho(x)$ is a weight function satisfying a Pearson equation.

In the continuous case it has the form

$$\frac{d}{dx}[\sigma(x)\rho(x)] = \tau(x)\rho(x).$$

They satisfy a second order differential equation of hypergeometric type

$$\sigma(x) P_n''(x) + \tau(x) P_n'(x) + \lambda_n P_n(x) = 0, \tag{2}$$

where $\tau(x)$ is a polynomial of degree 1 and $\sigma(x)$ is a polynomial of degree at most 2, such that $\sigma(x)$ vanishes at the ends of the interval of orthogonality. The polynomial solutions of equation (2) are uniquely determined, up to a normalized factor (R_n), by the Rodrigues formula (see [23] page 4 Eq.(1.2.8)):

$$P_n(x) = \frac{R_n}{\rho(x)} \frac{d^n}{d\,x^n} \left[\sigma^n(x)\rho(x)\right]. \tag{3}$$

In the discrete case, the Pearson-type difference equation has the form

$$\triangle[\sigma(x)\rho(x)] = \tau(x)\rho(x),$$

where

$$\nabla f(x) = f(x) - f(x-1), \qquad \triangle f(x) = f(x+1) - f(x).$$

The Pearson-type difference equation can be written in the equivalent form

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x)+\tau(x)}{\sigma(x+1)}.$$

In this case instead of a differential equation, they satisfy a second order difference equation of hypergeometric type

$$\sigma(x) \triangle \nabla P_n(x) + \tau(x) \triangle P_n(x) + \lambda_n P_n(x) = 0, \tag{4}$$

where $\tau(x)$ is also a polynomial of degree 1 and $\sigma(x)$ is a polynomial of degree at most 2, such that $\sigma(x)$ vanishes at one of the ends of the convex hull of the support and $\sigma(x)+\tau(x)$ vanishes in the other. The polynomial solutions of equation (4) are uniquely determined, up to a normalized factor (R_n), by the difference analog of the Rodrigues formula (see [23] page 24 Eq.(2.2.7)):

$$P_n(x) = \frac{R_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^{n} \sigma(x+k)\right]. \tag{5}$$

The orthogonality with respect to the linear funtional $\mathcal{C}$ means that

$$< \mathcal{C}, P_n P_m > \; = \begin{cases} 0 & m \neq n \\ d_n^2 & m = n \end{cases}. \tag{6}$$

In both cases, they satisfy a three term recurrence relation of the form

$$\begin{gathered} x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \ n \geq 0 \\ P_{-1}(x) = 0 \qquad \text{and} \qquad P_0(x) = 1 \end{gathered} \tag{7}$$

and the Christoffel-Darboux formula

$$\sum_{m=0}^{n-1} \frac{P_m(x)P_m(y)}{d_m^2} = \frac{1}{x-y} \frac{a_{n-1}}{a_n} \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{d_{n-1}^2} \qquad n = 1, 2, 3, \ldots \;. \tag{8}$$

Here a_n is the leading coefficient of the polynomial, i.e., the coefficient of the nth power of x in the expansion (in our cases since P_n is monic $a_n = 1$)

$$P_n(x) = a_n x^n + b_n x^{n-1} + \ldots = x^n + b_n x^{n-1} + \ldots \tag{9}$$

We will consider the modification of the following classical monic orthogonal polynomials.

1. **The discrete case.**

1. The Meixner polynomials $M_n^{\gamma,\mu}(x)$, orthogonal with respect to the weight function $\rho(x)$ supported on $[0,\infty)$, where

$$\sigma(x) = x, \quad \tau(x) = \gamma\mu - x(1-\mu) \quad, 0 < \mu < 1, \quad \gamma > 0, \quad \lambda_n = n(1-\mu),$$

and

$$R_n = \frac{1}{(\mu-1)^n}, \qquad \rho(x) = \frac{\mu^x (1-\mu)^\gamma \Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(1+x)}, \qquad d_n^2 = \frac{n!(\gamma)_n \mu^n}{(1-\mu)^{2n}}.$$

2. The Kravchuk polynomials $K_n^p(x)$, orthogonal with respect to the weight function $\rho(x)$ supported on $[0,N]$, with $n \leq N$

$$\sigma(x) = x, \quad \tau(x) = \frac{Np - x}{1-p} \quad, 0 < p < 1, \quad \lambda_n = \frac{n}{1-p},$$

and

$$R_n = (p-1)^n, \quad \rho(x) = \frac{p^x N!(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(1+x)}, \quad d_n^2 = \frac{n!N!p^n(1-p)^n}{(N-n)!}.$$

3. The Charlier polynomials $C_n^\mu(x)$, orthogonal with respect to the weight function $\rho(x)$ supported on $[0,\infty)$, where

$$\sigma(x) = x, \quad , \tau(x) = \mu - x \quad , \mu > 0, \quad \lambda_n = n,$$

and

$$R_n = (-1)^n, \qquad \rho(x) = \frac{\mu^x e^{-\mu}}{\Gamma(1+x)}, \qquad d_n^2 = n!\mu^n.$$

4. The Hahn polynomials $h_n^{\alpha,\beta}(x,N)$, orthogonal with respect to the weight function $\rho(x)$ supported on $[0,N)$, where $(\alpha > -1, \quad \beta > -1)$

$$\sigma(x) = x(x+\alpha-N), \quad \tau(x) = (\beta+1)(N-1) - x(\alpha+\beta+2) \quad , \quad \lambda_n = n(\alpha+\beta+N+1),$$

and

$$R_n = \frac{(-1)^n}{(\alpha+\beta+n+1)_n},$$

$$\rho(x) = \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+1+x)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(1+x)},$$

$$d_n^2 = \frac{\Gamma(N)\Gamma(\alpha+\beta+2)n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)(\alpha+\beta+n+1)_n^{-2}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)(\alpha+\beta+2n+1)(N-n-1)!\Gamma(\alpha+\beta+n+1)}.$$

They satisfy the symmetry property

$$h_n^{\beta,\alpha}(N-1-x,N) = (-1)^n h_n^{\alpha,\beta}(x,N). \tag{10}$$

2. **The continuous case.**

1. The Jacobi polynomials $P_n^{\alpha,\beta}(x)$, orthogonal with respect to the weight function $\rho(x)$ supported on $[-1,1]$, where

$$\sigma(x) = 1 - x^2, \quad \tau(x) = -(\alpha+\beta+2)x + \beta - \alpha \quad , \quad \lambda_n = n(n+\alpha+\beta+1),$$

$$R_n = \frac{(-1)^n}{(n+\alpha+\beta+1)_n},$$

$$\rho(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta \qquad \alpha > -1, \quad \beta > -1,$$

$$d_n^2 = \frac{2^{2n} n! \Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+1)_n^2}.$$

They satisfy the symmetry property

$$P_n^{\beta,\alpha}(-x) = (-1)^n P_n^{\alpha,\beta}(x). \tag{11}$$

2. The Laguerre polynomials $L_n^\alpha(x)$, orthogonal with respect to the weight function $\rho(x)$ supported on $[0,\infty)$, where

$$\sigma(x) = x, \quad \tau(x) = -x + \alpha + 1 \quad , \quad \lambda_n = n,$$

and

$$R_n = (-1)^n, \;\; \rho(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} \qquad \alpha > -1, \qquad d_n^2 = \frac{\Gamma(n+\alpha+1)n!}{\Gamma(\alpha+1)}.$$

In the above formulas we have scaled the weight functions $\rho(x)$ such that they becomes probability measures, i.e., total weight equal 1. This will be useful in order to obtain *the right limits* between the corresponding generalized polynomials.

