ZERO DISTRIBUTIONS OF DISCRETE AND CONTINUOUS POLYNOMIALS FROM THEIR RECURRENCE RELATION.

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Abstract

The hypergeometric polynomials in a continous or a discrete variable, whose canonical forms are the so-called classical orthogonal polynomial systems, are objects which naturally appear in a broad range of physical and mathematical fields from quantum mechanics, the theory of vibrating strings and the theory of group representations to numerical analysis and the theory of Sturm-Liouville differential and difference equations. Often, they are encountered in the form of a three term recurrence relation (TTRR) which connects a polynomial of a given order with the polynomial of the contiguous orders. This relation can be directly found, in particular, by use of Lanczos-type methods, tight-binding models or the application of the conventional discretisation procedures to a given differential operator. Here the distribution of zeros and its asymptotic limit, characterized by means of its moments around the origin, are found for the continuous classical (Hermite, Laguerre, Jacobi, Bessel) polynomials and for the discrete classical (Charlier, Meixner, Kravchuk, Hahn) polynomials by means of a general procedure which (i) only requires the three-term recurrence relation and (ii) avoids the often high-brow subleties of the potential theoretic considerations used in some recent approaches. The moments are given in an explicit manner which, at times, allows us to recognize the analytical form of the corresponding distribution.

1 Introduction.

The hypergeometric polynomials in a continous [10, 39] or a discrete [40, 39] variable are objects not only interesting *per se* and because of its abundant applications in many areas of mathematics ranging from angular momentum algebra and probability theory to numerical analysis and the theory of Sturm-Liouville differential and difference equations, the theory of random matrices and the study of speech signals, but also because they help us to interpret and characterize numerous natural phenomena encountered, e.g. in the quantum mechanical description of physical and chemical systems, the theory of vibrating strings and the study of random walks with discrete time processes, as pointed out by several authors [10, 2, 3, 4, 7, 18, 19, 35, 40, 39, 42, 43, 44].

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The global behaviour of the zeros of the discrete and continuous classical orthogonal polynomials in both finite and asymptotic cases has received a great deal of attention from the early times [22, 27, 45] of approximation theory up to now [8, 9, 20, 23, 24, 31, 32, 33, 34, 36, 37, 41, 42, 46, 47, 48, 50]. Indeed, numerous interesting results have been found from the different characterizations (explicit expression, weight function, recurrence relation, second order difference or differential equation) of the polynomial. See [16] for a survey of the published results up to 1977; more recent discoveries are collected in [49] and [31] for continuos and discrete polynomials, respectively. Still now, however, there are open problems which are very relevant by their own and because of its numerous applications to a great variety of classical systems [29, 35] as well as quantum-mechanical systems whose wavefunctions are governed by orthogonal polynomials in a *"discrete"* [2, 3, 40, 43, 44] or a *"continuous"* variable [5, 18, 19, 39].

In this paper the attention will be addressed to the problem of determination of the moments of the distribution density of zeros for a classical orthogonal polynomial of a given order n in both discrete and continuous cases as well as its asymptotic values (i.e, when $n \to \infty$), which fully characterize the limiting distribution of zeros of those polynomials, in an explicit and exact manner. At times, the analytical form of the distribution associated to the calculated moments is recognized. This problem is solved for a general system of polynomials, defined by the recurrence relation given by (2.1) and (2.2) below, which includes all classical orthogonal families in the discrete (Hahn, Meixner, Kravchuk, Charlier) and continuous (Hermite, Laguerre, Jacobi, Bessel) cases.

We have used a method [12, 16, 17] which is based only on the three-term recurrence relation satisfied by the involved polynomials. This method, which will be described in Section 2, is of general vality since no peculiar constraints are imposed upon the coefficients of the recurrence relation. It was found in a context of tridiagonal matrices [6, 13, 14, 15] and it has been already used for the study of the distribution of zeros of q-polynomials [1, 11, 16]. Some of the results found here have been previously obtained by other means and are dispersely published, what will be mentioned in the appropiate place; they are included here for completeness, for illustrating the goodness of our procedure or because they are not accessible for the general reader [16].