For all those monic polynomials we also know the values

$$M_n^{\gamma,\mu}(0) = \frac{\mu^n}{(\mu-1)^n}\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}, \quad K_n^p(0) = \frac{(-p)^n N!}{(N-n)!}, \quad C_n^\mu(0) = (-\mu)^n,$$

$$h_n^{\alpha,\beta}(0,N) = \frac{(-1)^n\Gamma(\beta+n+1)(N-1)!}{\Gamma(\beta+1)(N-n-1)!(n+\alpha+\beta+1)_n},$$

$$h_n^{\alpha,\beta}(N-1,N) = \frac{\Gamma(\alpha+n+1)(N-1)!}{\Gamma(\alpha+1)(N-n-1)!(n+\alpha+\beta+1)_n}, \tag{12}$$

$$P_n^{\alpha,\beta}(1) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}, \quad P_n^{\alpha,\beta}(-1) = \frac{2^n(-1)^n(\beta+1)_n}{(n+\alpha+\beta+1)_n},$$

$$L_n^\alpha(0) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}.$$

From the hypergeometric representation of Jacobi polynomials (see [23] - [25]) we can obtain the following two expressions [24]

$$P_{n-1}^{\alpha,\beta+1}(x) = \frac{(2n+\alpha+\beta)(1-x)}{2n(\alpha+n)}\frac{d\,P_n^{\alpha,\beta}}{dx}(x) + \frac{(2n+\alpha+\beta)}{2(\alpha+n)}P_n^{\alpha,\beta}(x) \tag{13}$$

and

$$P_{n-1}^{\alpha+1,\beta}(x) = \frac{(2n+\alpha+\beta)(x+1)}{2n(\beta+n)}\frac{d\,P_n^{\alpha,\beta}}{dx}(x) - \frac{(2n+\alpha+\beta)}{2(\beta+n)}P_n^{\alpha,\beta}(x). \qquad (14)$$

For the kernels of the Charlier, Meixner, Kravchuk, Hahn, Jacobi and Laguerre polynomials we have the following representation (see for instance [2]-[5] and [25])

1. Meixner case

$$Ker_{n-1}^{M}(x,0) \equiv \sum_{m=0}^{n-1}\frac{M_m^{\gamma,\mu}(x)M_m^{\gamma,\mu}(0)}{d_m^2} = \frac{(-1)^{n-1}(1-\mu)^{n-1}}{n!}\nabla M_n^{\gamma,\mu}(x), \qquad (15)$$

$$Ker_{n-1}^{M}(0,0) = \sum_{m=0}^{n-1}\frac{(\gamma)_m\mu^m}{m!}. \qquad (16)$$

2. Kravchuk case

$$Ker_{n-1}^{K}(x,0) \equiv \sum_{m=0}^{n-1}\frac{K_m^{p}(x)K_m^{p}(0)}{d_m^2} = \frac{(p-1)^{1-n}}{n!}\nabla K_n^{p}(x), \qquad (17)$$

$$Ker_{n-1}^{K}(0,0) = \sum_{m=0}^{n-1}\frac{p^m N!}{(1-p)^m m!(N-m)!}. \qquad (18)$$

3. Charlier case

$$Ker_{n-1}^{C}(x,0) \equiv \sum_{m=0}^{n-1}\frac{C_m^{\mu}(x)C_m^{\mu}(0)}{d_m^2} = \frac{(-1)^{n-1}}{n!}\nabla C_n^{\mu}(x), \qquad (19)$$

$$Ker_{n-1}^{C}(0,0) = \sum_{m=0}^{n-1}\frac{\mu^m}{m!}. \qquad (20)$$

4. Hahn case

$$\begin{aligned} Ker_{n-1}^{H,\alpha,\beta}(x,0) &\equiv \sum_{m=0}^{n-1}\frac{h_m^{\alpha,\beta}(x,N)h_m^{\alpha,\beta}(0,N)}{d_m^2} = \kappa_n(\alpha,\beta)\nabla h_n^{\alpha-1,\beta}(x,N), \\ Ker_{n-1}^{H,\alpha,\beta}(x,N-1) &= \kappa_n(\beta,\alpha)(-1)^{n+1}\triangle h_n^{\alpha,\beta-1}(x,N). \end{aligned} \qquad (21)$$

where $\kappa_n(\alpha,\beta)$ denotes the following quantity

$$\kappa_n(\alpha,\beta) = \frac{(-1)^{n-1}\Gamma(\alpha+\beta+2n)\Gamma(\alpha+1)\Gamma(\alpha+\beta+N+1)}{n!\Gamma(\alpha+n)\Gamma(\alpha+\beta+n+N)\Gamma(\alpha+\beta+2)}, \qquad (22)$$

$$\begin{aligned} Ker_{n-1}^{H,\alpha,\beta}(0,0) = & \sum_{m=0}^{n-1}\frac{\Gamma(m+\beta+1)\Gamma(m+\alpha+\beta+1)}{m!\Gamma(\beta+1)(N-m-1)!}\times \\ & \times\frac{(2m+\alpha+\beta+1)(N-1)!\Gamma(\alpha+1)\Gamma(\alpha+\beta+N+1)}{\Gamma(\alpha+m+1)\Gamma(\alpha+\beta+N+m+1)\Gamma(\alpha+\beta+2)}, \end{aligned} \qquad (23)$$

$$Ker_{n-1}^{H,\alpha,\beta}(0,N-1) =$$

$$= \sum_{m=0}^{n-1} \frac{(-1)^m \Gamma(m+\alpha+\beta+1)(2m+\alpha+\beta+1)(N-1)!\Gamma(\alpha+\beta+N+1}{m!(N-m-1)!\Gamma(\alpha+\beta+N+m+1)\Gamma(\alpha+\beta+2)}, \tag{24}$$

and, finally, from the symmetry of the Hahn polynomials (10) we obtain

$$Ker_{n-1}^{H,\alpha,\beta}(N-1,N-1) = Ker_{n-1}^{H,\beta,\alpha}(0,0).$$

5. Laguerre case

$$Ker_{n-1}^{L}(x,0) \equiv \sum_{m=0}^{n-1} \frac{L_m^\alpha(x) L_m^\alpha(0)}{d_m^2} = \frac{(-1)^{n-1}}{n!} (L_n^\alpha)'(x), \tag{25}$$

$$Ker_{n-1}^{L}(0,0) = \sum_{m=0}^{n-1} \frac{(\alpha+1)_m}{m!} = \frac{(\alpha+2)_{n-1}}{(n-1)!}. \tag{26}$$

6. Jacobi case

$$Ker_{n-1}^{J,\alpha,\beta}(x,-1) \equiv \sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x) P_m^{\alpha,\beta}(-1)}{d_m^2} = \eta_n^{\alpha,\beta} \frac{d}{dx} P_n^{\alpha-1,\beta}(x), \tag{27}$$

$$Ker_{n-1}^{J,\alpha,\beta}(x,1) \equiv \sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x) P_m^{\alpha,\beta}(1)}{d_m^2} = (-1)^{n+1} \eta_n^{\beta,\alpha} \frac{d}{dx} P_n^{\alpha,\beta-1}(x), \tag{28}$$

where $\eta_n^{\alpha,\beta}$, $\eta_n^{\beta,\alpha}$ denote the quantities

$$\begin{aligned} \eta_n^{\alpha,\beta} &= \frac{(-1)^{n-1}\Gamma(2n+\alpha+\beta)\Gamma(\alpha+1)}{2^{n-1} n! \Gamma(\alpha+n)\Gamma(\beta+1)\Gamma(\alpha+\beta+2)}, \\ \eta_n^{\beta,\alpha} &= \frac{(-1)^{n-1}\Gamma(2n+\alpha+\beta)\Gamma(\beta+1)}{2^{n-1} n! \Gamma(\beta+n)\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)}. \end{aligned} \tag{29}$$

$$\begin{aligned} Ker_{n-1}^{J,\alpha,\beta}(-1,-1) = & \sum_{m=0}^{n-1} \frac{\Gamma(\beta+m+1)\Gamma(\alpha+\beta+m+1)(2m+\alpha+\beta+1)\Gamma(\alpha+1)}{2^{n-1} m! \Gamma(\beta+1)\Gamma(\alpha+m+1)\Gamma(\alpha+\beta+2)} = \\ = & \frac{\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+1)}{2^{n-1}(n-1)!\Gamma(\beta+2)\Gamma(\alpha+n)\Gamma(\alpha+\beta+2)}, \end{aligned} \tag{30}$$

and

$$\begin{aligned} Ker_{n-1}^{J,\alpha,\beta}(-1,1) = & \sum_{m=0}^{n-1} \frac{(-1)^m \Gamma(\alpha+\beta+m+1)(2m+\alpha+\beta+1)}{2^{n-1} m! \Gamma(\alpha+\beta+2)} = \\ = & \frac{(-1)^{n-1}\Gamma(\alpha+\beta+n+1)}{2^{n-1}(n-1)!}. \end{aligned} \tag{31}$$

and, finally, from the symmetry property of the Jacobi (11) polynomials we have

$$Ker_{n-1}^{J,\alpha,\beta}(1,1) = Ker_{n-1}^{J,\beta,\alpha}(-1,-1).$$

Using the relations (13)-(14) we also obtain the following equivalent formulas for the kernels (27) and (28)

$$Ker_{n-1}^{J,\alpha,\beta}(x,-1) = \tilde{\eta}_n^{\alpha,\beta}\left[(1-x)\frac{dP_n^{\alpha,\beta}(x)}{dx} + n\,P_n^{\alpha,\beta}(x)\right], \tag{32}$$

$$Ker_{n-1}^{J,\alpha,\beta}(x,1) = (-1)^{n+1}\tilde{\eta}_n^{\beta,\alpha}\left[(1+x)\frac{dP_n^{\alpha,\beta}(x)}{dx} - n\,P_n^{\alpha,\beta}(x)\right], \tag{33}$$

where $\tilde{\eta}_n^{\alpha,\beta}$, $\tilde{\eta}_n^{\beta,\alpha}$ denotes the quantities

$$\begin{aligned} \tilde{\eta}_n^{\alpha,\beta} &= -\frac{(-1)^n\Gamma(2n+\alpha+\beta+1)\Gamma(\alpha+1)}{2^n n!\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+2)}, \\ \tilde{\eta}_n^{\beta,\alpha} &= -\frac{(-1)^n\Gamma(2n+\alpha+\beta+1)\Gamma(\beta+1)}{2^n n!\Gamma(\beta+n+1)\Gamma(\alpha+\beta+2)}. \end{aligned} \tag{34}$$

3 The definition and the representation.

Firstly, we will consider the case when we add a point mass at $x=0$. This case corresponds to the Laguerre, Charlier, Meixner and Kravchuk polynomials. Later on, we will consider the Jacobi and Hahn polynomials which involve two point masses at the ends of the interval of orthogonality. The reason of such a choice of the point in which we will add our positive mass will be clear from formulas (39) and (41) from below, because in such formulas appears the value of the kernel polynomials $K_n(x,y)$ and they have a very simple analytical expression in the case when y takes the values of the zeros of $\sigma(x)$ (for the continuous case) or one of the zeros of $\sigma(x)$ and $\sigma(x)+\tau(x)$ (for the discrete case). In fact this gives us a simple expression for the kernels in terms of the same polynomials, its derivatives or difference-derivatives (see (15)-(31)).

3.1 The Case of one point mass.

Consider the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients defined as

$$< \mathcal{U}, P > \; = < \mathcal{C}, P > + AP(0), \quad , \quad A \geq 0 \;\; , \tag{35}$$

where $\mathcal{C}$ is a classical moment functional (1) associated to some Meixner, Charlier and Kravchuk polynomials of a discrete variable and Laguerre polynomials, respectively.

We will determine the monic polynomials $P_n^A(x)$ which are orthogonal with respect to the functional $\mathcal{U}$ and we will prove that they exist for all positive A (see (40) from below). To obtain this, we can write the Fourier expansion of such generalized polynomials

$$P_n^A(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k}P_k(x), \tag{36}$$

where P_n denotes the classical monic orthogonal polynomial (CMOP) of degree n.

In order to find the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $P_n^A(x)$ with respect to $\mathcal{U}$, i.e.,

$$< \mathcal{U}, P_n^A(x) P_k(x) >= 0 \qquad \forall k < n.$$

Now putting (36) in (35) we find:

$$< \mathcal{U}, P_n^A(x) P_k(x) > \; = \; < \mathcal{C}, P_n^A(x) P_k(x) > \quad + \quad A P_n^A(0) P_k(0). \tag{37}$$

If we use the decomposition (36) and taking into account the orthogonality of the classical orthogonal polynomials with respect to the linear functional $\mathcal{C}$, then the coefficients $a_{n,k}$ are given by

$$a_{n,k} = -A \; \frac{P_n^A(0) P_k(0)}{d_k^2}. \tag{38}$$

Finally the equation (36) provides us the expression

$$P_n^A(x) = P_n(x) - A P_n^A(0) \sum_{k=0}^{n-1} \frac{P_k(0) P_k(x)}{d_k^2} = P_n(x) - A P_n^A(0) Ker_{n-1}(x, 0). \tag{39}$$

From (39) we can conclude that the representation of $P_n^A(x)$ exists for any positive value of the mass A. To obtain this it is enough to evaluate (39) in $x = 0$,

$$\left(1 + A \sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2} \right) P_n^A(0) = P_n(0) \neq 0, \tag{40}$$

and use the fact that

$$1 + A \sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2} \quad > \quad 0 \qquad\qquad n = 1, 2, 3, ...$$

From (40) we can deduce the values of $P_n^A(0)$ as follows

$$P_n^A(0) = \frac{P_n(0)}{1 + A \displaystyle\sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2}}. \tag{41}$$

From (39) and taking into account formulas (15)-(25) as well as (41), we obtain the following expressions for the generalized polynomials (for more details see [2],[3], [5] and [13])

For Meixner polynomials

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) + B_n \bigtriangledown M_n^{\gamma,\mu}(x) = (I + B_n \bigtriangledown) M_n^{\gamma,\mu}(x), \tag{42}$$

$$B_n = A \frac{\mu^n (1-\mu)^{-1} (\gamma)_n}{n! (1 + A Ker_{n-1}^M(0,0))}.$$

For Kravchuk polynomials

$$K_n^{p,A}(x) = K_n^p(x) + A_n \bigtriangledown K_n^p(x) = (I + A_n \bigtriangledown) K_n^p(x), \tag{43}$$

$$A_n = A \frac{N!}{n!(N-n)!} \frac{p^n (1-p)^{1-n}}{(1 + A Ker_{n-1}^K(0,0))}.$$

For Charlier polynomials

$$C_n^{\mu,A}(x) = C_n^\mu(x) + D_n \bigtriangledown C_n^\mu(x) = (I + D_n \bigtriangledown) C_n^\mu(x), \tag{44}$$

$$D_n = A \frac{\mu^n}{n!(1 + A Ker_{n-1}^C(0,0))}.$$

For Laguerre polynomials

$$L_n^{\alpha,A}(x) = L_n^\alpha(x) + \Gamma_n \frac{d}{dx} L_n^\alpha(x) = (I + \Gamma_n \frac{d}{dx}) L_n^\alpha(x), \tag{45}$$

$$\Gamma_n = \frac{A(\alpha+1)_n}{n!(1 + A Ker_{n-1}^L(0,0))} = \frac{A(\alpha+1)_n}{n!\left(1 + A\frac{(\alpha+2)_{n-1}}{(n-1)!}\right)}.$$

3.2 The Case of two point masses.

Consider the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients defined as $(A, B \geq 0)$

$$< \mathcal{U}, P > \; = \begin{cases} < \mathcal{C}, P > + AP(0) + BP(N-1), & \text{Hahn case} \\ < \mathcal{C}, P > + AP(1) + BP(-1), & \text{Jacobi case} \end{cases}. \tag{46}$$

where $\mathcal{C}$ is a classical moment functional (1) associated with the classical Hahn and Jacobi polynomials, respectively.

We will determine the monic polynomials $P_n^{A,B}(x)$ which are orthogonal with respect to the functional $\mathcal{U}$ and prove that they exist for all positive values of the masses A and B.

Let us write the Fourier expansion of such generalized polynomials in terms of the classical monic orthogonal polynomials under consideration (Hahn or Jacobi).

$$P_n^{A,B}(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k} P_k(x). \tag{47}$$

In order to obtain the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $P_n^{A,B}(x)$ with respect to $\mathcal{U}$, i.e.,

$$< \mathcal{U}, \, P_n^{A,B}(x) \, P_k(x) >= 0 \qquad 0 \leq k < n.$$

Now putting (47) in (46) we find

$$0 = \; < \mathcal{C}, P_n^{A,B}(x) \, P_k(x) > \; + \\ + \begin{cases} AP_n^{A,B}(0) \, P_k(0) + BP_n^{A,B}(N-1) P_k(N-1), & \text{Hahn case} \\ AP_n^{A,B}(-1) \, P_k(-1) + BP_n^{A,B}(1) P_k(1), & \text{Jacobi case} \end{cases}. \tag{48}$$

In order to obtain the coefficients $a_{n,k}$ of the Fourier expansion (47) we can use, as before, the orthogonality of the classical orthogonal polynomials with respect to the linear functional $\mathcal{C}$ and from equation (47) we obtain

$$P_n^{A,B}(x) = P_n(x) + \left\{ \begin{array}{ll} -AP_n^{A,B}(0)\,Ker_{n-1}(x,0) - BP_n^{A,B}(N-1)\,Ker_{n-1}(x,N-1), & \text{Hahn case} \\ -AP_n^{A,B}(-1)\,Ker_{n-1}(x,-1) - BP_n^{A,B}(1)\,Ker_{n-1}(x,1), & \text{Jacobi case} \end{array} \right. . \tag{49}$$