Then, in Section 3, expresions for the moments of the discrete distribution of zeros of any discrete and continuous classical polynomials of arbitrary, but fixed, degree n are given in a closed and compact form; the explicit values for the first few moments of lowest order are also shown. Finally, in Section 4, the limiting distribution of zeros of all classical polynomials is described by means of its moments and, at times, its analytical form is shown.

2 Density of zeros of a general polynomial system from its recurrence relation. Basic tools

We will consider here a general system of polynomials $\{P_n\}_{n=0}^{\infty}$ defined by the following three-term recurrence relation

$$P_n(x) = (x - a_n)P_{n-1}(x) - b_{n-1}^2 P_{n-2}(x)$$
$$P_{-1}(x) = 0, \quad P_0(x) = 1, \quad n \geq 1 \tag{2.1}$$

where the coefficients a_n and b_{n-1}^2 are rational functions in n defined by

$$a_n = \frac{\sum_{i=0}^{\theta} c_i n^{\theta-i}}{\sum_{i=0}^{\beta} d_i n^{\beta-i}} \equiv \frac{Q_\theta(n)}{Q_\beta(n)}, \qquad b_n^2 = \frac{\sum_{i=0}^{\alpha} e_i n^{\alpha-i}}{\sum_{i=0}^{\gamma} f_i n^{\gamma-i}} \equiv \frac{Q_\alpha(n)}{Q_\gamma(n)}. \tag{2.2}$$

The parameters defining a_n and b_n^2 are supposed to be real. In the case when the e_i and f_i are such that $b_n^2 > 0$ for $n \geq 1$, then, Favard's theorem [10] assures the orthogonality of the polynomials $\{P_n\}_{n=0}^{\infty}$ and we will say that the relation (2.1) defines a sequence of orthogonal polynomials.

Here we will collect the tools which allow us to find the moments of the distribution of zeros (Theorem 1), and its asymptotic values (Theorem 2), of the polynomials which obey a three-term recurrence relation of the form (2.1). These results, previously found in a context of tridiagonal matrices [12, 16, 17], constitute an alternative method to compute the properties of the spectral moments of the orthogonal polynomials directly from the three-term recurrence coefficients (a_n, b_n). They are used to obtain the distribution of zeros of the discrete and continuous classical orthogonal polynomials for both finite and asymptotic cases in Section 3 and 4, respectively.

Theorem 1 The spectral moments *(Dehesa [12, 16])*
Let $\{P_k,\ k = 0, 1, ..., n, ...\}$, be a system of polynomials defined by the recurrence relation (2.1), which is characterized by the sequences of numbers $\{a_n\}$ and $\{b_n\}$. Let the quantities

$$\mu_0 = 1, \quad \mu_m'^{(n)} = \frac{1}{n}\int_a^b x^m \rho_n(x)\, dx, \quad m = 1, 2, ..., n \tag{2.3}$$

be the normalized-to-unity spectral moments of the polynomial P_n, i.e., the moments around the origin of the discrete density of zeros ρ_n defined by

$$\rho_n(x) = \frac{1}{n}\sum_{i=1}^{n} \delta(x - x_{n,i}), \tag{2.4}$$

$\{x_{n,i},\ i = 1, 2, ..., n\}$ being the zeros of that polynomial. It is fulfilled that

$$\mu_m'^{(n)} = \frac{1}{n}\sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} a_i^{r_1'} b_i^{2r_1} a_{i+1}^{r_2'} b_{i+1}^{2r_2} \ldots b_{i+j-1}^{2r_j} a_{i+j}^{r_{j+1}'}, \qquad m = 1, 2, ..., n. \tag{2.5}$$

Or, in a compact form,

$$\mu_m'^{(n)} = \frac{1}{n}\sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \left[\prod_{k=1}^{j+1} a_{i+k-1}^{r_k'}\right]\left[\prod_{k=1}^{j} (b_{i+k-1}^2)^{r_k}\right], \qquad m = 1, 2, ..., n, \tag{2.6}$$

where s denotes the number of non-vanishing r_i which are involved in each partition of m. The first summation runs over all partitions $(r_1', r_1, ..., r_{j+1}')$ of the number m such that

1. *$R' + 2R = m$, where R and R' denote the sums $R = \sum_{i=1}^{[\frac{m}{2}]} r_i$ and $R' = \sum_{i=1}^{[\frac{m}{2}]-1} r_i'$, or*

$$\sum_{i=1}^{[\frac{m}{2}]-1} r_i' + 2\sum_{i=1}^{[\frac{m}{2}]} r_i = m\,. \tag{2.7}$$