From the last expression and using the Eqs. (21)-(24) for the Hahn polynomials we find (for more details see [4])

$$\begin{aligned} h_n^{A,B,\alpha,\beta}(x,N) = \quad & h_n^{\alpha,\beta}(x,N) - Ah_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta) \nabla h_n^{\alpha-1,\beta}(x,N) - \\ & - Bh_n^{A,B,\alpha,\beta}(N-1,N)\kappa_n(\beta,\alpha)(-1)^{n+1} \triangle h_n^{\alpha,\beta-1}(x,N). \end{aligned} \tag{50}$$

where $\kappa_n(\alpha,\beta)$ is given in (22), $h_n^{A,B,\alpha,\beta}(0,N)$ and $h_n^{A,B,\alpha,\beta}(N-1,N)$ are given by formulas

$$h_n^{A,B,\alpha,\beta}(0,N) = \frac{\begin{vmatrix} h_n^{\alpha,\beta}(0,N) & BKer_{n-1}^{H,\alpha,\beta}(0,N-1) \\ h_n^{\alpha,\beta}(N-1,N) & 1 + BKer_{n-1}^{H,\alpha,\beta}(N-1,N-1) \end{vmatrix}}{\begin{vmatrix} 1 + AKer_{n-1}^{H,\alpha,\beta}(0,0) & BKer_{n-1}^{H,\alpha,\beta}(0,N-1) \\ AKer_{n-1}^{H,\alpha,\beta}(0,N-1) & 1 + BKer_{n-1}^{H,\alpha,\beta}(N-1,N-1) \end{vmatrix}}, \tag{51}$$

and

$$h_n^{A,B,\alpha,\beta}(N-1,N) = \frac{\begin{vmatrix} 1 + AKer_{n-1}^{H,\alpha,\beta}(0,0) & h_n^{H,\alpha,\beta}(0,N) \\ AKer_{n-1}^{H,\alpha,\beta}(0,N-1) & h_n^{H,\alpha,\beta}(N-1,N) \end{vmatrix}}{\begin{vmatrix} 1 + AKer_{n-1}^{H,\alpha,\beta}(0,0) & BKer_{n-1}^{H,\alpha,\beta}(0,N-1) \\ AKer_{n-1}^{H,\alpha,\beta}(0,N-1) & 1 + BKer_{n-1}^{H,\alpha,\beta}(N-1,N-1) \end{vmatrix}}, \tag{52}$$

respectively, or

$$h_n^{A,B,\alpha,\beta}(x,N) = h_n^{\alpha,\beta}(x,N) + \tau_{A,B}^{n,\alpha,\beta} \nabla h_n^{\alpha-1,\beta}(x,N) - \tau_{B,A}^{n,\beta,\alpha} \triangle h_n^{\alpha,\beta-1}(x,N), \tag{53}$$

where $\tau_{A,B}^{n,\alpha,\beta} = -Ah_n^{A,B,\alpha,\beta}(0,N)\kappa_n(\alpha,\beta)$ and $\tau_{B,A}^{n,\beta,\alpha} = -Bh_n^{B,A,\beta,\alpha}(0,N)\kappa_n(\beta,\alpha)$. In the case when $B=0$ we obtain $\tau_{0,A}^{n,\beta,\alpha} \equiv 0$ and

$$\begin{aligned} \tau_A^{n,\alpha,\beta} \equiv \tau_{A,0}^{n,\alpha,\beta} = A & \frac{\Gamma(N)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+1)}{\Gamma(\beta+1)n!(N-n-1)!\Gamma(\alpha+n)\Gamma(\alpha+\beta+n+N)} \times \\ & \times \frac{\Gamma(\alpha+\beta+N+1)}{\Gamma(\alpha+\beta+2)(\alpha+\beta+2n)(1+AKer_{n-1}^{H,\alpha,\beta}(0,0))}. \end{aligned} \tag{54}$$

For Jacobi polynomials from Eq. (49) by using (27), (28), (29) we obtain (for more details see [19])

$$P_n^{A,B,\alpha,\beta}(x) = \quad P_n^{\alpha,\beta}(x) - AP_n^{A,B,\alpha,\beta}(-1)\eta_n^{\alpha,\beta}\frac{d}{dx}P_n^{\alpha-1,\beta}(x) - $$
$$ -BP_n^{A,B,\alpha,\beta}(1)\eta_n^{\beta,\alpha}(-1)^{n-1}\frac{d}{dx}P_n^{\alpha,\beta-1}(x), \tag{55}$$

where $\eta_n^{\alpha,\beta}$, $\eta_n^{\beta,\alpha}$ are given in (29) and $P_n^{A,B,\alpha,\beta}(-1)$ and $P_n^{A,B,\alpha,\beta}(1,N)$ are given by

$$P_n^{A,B,\alpha,\beta}(-1) = \frac{\begin{vmatrix} P_n^{\alpha,\beta}(-1) & BKer_{n-1}^{J,\alpha,\beta}(-1,1) \\ P_n^{\alpha,\beta}(1) & 1+BKer_{n-1}^{J,\alpha,\beta}(1,1) \end{vmatrix}}{\begin{vmatrix} 1+AKer_{n-1}^{J,\alpha,\beta}(-1,-1) & BKer_{n-1}^{J,\alpha,\beta}(-1,1) \\ AKer_{n-1}^{J,\alpha,\beta}(-1,1) & 1+BKer_{n-1}^{J,\alpha,\beta}(1,1) \end{vmatrix}}, \tag{56}$$

and

$$P_n^{A,B,\alpha,\beta}(1) = \frac{\begin{vmatrix} 1+AKer_{n-1}^{J,\alpha,\beta}(-1,-1) & P_n^{\alpha,\beta}(-1) \\ AKer_{n-1}^{J,\alpha,\beta}(-1,1) & P_n^{\alpha,\beta}(1) \end{vmatrix}}{\begin{vmatrix} 1+AKer_{n-1}^{J,\alpha,\beta}(-1,-1) & BKer_{n-1}^{J,\alpha,\beta}(-1,1) \\ AKer_{n-1}^{J,\alpha,\beta}(-1,1) & 1+BKer_{n-1}^{J,\alpha,\beta}(1,1) \end{vmatrix}}, \tag{57}$$

respectively, or

$$P_n^{A,B,\alpha,\beta}(x) = P_n^{\alpha,\beta}(x) + \chi_{A,B}^{n,\alpha,\beta}\frac{d}{dx}P_n^{\alpha-1,\beta}(x) - \chi_{B,A}^{n,\beta,\alpha}\frac{d}{dx}P_n^{\alpha,\beta-1}(x), \tag{58}$$

where $\chi_{A,B}^{n,\alpha,\beta} = -AP_n^{A,B,\alpha,\beta}(-1)\eta_n^{\alpha,\beta}$ and $\chi_{B,A}^{n,\beta,\alpha} = -BP_n^{B,A,\beta,\alpha}(-1)\eta_n^{\beta,\alpha}$.

Using the expressions (32), (33), (34) and (49) we obtain an equivalent representation, similar to the representation obtained in [19] for the monic generalized polynomials

$$P_n^{A,B,\alpha,\beta}(x) \quad = (1 - nJ_{A,B}^{n,\alpha,\beta} - nJ_{B,A}^{n,\beta,\alpha})P_n^{\alpha,\beta}(x) +$$
$$+[J_{A,B}^{n,\alpha,\beta}(x-1) + J_{B,A}^{n,\beta,\alpha}(1+x)]\frac{d}{dx}P_n^{\alpha,\beta}(x), \tag{59}$$

where $J_{A,B}^{n,\alpha,\beta} = -AP_n^{A,B,\alpha,\beta}(-1)\tilde{\eta}_n^{\alpha,\beta}$, $J_{B,A}^{n,\beta,\alpha} = -BP_n^{B,A,\beta,\alpha}(-1)\tilde{\eta}_n(\beta,\alpha)$ and $\tilde{\eta}_n^{\beta,\alpha}$ are defined in (34).

Remark I From the last expressions for generalized Hahn and Jacobi polynomials we can conclude their existence for all positive values of the masses. In fact, if we expand the denominators in (51), (52), (56) and (57) and use the symmetry properties (10) and (11), as well as the Cauchy-Schwarz inequality the desired result follows.

Remark II From the representation formulas (53) and (58), as well as the symmetry properties (10) and (11) we can obtain the following symmetry properties for generalized polynomials

$$h_n^{B,A,\beta,\alpha}(N-1-x) = (-1)^n h_n^{A,B,\alpha,\beta}(x), \tag{60}$$

$$P_n^{B,A,\beta,\alpha}(-x) = (-1)^n P_n^{A,B,\alpha,\beta}(x). \tag{61}$$

The second order differential ecuation for generalized Jacobi and Laguerre polynomials.

Before to obtain the limit relations between these generalized orthogonal polynomials, let us to obtain explicitly the differential equation that the Koornwinder-Jacobi polynomials $P_n^{A,B,\alpha,\beta}(x)$ satisfy. In [19] Koornwinder proved that the generalized polynomials satisfy a second order differential equation, but he did not write it explicitly. The existence of such a differential equation is a straightforward consequence of the semiclassical character of such polynomials [22]. We will present an algorithm to obtain the differential equation for the both Laguerre and Jacobi generalized polynomials.

First of all, we will rewrite (45) and (59) in the unique form

$$\tilde{P}_n(x) = C_p P_n(x) + q_p(x) \tfrac{d}{dx} P_n(x), \tag{62}$$

where $\tilde{P}_n(x)$ denotes the generalized Laguerre or Jacobi polynomials and $P_n(x)$ denotes the corresponding classical polynomials, respectively. Here, $C_l = 1$ and $q_l(x) = \Gamma_n$ for the Laguerre polynomials (45) and $C_j = (1 - nJ_{A,B}^{n,\alpha,\beta} - nJ_{n,\beta,\alpha}^{B,A})$ and $q_j(x) = (x-1)J_{\alpha,\beta}^{A,B} + (1+x)J_{\beta,\alpha}^{B,A}$ for the Jacobi ones.

Taking derivatives in (62), multiplying by $\sigma(x)$ and using the second order differential equation that classical polynomials satisfy (2) $\sigma(x)P_n''(x) = -\tau(x)P_n'(x) - \lambda_n P_n(x)$ we obtain

$$\begin{aligned} &\sigma(x) \tfrac{d}{dx} \tilde{P}_n(x) = c(x) P_n(x) + d(x) \tfrac{d}{dx} P_n(x), \\ &\text{where } c(x) = -q_p(x)\lambda_n \text{ y } d(x) = \sigma(x)[C_p + q'] - \tau(x) q_p(x). \end{aligned} \tag{63}$$

Now, taking second derivatives in (62), multiplying by $\sigma(x)^2$ and using again (2), as well as their derivatives we find the following

$$\begin{aligned} &\sigma(x)^2 \tfrac{d^2}{dx^2} \tilde{P}_n(x) = e(x) P_n(x) + f(x) \tfrac{d}{dx} P_n(x), \\ &e(x) = \lambda_n \{[\tau(x) + \sigma'(x)] q_p(x) - [C_p + 2q_p'] \sigma(x)\} \\ &f(x) = q_p(x)\{\tau(x)[\tau(x) + \sigma'(x)] - \sigma(x)[\lambda_n + \tau']\} - [C_p + 2q_p'] \sigma(x) \tau(x). \end{aligned} \tag{64}$$

Then the following determinant vanishes

$$\begin{vmatrix} \tilde{P}_n(x) & a(x) & b(x) \\ \sigma(x)\tilde{P}_n'(x) & c(x) & d(x) \\ \sigma(x)^2 \tilde{P}_n''(x) & e(x) & f(x) \end{vmatrix} = 0\,, \tag{65}$$

where $a(x) = C_p$ and $b(x) = q_p(x)$. Expanding the determinant in (65) by the first column we obtain that the Laguerre and Jacobi polynomials satisfy the following equation:

$$\tilde{\sigma}_n(x) \frac{d^2}{dx^2} \tilde{P}_n(x) + \tilde{\tau}_n(x) \frac{d}{dx} \tilde{P}_n(x) + \tilde{\lambda}_n(x) \tilde{P}_n(x),$$

where

$$\begin{aligned} &\tilde{\sigma}_n(x) = \sigma(x)^2 [a(x)d(x) - c(x)b(x)], \\ &\tilde{\tau}_n(x) = \sigma(x)[e(x)b(x) - a(x)f(x)], \\ &\tilde{\lambda}_n(x) = c(x)f(x) - e(x)d(x). \end{aligned} \tag{66}$$

To obtain the explicit form of the coefficients $\tilde{\sigma}_n(x)$, , $\tilde{\tau}_n(x)$ and $\tilde{\lambda}_n(x)$ we implement a little program using the well-known program *Mathematica* [26]. Here will apply it to obtain the Koornwinder-Jacobi's differential equation.

```
In[1]:=
Remove["Global`*"]
In[2]:=
p[x_]:= CBA (x+1) + CAB (x-1)
dp=D[p[x],x];
const=1- n*CAB - n*CBA;
sig[x_]:= 1-x^2;
delsig[x_]=D[sig[x],x];
tau[x_]:= (beta-alpha) - (alpha+beta+2)x
deltau=D[tau[x],x];
ln=n(alpha+beta+n+1);
```

The functions $a(x), ..., f(x)$, defined in (63)-(64) are denoted by $a, ..., f$, respectively.

```
In[10]:=
a= Expand[ const ];
b =Expand[  p[x]   ];
c= Expand[ -ln p[x]   ];
d= Expand[ (const  + dp) sig[x] - tau[x] p[x] ];
e= Expand[ p[x]ln(delsig[x]+tau[x])-
    sig[x] ln (const+ 2 dp)];
f= Expand[ - (const + 2 dp) tau[x] sig[x] +
   p[x](tau[x]( tau[x]+delsig[x])-sig[x](deltau+ln))];
In[16]:=
newsigma=sig[x]^2 Simplify[Expand[a d - c b]];
newtau=Expand[sig[x] ( e b - a f)];
lambda=Expand[(c f - e d) ];
p=Simplify[{lambda ,  newtau , newsigma}/sig[x]];

In[20]:=
Simplify[p/sig[x]-{ln,tau[x],sig[x]} //.{CAB->0,CBA->0}]
Out[20]=
{0, 0, 0}
```

Using the above algorithm and the *Mathematica* program we obtain

- **Generalized Laguerre polynomials.** [13]

$$\begin{aligned}
\tilde{\sigma}_n(x) &= x\left(-\Gamma_n - \alpha\,\Gamma_n + {\Gamma_n}^2\,n + x + \Gamma_n\,x\right), \\
\tilde{\tau}_n(x) &= \left(-2\,\Gamma_n - 3\,\alpha\,\Gamma_n - \alpha^2\,\Gamma_n + 2\,{\Gamma_n}^2\,n + \alpha\,{\Gamma_n}^2\,n + x + \alpha\,x + \right. \\
&\quad \left. + 2\,\Gamma_n\,x + 2\,\alpha\,\Gamma_n\,x - {\Gamma_n}^2\,n\,x - x^2 - \Gamma_n\,x^2\right), \\
\tilde{\lambda}_n(x) &= n\left(-2\,\Gamma_n - \alpha\,\Gamma_n - {\Gamma_n}^2 + {\Gamma_n}^2\,n + x + \Gamma_n\,x\right).
\end{aligned}$$

Taking the limit $A \to 0$ we obtain

$$\lim_{A\to 0} \tilde\sigma_n(x) = x^2 = \sigma(x)^2,$$

$$\lim_{A\to 0} \tilde\tau_n(x) = (1+\alpha-x)\; x = \sigma(x)\tau(x),$$

$$\lim_{A\to 0} \tilde\lambda_n(x) = n\,x = \sigma(x)\lambda_n.$$

- **Generalized Jacobi polynomials.** [19]

$$\begin{aligned}
\tilde\sigma_n(x) \;&= \left(1-x^2\right)\,(\;1 + J_{A,B}^{n,\alpha,\beta} - \alpha\,J_{A,B}^{n,\alpha,\beta} + \beta\,J_{A,B}^{n,\alpha,\beta} + J_{B,A}^{n,\beta,\alpha} + \alpha\,J_{B,A}^{n,\beta,\alpha} - \beta\,J_{B,A}^{n,\beta,\alpha} - 2\,J_{A,B}^{n,\alpha,\beta}\,n + \\
&+2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n - 2\,J_{B,A}^{n,\beta,\alpha}\,n - 4\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n + \\
&+2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n + 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2 + 2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2 - 2\,J_{A,B}^{n,\alpha,\beta}\,x - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x + 2\,J_{B,A}^{n,\beta,\alpha}\,x + 2\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,x - \\
&-2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x - 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x - 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x + \\
&+2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x - x^2 + J_{A,B}^{n,\alpha,\beta}\,x^2 + \alpha\,J_{A,B}^{n,\alpha,\beta}\,x^2 + \beta\,J_{A,B}^{n,\alpha,\beta}\,x^2 + J_{B,A}^{n,\beta,\alpha}\,x^2 + \alpha\,J_{B,A}^{n,\beta,\alpha}\,x^2 + \beta\,J_{B,A}^{n,\beta,\alpha}\,x^2 \\
&+ + 2\,J_{A,B}^{n,\alpha,\beta}\,n\,x^2 + 2\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2\;)
\end{aligned}$$