2. If $r_i = 0,\ 1 < i < [\frac{m}{2}]$, *then* $r_k = r'_k = 0$ *for each* $k > i$ *and*

3. $[\frac{m}{2}] = \frac{m}{2}$ *or* $[\frac{m}{2}] = \frac{m-1}{2}$ *for* m *even or odd, respectively.*

The factorial coefficient F *is defined by*

$$F(r'_1, r_1, r'_2, ..., r'_{p-1}, r_{p-1}, r'_p) =$$

$$= m\frac{(r'_1 + r_1 - 1)!}{r'_1! r_1!}\left[\prod_{i=2}^{p-1}\frac{(r_{i-1} + r'_i + r_i - 1)!}{(r_{i-1} - 1)! r_i! r'_i!}\right]\frac{(r_{p-1} + r'_p - 1)!}{(r_{p-1} - 1)! r'_p!}, \tag{2.8}$$

with the convention $r_0 = r_p = 1$. *For the evaluation of these coefficients, we must take into account the following convention*

$$F(r'_1, r_1, r'_2, r_2 ..., r'_{p-1}, 0, 0) = F(r'_1, r_1, r'_2, r_2 ..., r'_{p-1}).$$

This theorem was initially found in a context of Jacobi matrices [12, 16]. A straightforward calculation gives

$$\mu'_1 = \frac{1}{n}\left[\sum_{i=1}^{n} a_i\right], \qquad \mu'_2 = \frac{1}{n}\left[\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} b_i^2\right],$$

$$\mu'_3 = \frac{1}{n}\left[\sum_{i=1}^{n} a_i^3 + 3\sum_{i=1}^{n-1} b_i^2(a_i + a_{i+1})\right], \tag{2.9}$$

$$\mu'_4 = \frac{1}{n}\left[\sum_{i=1}^{n} a_i^4 + 4\sum_{i=1}^{n-1} b_i^2(a_i^2 + a_i a_{i+1} + a_{i+1}^2 + \frac{1}{2}b_i^2) + 4\sum_{i=1}^{n-2} b_i^2 b_{i+1}^2\right],$$

for the the first four spectral moments.

Recently, it has been shown [25, 26, 34] that the moments given by Eq. (2.6) may be represented as the so-called Lucas polynomials of the first kind in several variables, each depending on the recurrence coefficients (a_n, b_n) in a certain manner.

Theorem 2 The asymptotic values for the moments *(Dehesa [16, 17])*
Let $\{P_k,\ k = 0, 1, ..., n, ...\}$ *be a system of polynomials defined by the recurrence relation (2.1), which is characterized by the sequences of numbers* $\{a_n\}$ *and* $\{b_n\}$. *Let* ρ, ρ^* *and* ρ^{**} *the asymptotic zero distribution functions of the polynomial* P_n *defined as follows*

$$\rho(x) = \lim_{n\to\infty}\rho_n(x), \quad \rho^*(x) = \lim_{n\to\infty}\rho_n\left(\frac{x}{n^{\frac{1}{2}(\alpha-\gamma)}}\right),$$

$$\rho^{**}(x) = \lim_{n\to\infty}\rho_n\left(\frac{x}{n^{(\theta-\beta)}}\right). \tag{2.10}$$

Here, ρ_n *is given by Eq. (2.4), and the moments of the functions* ρ, ρ^*, *and* ρ^{**} *are*

$$\mu'_m = \lim_{n\to\infty}\mu'^{(n)}_m, \quad \mu''_m = \lim_{n\to\infty}\frac{\mu'^{(n)}_m}{n^{\frac{m}{2}(\alpha-\gamma)}}, \quad \mu'''_m = \lim_{n\to\infty}\frac{\mu'^{(n)}_m}{n^{m(\theta-\beta)}}. \tag{2.11}$$

Then, according to the different behaviour of the asymptotic zero distribution, the polynomial system $\{P_k\}_{k=0}^{\infty}$ *may be subdivided in the seven following classes*