$$\begin{aligned}
\tilde\lambda_n(x) \;&= n\;(1+\alpha+\beta+n)\,(\;1 + 3\,J_{A,B}^{n,\alpha,\beta} - \alpha\,J_{A,B}^{n,\alpha,\beta} + \beta\,J_{A,B}^{n,\alpha,\beta} - 2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2 + 3\,J_{B,A}^{n,\beta,\alpha} + \\
&+\alpha\,J_{B,A}^{n,\beta,\alpha} - \beta\,J_{B,A}^{n,\beta,\alpha} + 8\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha} + 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha} + 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha} - \\
&2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2 - 2\,J_{A,B}^{n,\alpha,\beta}\,n - 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n + 2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n - 2\,J_{B,A}^{n,\beta,\alpha}\,n - 8\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - \\
&-2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n + 2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n + 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2 + \\
&+2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2 - 4\,J_{A,B}^{n,\alpha,\beta}\,x - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x + 2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,x + 4\,J_{B,A}^{n,\beta,\alpha}\,x + 2\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,x + \\
&+2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,x - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,x - 2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,x + 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x - 2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x - \\
&-2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x - 2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x + 2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x - 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x + \\
&+2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x - x^2 + J_{A,B}^{n,\alpha,\beta}\,x^2 + \alpha\,J_{A,B}^{n,\alpha,\beta}\,x^2 + \beta\,J_{A,B}^{n,\alpha,\beta}\,x^2 + J_{B,A}^{n,\beta,\alpha}\,x^2 + \alpha\,J_{B,A}^{n,\beta,\alpha}\,x^2 + \\
&+\beta\,J_{B,A}^{n,\beta,\alpha}\,x^2 + 2\,J_{A,B}^{n,\alpha,\beta}\,n\,x^2 + 2\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2\;)
\end{aligned}$$

$$\begin{aligned}
\tilde\tau_n(x) = \;& -\alpha + \beta + 2\,J_{A,B}^{n,\alpha,\beta} - \alpha\,J_{A,B}^{n,\alpha,\beta} + \alpha^2\,J_{A,B}^{n,\alpha,\beta} + 3\,\beta\,J_{A,B}^{n,\alpha,\beta} - 2\,\alpha\,\beta\,J_{A,B}^{n,\alpha,\beta} + \beta^2\,J_{A,B}^{n,\alpha,\beta} - 2\,J_{B,A}^{n,\beta,\alpha} - \\
&-3\,\alpha\,J_{B,A}^{n,\beta,\alpha} - \alpha^2\,J_{B,A}^{n,\beta,\alpha} + \beta\,J_{B,A}^{n,\beta,\alpha} + 2\,\alpha\,\beta\,J_{B,A}^{n,\beta,\alpha} - \beta^2\,J_{B,A}^{n,\beta,\alpha} + 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,n - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,n + \\
&+2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n - 2\,\alpha^2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n + 2\,\alpha\,\beta\,{J_{A,B}^{n,\alpha,\beta}}^2\,n + 2\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,\beta\,J_{B,A}^{n,\beta,\alpha}\,n + 6\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n + \\
&+2\,\alpha^2\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 6\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,\beta^2\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n - 2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n - 2\,\alpha\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n + \\
&+2\,\beta^2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n + 2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2 - 2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2 + 2\,\beta\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2 - 2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2 - 2\,\alpha\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2 + \\
&+2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2 - 2\,x - \alpha\,x - \beta\,x - 6\,J_{A,B}^{n,\alpha,\beta}\,x + 3\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,x + \alpha^2\,J_{A,B}^{n,\alpha,\beta}\,x - 9\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x + 2\,\alpha\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x - \\
&-3\,\beta^2\,J_{A,B}^{n,\alpha,\beta}\,x - 6\,J_{B,A}^{n,\beta,\alpha}\,x - 9\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,x - 3\,\alpha^2\,J_{B,A}^{n,\beta,\alpha}\,x + 3\,\beta\,J_{B,A}^{n,\beta,\alpha}\,x + 2\,\alpha\,\beta\,J_{B,A}^{n,\beta,\alpha}\,x + \beta^2\,J_{B,A}^{n,\beta,\alpha}\,x + \\
&+4\,J_{A,B}^{n,\alpha,\beta}\,n\,x + 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,n\,x + 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,n\,x - 8\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x - 4\,\alpha\,\beta\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x + 4\,J_{B,A}^{n,\beta,\alpha}\,n\,x + \\
&+2\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 2\,\beta\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 16\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 12\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 4\,\alpha^2\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + \\
&+12\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x + 4\,\beta^2\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x - 8\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x - 4\,\alpha\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x - 8\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x - \\
&-4\,\beta\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x - 8\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x - 4\,\alpha\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x + \alpha\,x^2 - \beta\,x^2 + 6\,J_{A,B}^{n,\alpha,\beta}\,x^2 + \alpha\,J_{A,B}^{n,\alpha,\beta}\,x^2 - \\
&-\alpha^2\,J_{A,B}^{n,\alpha,\beta}\,x^2 + 9\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x^2 + 2\,\alpha\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x^2 + 3\,\beta^2\,J_{A,B}^{n,\alpha,\beta}\,x^2 - 6\,J_{B,A}^{n,\beta,\alpha}\,x^2 - 9\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,x^2 - \\
&-3\,\alpha^2\,J_{B,A}^{n,\beta,\alpha}\,x^2 - \beta\,J_{B,A}^{n,\beta,\alpha}\,x^2 - 2\,\alpha\,\beta\,J_{B,A}^{n,\beta,\alpha}\,x^2 + \beta^2\,J_{B,A}^{n,\beta,\alpha}\,x^2 - 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,n\,x^2 + 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,n\,x^2 + \\
&+6\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x^2 + 2\,\alpha^2\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x^2 + 2\,\alpha\,\beta\,{J_{A,B}^{n,\alpha,\beta}}^2\,n\,x^2 - 2\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2 + 2\,\beta\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2 + \\
&+6\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2 + 2\,\alpha^2\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2 - 6\,\beta\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2 - 2\,\beta^2\,J_{A,B}^{n,\alpha,\beta}\,J_{B,A}^{n,\beta,\alpha}\,n\,x^2 - \\
&-6\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x^2 - 2\,\alpha\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x^2 - 2\,\beta^2\,{J_{B,A}^{n,\beta,\alpha}}^2\,n\,x^2 + 6\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x^2 + 2\,\alpha\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x^2 + \\
&+2\,\beta\,{J_{A,B}^{n,\alpha,\beta}}^2\,n^2\,x^2 - 6\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x^2 - 2\,\alpha\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x^2 - 2\,\beta\,{J_{B,A}^{n,\beta,\alpha}}^2\,n^2\,x^2 + 2\,x^3 + \alpha\,x^3 + \beta\,x^3 - \\
&-2\,J_{A,B}^{n,\alpha,\beta}\,x^3 - 3\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,x^3 - \alpha^2\,J_{A,B}^{n,\alpha,\beta}\,x^3 - 3\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x^3 - 2\,\alpha\,\beta\,J_{A,B}^{n,\alpha,\beta}\,x^3 - \beta^2\,J_{A,B}^{n,\alpha,\beta}\,x^3 - \\
&-2\,J_{B,A}^{n,\beta,\alpha}\,x^3 - 3\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,x^3 - \alpha^2\,J_{B,A}^{n,\beta,\alpha}\,x^3 - 3\,\beta\,J_{B,A}^{n,\beta,\alpha}\,x^3 - 2\,\alpha\,\beta\,J_{B,A}^{n,\beta,\alpha}\,x^3 - \beta^2\,J_{B,A}^{n,\beta,\alpha}\,x^3 - \\
&-4\,J_{A,B}^{n,\alpha,\beta}\,n\,x^3 - 2\,\alpha\,J_{A,B}^{n,\alpha,\beta}\,n\,x^3 - 2\,\beta\,J_{A,B}^{n,\alpha,\beta}\,n\,x^3 - 4\,J_{B,A}^{n,\beta,\alpha}\,n\,x^3 - 2\,\alpha\,J_{B,A}^{n,\beta,\alpha}\,n\,x^3 - 2\,\beta\,J_{B,A}^{n,\beta,\alpha}\,n\,x^3
\end{aligned}$$

Taking the limit $A, B \to 0$ we obtain

$$\lim_{A,B\to 0} \tilde{\sigma}_n(x) = \left(1 - x^2\right)^2 = \sigma(x)^2,$$

$$\lim_{A,B\to 0} \tilde{\lambda}_n(x) = n\,(1 + \alpha + \beta + n)\,\left(1 - x^2\right) = \sigma(x)\lambda_n,$$

$$\lim_{A,B\to 0} \tilde{\tau}_n(x) = \left(1 - x^2\right)(-\alpha + \beta - 2\,x - \alpha\, x - \beta\, x) = \sigma(x)\tau(x).$$

4 Limit relations between modifications of orthogonal polynomials.

In this Section we will study limit relations involving the modifications of the Jacobi and Laguerre polynomials as well as the modifications of the classical polynomials of discrete variables. In some way we will obtain an analogue of the Askey-scheme of hypergeometric polynomials (for a review see [18]). Results are predictible but we have found nothing of this kind in the literature. Anyway, we want to remark that the main difference with respect to the classical case is the fact, as we will show below, that the point masses change.