1. *Class $\theta < \beta$ and $\alpha < \gamma$. The polynomials belonging to this class has a spectrum of zeros characterized by the quantities*

$$\mu'_0 = 1, \qquad \mu'_m = 0, \qquad m = 1, 2, ...$$

2. *Class $\theta < \beta$ and $\alpha = \gamma$. The polynomials in this class are such that*

$$\begin{cases} \mu'_{2m} = \left(\frac{e_0}{f_0}\right)^m \binom{2m}{m}, & \\ & m = 0, 1, 2, ... \\ \mu'_{2m+1} = 0, & \end{cases}$$

3. *Class $\theta \leq \beta$ and $\alpha > \gamma$. The polynomials in this class are such that*

$$\begin{cases} \mu''_{2m} = \frac{1}{m(\alpha - \gamma) + 1} \left(\frac{e_0}{f_0}\right)^m \binom{2m}{m}, & \\ & m = 0, 1, 2, ... \\ \mu''_{2m+1} = 0, & \end{cases}$$

4. *Class $\theta = \beta$ and $\alpha < \gamma$. The polynomials in this class are such that*

$$\mu'_m = \left(\frac{c_0}{d_0}\right)^m, \quad m = 0, 1, 2, ...$$

5. *Class $\theta = \beta$ and $\alpha = \gamma$. The polynomials in this class are such that*

$$\mu'_m = \sum_{i=0}^{[\frac{m}{2}]} \left(\frac{c_0}{d_0}\right)^{m-2i} \left(\frac{e_0}{f_0}\right)^i \binom{2i}{i} \binom{m}{2i}, \quad m = 0, 1, 2, ...$$

6. *Class $\theta > \beta$ and $\alpha \leq \gamma$. The polynomials in this class are such that*

$$\mu'''_m = \frac{1}{m(\theta - \beta) + 1} \left(\frac{c_0}{d_0}\right)^m, \quad m = 0, 1, 2, ...$$

7. *Class $\theta > \beta$ and $\alpha > \gamma$. Here three cases may be distinguished:*
   (a) *Case $\theta - \beta > \frac{1}{2}(\alpha - \gamma)$. The polynomials in this subclass are such that (see case 6)*

   $$\mu'''_m = \frac{1}{m(\theta - \beta) + 1} \left(\frac{c_0}{d_0}\right)^m, \quad m = 0, 1, 2, ...$$

   (b) *Case $\theta - \beta = \frac{1}{2}(\alpha - \gamma)$. The polynomials in this subclass are such that*

   $$\mu'''_m = \frac{1}{m(\theta - \beta) + 1} \sum_{i=0}^{[\frac{m}{2}]} \left(\frac{c_0}{d_0}\right)^{m-2i} \left(\frac{e_0}{f_0}\right)^i \binom{2i}{i} \binom{m}{2i}, \quad m = 0, 1, 2, ...$$

   (c) *Case $\theta - \beta < \frac{1}{2}(\alpha - \gamma)$. The polynomials in this subclass are such that (see case 3)*

   $$\begin{cases} \mu''_{2m} = \frac{1}{m(\alpha - \gamma) + 1} \left(\frac{e_0}{f_0}\right)^m \binom{2m}{m}, & \\ & m = 0, 1, 2, ... \\ \mu''_{2m+1} = 0, & \end{cases}$$

3 The spectral moments of the classical polynomials.

3.1 Classical discrete polynomials.

Let us compute the moments around the origin of the distribution of zeros of a polynomial of a given degree, which belong to one of the four classical families (Charlier, Meixner, Kravchuk and Hahn) of orthogonal polynomials in a discrete variable. They are given in terms of the parameters which characterize the three-term recurrence relation of the corresponding family. Alternative expressions for these quantities may be obtained from the explicit expressions of the polynomial [49].