4.1 Limit Meixner $\longrightarrow$ Laguerre.

The limit relation between the classical Meixner and Laguerre polynomials is well known

$$\lim_{h\to 0} h^n M_n^{\alpha+1,1-h}\left(\frac{x}{h}\right) = L_n^{\alpha}(x). \tag{67}$$

In order to obtain the analogues of this relation for generalized polynomials we notice that (see (16) and (26))

$$Ker_{n-1}^M(0,0) = \sum_{k=0}^{n-1} \frac{[M_k^{\alpha+1,1-h}(0)]^2}{d_k^2} = \sum_{k=0}^{n-1} \frac{(\alpha+1)_k(1-h)^k}{k!}.$$

Then $Ker_{n-1}^M(0,0) = Ker_{n-1}^L(0,0) + O(h)$. Now from the representation formulas (42) we find

$$M_n^{\alpha+1,1-h,A}\left(\tfrac{x}{h}\right) = M_n^{\alpha+1,1-h}\left(\frac{x}{h}\right) + A\,\frac{(\alpha+1)_n(1-h)^n}{n!\,(1 + A Ker_{n-1}^L(0,0))} \times$$

$$\times \frac{M_n^{\alpha+1,1-h,A}\left(\frac{x}{h}\right) - M_n^{\alpha+1,1-h,A}\left(\frac{x-h}{h}\right)}{h}.$$

Multiplying this expression by the factor h^n, taking the limit when $h \to 0$ and using (67) we notice that the right side of the last expression becomes into the right side of (45). Then, the following relation holds

$$\lim_{h\to 0} h^n M_n^{\alpha+1,1-h,A}\left(\frac{x}{h}\right) = L_n^{\alpha,A}(x). \tag{68}$$

4.2 Limit Meixner $\longrightarrow$ Charlier.

We start again from the classical limit relation for monic Meixner and Charlier polynomials

$$\lim_{\gamma\to\infty} M_n^{\gamma,\frac{\mu}{\mu+\gamma}}(x) = C_n^{\mu}(x). \tag{69}$$

For the kernels of Meixner polynomials we have (see (16) and (20))

$$\lim_{\gamma\to\infty} Ker^M_{n-1}(0,0) = \sum_{k=0}^{n-1} \frac{\mu^k}{k!} = Ker^C_{n-1}(0,0).$$

Now from formula (42) we find that

$$\lim_{\gamma\to\infty} B_n = A\frac{\mu^n}{n!(1+AKer^C_{n-1}(0,0))},$$

which agrees with D_n in the representation formula for Charlier polynomials (44). Now, like in the previous case, we take the limit $\gamma\to\infty$. Hence, using (69) the following relation holds

$$\lim_{\gamma\to\infty} M_n^{\gamma,\frac{\mu}{\mu+\gamma},A}(x) = C_n^{\mu,A}(x). \tag{70}$$

4.3 Limit Kravchuk $\longrightarrow$ Charlier.

In this case the limit relation takes the form

$$\lim_{N\to\infty} K_n^{\frac{\mu}{N}}(x) = C_n^{\mu}(x). \tag{71}$$

First of all, since $\dfrac{N!}{(N-n)!} \sim N^n$ then $\lim\limits_{N\to\infty} \dfrac{N^n(N-n)!}{N!} = 1$. Using these two relations we find that $\lim\limits_{N\to\infty} Ker^K_{n-1}(0,0) = Ker^C_{n-1}(0,0)$, and also from (43) we have

$$\lim_{N\to\infty} A_n = A\frac{\mu^n}{n!(1+AKer^C_{n-1}(0,0))}.$$

Then from (44) we conclude that

$$\lim_{N\to\infty} K_n^{\frac{\mu}{N},A}(x) = C_n^{\mu,A}(x). \tag{72}$$

4.4 Limit Hahn $\longrightarrow$ Meixner.

From the hypergeometric representation of the Hahn and Meixner polynomials

$$h_n^{\alpha,\beta}(x,N) = \frac{(-1)^n(N-1)!\Gamma(\beta+n+1)}{n!(N-n-1)!\Gamma(\beta+1)} {}_3\mathrm{F}_2\left({-x,\alpha+\beta+n+1,-n \atop 1-N,\beta+1};1\right)$$

$$M_n^{\gamma,\mu}(x) = (\gamma)_n\frac{\mu^n}{(\mu-1)^n}\quad {}_2\mathrm{F}_1\left({-n,-x \atop \gamma};1-\frac{1}{\mu}\right),$$

it is easy to check that the following limit relation holds

$$\lim_{N\to\infty} h_n^{\frac{(1-\mu)}{\mu}N,\gamma-1}(x,N) = M_n^{\gamma,\mu}(x). \tag{73}$$

By using the well-known asymptotic formula for the Γ function (see for instance [1], Eq.(6.1.39) in page 257).

$$\Gamma(aN+b) \sim \sqrt{2\pi}e^{-aN}(aN)^{aN+b-\frac{1}{2}},$$

and doing some straightforward, but tedious, calculation we obtain for the kernels $Ker_{n-1}^{H,\alpha,\beta}(0,0)$ of the Hahn polynomials the following expression in terms of the kernels of the Meixner ones

$$\lim_{N\to\infty} Ker_{n-1}^{H,\frac{(1-\mu)}{\mu}N,\gamma-1}(0,0) = Ker_{n-1}^{M}(0,0).$$

From (54) we also notice that the constant $\tau_A^{n,\alpha,\beta} \equiv \tau_{A,0}^{n,\alpha,\beta}$ of the representation formula (53) (here we are interested in the case when $B=0$) is equal to

$$\lim_{N\to\infty} \tau_A^{n,\alpha,\beta} = A\frac{\mu^n(1-\mu)^{-1}(\gamma)_n}{n!(1+AKer_{n-1}^{M}(0,0))}.$$

From the last two expressions and taking into account Eqs. (73) and (42) we conclude that the following limit transition between Hahn and Meixner generalized polynomials holds

$$\lim_{N\to\infty} h_n^{\frac{(1-\mu)N}{\mu},\gamma-1,A}(x,N) = M_n^{\gamma,\mu,A}(x). \tag{74}$$

4.5 Limit Hahn $\longrightarrow$ Kravchuk.

In a similar way, in this case we start from the classical relation

$$\lim_{t\to\infty} h_n^{(1-p)t,pt}(x,N) = K_n^p(x,N-1). \tag{75}$$

Notice that in this relation the Hahn polynomials are defined for $n<N$, as well as the Kravchuk polynomials are defined for $n<N-1$, i.e., the interval of orthogonality is reduced in one unit. Besides, for the kernels we have the expression

$$\lim_{t\to\infty} Ker_{n-1}^{H,(1-p)t,pt}(0,0) = Ker_{n-1}^{K}(0,0),$$

and for the constant of the representation formula (53)

$$\lim_{t\to\infty} \tau_A^{n,(1-p)t,pt} = A\frac{p^n(1-p)^{1-n}(N-1)!}{n!(N-n-1)!(1+AKer_{n-1}^{K}(0,0))}.$$

Finally, using the last two expressions from (53) and (43) we obtain the limit relation

$$\lim_{t\to\infty} h_n^{(1-p)t,pt,A}(x,N) = K_n^{p,A}(x,N-1). \tag{76}$$

4.6 Limit Hahn $\longrightarrow$ Jacobi.

In this section we will analyze the limit relation involving Hahn and Jacobi polynomials. As before we start from the classical relation

$$\lim_{N\to\infty} \frac{2^n}{N^n} h_n^{\alpha,\beta}((N-1)x,N) = P_n^{\alpha,\beta}(2x-1). \tag{77}$$

In order to obtain the limit relation we will use the Eq. (49) for Hahn and Jacobi polynomials. First of all, notice that

$$\lim_{N\to\infty} Ker_{n-1}^{H,\alpha,\beta}(0,0) = Ker_{n-1}^{J,\alpha,\beta}(-1,-1),$$

$$\lim_{N\to\infty} Ker_{n-1}^{H,\alpha,\beta}(N-1,N-1) = Ker_{n-1}^{J,\alpha,\beta}(1,1),$$

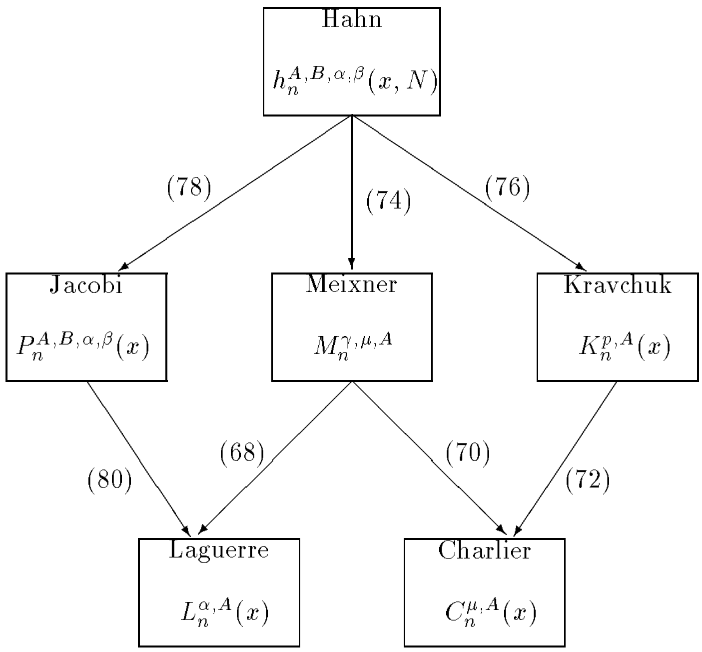

Figure 1: Limit relations involving the generalized polynomials.

and

$$\lim_{N\to\infty} Ker_{n-1}^{H,\alpha,\beta}(0,N-1) = Ker_{n-1}^{J,\alpha,\beta}(-1,1).$$

If we use now Eqs. (51), (52),(56) and (57), we conclude that

$$\lim_{N\to\infty} \frac{2^n}{N^n} h_n^{A,B,\alpha,\beta}(0,N) = P_n^{A,B,\alpha,\beta}(-1),$$

and

$$\lim_{N\to\infty} \frac{2^n}{N^n} h_n^{A,B,\alpha,\beta}(N-1,N) = P_n^{A,B,\alpha,\beta}(1).$$

The following limit relation between the norms of the Hahn $(d_n^H)^2$ and Jacobi $(d_n^J)^2$ polynomials is also valid

$$\lim_{N\to\infty} \frac{2^n}{N^n}(d_n^H)^2 = (d_n^J)^2.$$

Putting all these formulas in Eq. (49), taking the limit $N\to\infty$ and using the classical relation (77) we finally obtain the limit relation between the generalized polynomials, i.e.,

$$\lim_{N\to\infty} \frac{2^n}{N^n} h_n^{A,B,\alpha,\beta}((N-1)x,N) = P_n^{A,B,\alpha,\beta}(2x-1). \tag{78}$$

4.7 Limit Jacobi $\longrightarrow$ Laguerre.

Finally, we stablish the limit relation between Jacobi and Laguerre polynomials. Like in the previous cases we start from the classical relation

$$\lim_{\beta\to\infty} \frac{(-1)^n \beta^n}{2^n} P_n^{\alpha,\beta}\left(1-\frac{2x}{\beta}\right) = L_n^{\alpha}(x). \tag{79}$$

From the last relation we notice that it is reasonable that the connection should be between the Jacobi polynomials with a mass point at $x=1$ (i.e., A=0, B=A) and the generalized Laguerre polynomials. In fact putting $x=0$ we obtain

$$\lim_{\beta\to\infty} \frac{(-1)^n \beta^n}{2^n} P_n^{\alpha,\beta}(1) = L_n^{\alpha}(0).$$

Some straightforward calculations provide the relations

$$\lim_{\beta\to\infty} Ker_{n-1}^{J,\alpha,\beta}(1,1) = Ker_{n-1}^{L}(0,0),$$

and for the norms of the Jacobi $(d_n^J)^2$ and Laguerre $(d_n^L)^2$ polynomials

$$\lim_{\beta\to\infty} \frac{\beta^{2n}}{2^{2n}}(d_n^J)^2 = (d_n^L)^2.$$

Then, from (39), (49) and (79) we obtain

$$\lim_{\beta\to\infty} \frac{(-1)^n \beta^n}{2^n} P_n^{0,A\,\alpha,\beta}\left(1-\frac{2x}{\beta}\right) = L_n^{\alpha,A}(x). \tag{80}$$

ACKNOWLEDGEMENTS

The research of the authors was supported by Comisión Interministerial de Ciencia y Tecnología (CICYT) of Spain under grant PB 93-0228-C02-01 and by the INTAS. The authors thank to Professor Jessus Sánchez Dehesa (Universidad de Granada, Spain) for its useful suggestions and remarks and to the referees for they useful comments and remarks.

References

[1] M. Abramowitz and I.Stegun Editors: **Handbook of Mathematical Functions.** Dover Publications, Inc., New York, 1972.

[2] R. Álvarez-Nodarse and F. Marcellán: *Difference equation for modification of Meixner polynomials.* J. Math. Anal. and Appl. **194**(3), (1995), 250-258.

[3] R. Álvarez-Nodarse, A.G. García and F. Marcellán: *On the properties for modification of classical orthogonal polynomials of discrete variables.* J. Comput. and Appl. Math. **65**, (1995) 3-18.

[4] R. Álvarez-Nodarse and F. Marcellán: *The modification of classical Hahn polynomials of a discrete variable.* Integral Transforms and Special Functions **4**(4), (1995), 243-262.

[5] R. Álvarez-Nodarse and F. Marcellán: *A generalization of the classical Laguerre polynomials.* Rend. Circ. Mat. Palermo. Serie II, **44**, (1995), 315-329.

[6] R. Askey: *Difference equation for modifications of Meixner polynomials.* In **Orthogonal Polynomials and their Applications.** C. Brezinski et al. Editors. Annals on Computing and Applied Mathematics. J.C. Baltzer AG, Scientific Publishing Company, Basel. Volume 9, (1991), 418.

[7] H. Bavinck and H. van Haeringen: *Difference equations for generalized Meixner polynomials.* J. Math. Anal. and Appl.**184** (3), (1994), 453-463.

[8] H. Bavinck and R. Koekoek: *On a difference equation for generalizations of Charlier polynomials.* J. Approx. Th. **81**, (1995), 195-206.

[9] T. S. Chihara: *Orthogonal polynomials and measures with end point masses.* Rocky Mount. J. of Math.,**15**(3),(1985),705-719.

[10] T. S. Chihara: **An introduction to orthogonal polynomials.** Gordon and Breach. New York. 1978.

[11] N. Draïdi: *Sur l'adjonction de deux masses de Dirac à une forme linéaire régulière quelconque.*THÈSE DOCTORAT DE L'UNIVERSITÉ PIERRE ET MARIE CURIE. Paris, 1990.

[12] E. Godoy, F. Marcellán, L. Salto and A. Zarzo: *Classical-type orthogonal polynomials: The discrete case.* (manuscript submitted).

[13] J. Koekoek and R.Koekoek : *On a differential equation for Koornwinder's generalized Laguerre polynomials.* Proc. Amer. Math. Soc. **112**, (1991), 1045-1054.

[14] R. Koekoek and H.G. Meijer: *A generalization of Laguerre polynomials.* SIAM J. Mat. Anal. **24** (3), (1993), 768-782.

[15] R. Koekoek: *Koornwinder's Laguerre Polynomials.* Delft Progress Report. **12**, (1988), 393-404.

[16] R. Koekoek: *Differential equations for symmetric generalized ultraspherical polynomials.* Transact. of the Amer. Math. Soc. **345**, (1994), 47-72.

[17] R. Koekoek: *Generalizations of the classical Laguerre Polynomials and some q-analogues.* Thesis, Delft University of Technology. 1990.

[18] R. Koekoek and R.F. Swarttouw: *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue.* Reports of the Faculty of Technical Mathematics and Informatics No. 94-05. Delft University of Technology, Delft, 1994.

[19] T. H. Koornwinder: *Orthogonal polynomials with weight function* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$. Canad.Math.Bull, **27** (2), (1984), 205-214.

[20] H. L. Krall: *On Orthogonal Polynomials satisfying a certain fourth order differential equation.* The Pennsylvania State College Bulletin, **6**, (1940), 1-24.

[21] H.K. Kwon and S.B. Park: *Two points masses perturbation of regular moment functionals.* Indag. Math. (In press)

[22] F. Marcellán and P. Maroni: *Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique.* Ann.Mat.Pura ed Appl., **IV**, CLXII, (1992), 1-22.

[23] A.F.Nikiforov, S.K.Suslov and V.B.Uvarov: **Orthogonal Polynomials in Discrete Variables.** Springer Series in Computational Physics. Springer-Verlag,Berlin, 1991.

[24] E.D. Rainville: **Special Functions.** Chelsea Publishing Company, New York, 1971.

[25] G. Szegö: **Orthogonal Polynomials.** Amer. Math. Soc. Colloq. Publ., **23**, Amer. Math. Soc., Providence, RI, 1975 (4th edition).

[26] S. Wolfram: ***MATHEMATICA*. A system for doing Mathematics by Computer**. Addison-Wesley Publishing Co., New York, 1991.

[27] Wolfram Research Inc. **Guide to Standard *Mathematica* Packages**, version 2.2, Wolfram Research, Inc., Champaign, Illinois, (1993).