3.1.1 Charlier Polynomials.

The Charlier polynomials $c_n^{\mu}(x)$ satisfy a three-term recurrence relation (2.1) with the coefficients [40]

$$a_n = n + \mu - 1, \quad b_n^2 = n\mu. \tag{3.1}$$

Then, Theorem 1 leads to the expression

$$\mu_m'^{(n)} = \frac{1}{n} \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} [i + k - 2 + \mu]^{r_k'} \prod_{k=1}^{j} [(i + k - 1)\mu]^{r_k},$$

for the spectral moments of the polynomial $c_n^{\mu}(x)$. The first four moments are

$$\mu_1'^{(n)} = \frac{n + 2\mu - 1}{2}, \quad \mu_2'^{(n)} = \frac{(n-1)(2n-1)}{6} + 2(n-1)\mu + \mu^2,$$

$$\mu_3'^{(n)} = \frac{(n-1)^2 n}{4} + 3(n-1)^2\mu + \frac{9(n-1)\mu^2}{2} + \mu^3,$$

$$\mu_4'^{(n)} = \frac{n^2(10 + 3n(2n-5)) - 1}{30} + 4(n-1)^3\mu + 2(n-1)(6n-7)\mu^2 + 8(n-1)\mu^3 + \mu^4.$$

3.1.2 Meixner Polynomials.

The Meixner polynomials $m_n^{\gamma,\mu}(x)$ are defined by the three-term recurrence relation (2.1) with coefficients [40]

$$a_n = \frac{(n-1)(1+\mu) + \mu\gamma}{1-\mu}, \quad b_n^2 = \frac{n\mu(n-1+\gamma)}{(1-\mu)^2}. \tag{3.2}$$

Application of Theorem 1 gives the value

$$\mu_m'^{(n)} = \frac{1}{n} \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} \left[\frac{(i+k-2)(1+\mu) + \mu\gamma}{1-\mu}\right]^{r_k'} \times$$

$$\times \prod_{k=1}^{j} \left[\frac{(i+k-1)\mu(i+k-2+\gamma)}{(1-\mu)^2}\right]^{r_k},$$

for the m-th order spectral moment of the Meixner polynomial of n-th degree. A simple calculation gives

$$\mu_1'^{(n)} = \frac{1 + \mu - 2\gamma\mu - n(1+\mu)}{2(\mu - 1)},$$

$$\mu_2^{\prime(n)} = \frac{(1-3n+2n^2+\mu^2\ (1+6\ (\gamma-1)\ \gamma-3n+6\gamma n+2n^2)+2\mu\ (n-1)\ (6\gamma+4n-5))}{6\,(\mu-1)^2},$$

$$\mu_3^{\prime(n)} = \frac{1}{4\ (1-\mu)^3}\Big[\Big(2\gamma^2\mu^2+2\gamma\mu\ (1+\mu)\ (n-1)+(1+\mu)^2\ (n-1)\ n\Big)\ (n-1+\mu\ (2\gamma+n-1))+$$
$$+2\mu\ (n-1)\ \Big(6\gamma^2\mu+3\ (1+\mu)\ (n-2)\ (n-1)+\gamma\ (4n-5+\mu\ (8n-13))\Big)\Big],$$

for the three spectral moments of lowest order.

3.1.3 Kravchuk Polynomials.

The Kravchuk polynomials $k_n(x,p,N)$ satisfy a three-term recurrence relation of the type (2.1) with coefficients [40]

$$a_n = Np+(1-2p)(n-1), \quad b_n^2 = np(1-p)(N-n+1). \tag{3.3}$$

In this case, Theorem 1 leads to

$$\mu_m^{\prime(n)} = \frac{1}{n}\sum_{(m)} F(r'_1,r_1,...,r_j,r'_{j+1})\sum_{i=1}^{n-s}\prod_{k=1}^{j+1}[Np+(1-2p)(i+k-2)]^{r'_k}\times$$
$$\times\prod_{k=1}^{j}[(i+k-1)p(1-p)(N-i-k+2)]^{r_k}\ , \tag{3.4}$$

for the m-th order spectral moment of the n-th degree Kravchuk polynomial. From this general expression, it is straightforward to find the values

$$\mu_1^{\prime(n)} = -\frac{1}{2}+n\left(\frac{1}{2}-p\right)+p+Np,$$

$$\mu_2^{\prime(n)} = \frac{(n-1)\,(2n-1)}{6}+2\,(n-1)\,(N-n+1)\,p+(N-n+1)\,(N-2n+2)\,p^2,$$

$$\mu_3^{\prime(n)} = \tfrac{1}{4}\Big((n-1)^2 n+12(n-1)^2\,(N-n+1)\,p-6\,(n-1)\,(N-n+1)\,(5n-3\,(N+2))\,p^2-$$
$$-\,4\,(N-n+1)\,\big(6+n\,(5n-11)+5N-5nN+N^2\big)\,p^3\Big)\,,$$

$$\mu_4^{\prime(n)} = \frac{1}{30}\left[n^2\,(10+3n\,(-5+2n))-1\right]+4(n-1)^3\,(N-n+1)\,p+$$
$$+2\,(n-1)\,(N-n+1)\,(n\,(6N-9n+22)+7\,(2+N))\,p^2-$$
$$-4\,(n-1)\,(N-n+1)\,\big(7n^2+2\,(2+N)\,(3+N)-2n\,(9+4N)\big)\,p^3+$$
$$+\,(N-n+1)\,\big(14n^3-(2+N)\,(3+N)\,(4+N)+n\,(3+N)\,(20+9N)-n^2\,(50+21N)\big)\,p^4\,,$$

for the four moments of lowest order of the distribution of zeros of the polynomials $k_n(x,p,N)$.

3.1.4 Hahn Polynomials.

The Hahn polynomials $h_n^{\alpha,\beta}(x,N)$ satisfy a three-term recurrence relation of the form (2.1) with coefficients [40]

$$a_n = \frac{(\beta+1)(N-1)(\alpha+\beta)+(n-1)(2N+\alpha-\beta-2)(\alpha+\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n-2)},$$
$$b_n^2 = \frac{n(N-n)(\alpha+\beta+n)(\alpha+n)(\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}. \tag{3.5}$$

They constitute a finite family of orthogonal polynomials, defined for the degrees $n < N$ (N is the number of points in the discrete set). Applying Theorem 1, one obtains

$$\mu_m'^{(n)} = \frac{1}{n} \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \times$$

$$\times \prod_{k=1}^{j+1} \left[\frac{(\beta+1)(N-1)(\alpha+\beta) + (i+k-2)(2N+\alpha-\beta-3)(\alpha+\beta+i+k-1)}{(\alpha+\beta+2i+2k-4)(\alpha+\beta+2i+2k-2)} \right]^{r_k'} \times$$

$$\times \prod_{k=1}^{j} \left[\frac{(i+k-1)(N-i-k+1)(\alpha+\beta+i+k-1)(\alpha+i+k-1)(\beta+i+k-1)(\alpha+\beta+N+i+k-1)}{(\alpha+\beta+2i+2k+1)(\alpha+\beta+2i+2k-2)^2(\alpha+\beta+2i+2k-3)} \right]^{r_k},$$

for the m-th order spectral moment of the Hahn polynomial of degree n. This general expression for $m = 1, 2$ reduces to

$$\mu_1'^{(n)} = \frac{-\alpha + n\,(-2+2N+\alpha-\beta) + (-1+2N)\,\beta}{2\,(2n+\alpha+\beta)},$$

$$\mu_2'^{(n)} = \frac{1}{6\,(-1+2n+\alpha+\beta)\,(2n+\alpha+\beta)^2}$$

$$\Big[-2n^5 + n^4(4-6\alpha-6\beta) - 2n^3(-4+\alpha-3N(3N+3-4\alpha) - 11\beta + 3(N+3\alpha)\beta) +$$
$$+(\alpha+\beta)(\alpha^2 + (1+6(N-1)N)(\beta-1)\beta + \alpha(2\beta-12N-1\beta)) +$$
$$+2n^2(-2+\alpha(9+(\alpha-4)\alpha) + 3\beta - \alpha(-10+3\alpha)\beta - (3\alpha-8)\beta^2 +$$
$$+\beta^3 + 6N^2(\alpha+3\beta-1) + 6N(1-3\alpha+\alpha^2+2(\alpha-2)\beta-\beta^2)) +$$
$$+n(-3\alpha^3 - 4\beta + 3\alpha^2(3+4N(\beta-1)+\beta) + 3\beta(6(1-N)N+\beta +$$
$$+2N(4N-5)\beta + (1-2N)\beta^2) - \alpha(4+6(N-1)N + 6(2+3(N-3)N)\beta +$$
$$+3(3+2N)\beta^2)) \Big]$$

A very important special subclass of the Hahn polynomials is when $\alpha = \beta = 0$; it is the Chebyshev system of discrete polynomials $t_n(x, N)$. In this case the recurrence coefficients (2.2) become $a_n = \frac{N-1}{2}$ and $b_n^2 = \frac{n^2(N^2-n^2)}{4(4n^2-1)}$, and the spectral moments have the simpler expression

$$\mu_m'^{(n)} = \frac{1}{n} \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} \left[\frac{(N-1)}{2} \right]^{r_k'} \prod_{k=1}^{j} \left[\frac{(i+k-1)^2[N^2-(i+k-1)^2]}{4[4(2i+2k-1)^2-1]} \right]^{r_k},$$

and

$$\mu_1'^{(n)} = \frac{N-1}{2}, \quad \mu_2'^{(n)} = \frac{2\,n^2 - n^3 + n(3N-2)^2 - 6\,(N-1)\,N - 2}{24n-12},$$

$$\mu_3'^{(n)} = \frac{(N-1)\,((2-n)\,n^2 + (2-4n)\,N + (5n-4)\,N^2)}{16n-8},$$

$$\mu_4'^{(n)} = \frac{1}{240(2n-1)^2\,(2n-3)} \Big[24 - 112n + 280n^2 - 414n^3 + 206n^4 + 52n^5 - 60n^6 +$$
$$+9n^7 - 360n^2N + 1140n^3N - 960n^4N + 240n^5N - 360N^2 + 1470nN^2 -$$
$$-1590n^2N^2 - 90n^3N^2 + 630n^4N^2 - 150n^5N^2 + 720N^3 - 2820nN^3 + 3360n^2N^3 -$$
$$-1200n^3N^3 - 360N^4 + 1350nN^4 - 1530n^2N^4 + 525n^3N^4 \Big],$$

for $m = 1, 2, 3, 4$.

For convenience let us also consider the rescaled polynomials

$$T_n(x,N) \equiv \left(\frac{N-1}{2}\right)^{-n} t_n\left(\frac{N-1}{2}(x+1), N\right), \tag{3.6}$$

which form an orthogonal system with respect to the discrete set of the points

$$\left\{x_k = -1 + 2\frac{k-1}{N-1}; \quad k = 0, 1, 2, ...\right\} \subset [-1, 1] \ .$$

They satisfy a recurrence relation of the form (2.1) with coefficients

$$a_n = 0, \qquad b_n^2 = \frac{n^2(N^2-n^2)}{(N-1)^2(4n^2-1)}. \tag{3.7}$$

Then, the moments of its distribution of zeros are given by

$$\mu_m'^{(n)} = \begin{cases} \frac{2}{n}\sum_{p=1}^{k}\left(\sum_{(m)} F(0, r_1, 0, r_2, ..., 0, r_p)\right)\sum_{i=1}^{n-p}\prod_{k=1}^{p}\left[\frac{(i+k-1)^2[N^2-(i+k-1)^2]}{(N-1)^2[4(i+k-1)^2-1]}\right]^{r_k}, & m = 2k \\ 0 \ , & m = 2k-1 \end{cases},$$

so that the firts few non-vanishing moments are given by

$$\mu_2'^{(n)} = \frac{(n-1)\left(3N^2-n^2+n-1\right)}{3\left(2n-1\right)\left(N-1\right)^2},$$

$$\mu_4'^{(n)} = \frac{1}{15\left(2n-3\right)\left(2n-1\right)^2\left(N-1\right)^4}\Big[45n^3N^4 + 98n - 200n^2 + 276n^3 - 274n^4 + 172n^5 - 60n^6 + 9n^7 + \\ +90N^2 - 360nN^2 + 510n^2N^2 - 360n^3N^2 + 150n^4N^2 - 30n^5N^2 - 45N^4 + 150nN^4 - 150n^2N^4 - 21\Big].$$

3.2 Classical continuous polynomials.

Let us now consider the classical orthogonal polynomials in the a continuous variable: Hermite, Laguerre, Jacobi and Bessel. Here we compute the spectral moments of a polynomial of a given degree belonging to one of these classical continuous families. Let us mention that the first few moments of lowest order were previously obtained by use of a general highly-non-linear recurrence relationship generated from the second-order differential equation satisfied by the polynomials under consideration [8, 9, 23]. As well, they can be also found by means of the explicit expression of the polynomials [34, 49].

3.2.1 Hermite Polynomials.

The Hermite polynomials $H_n(x)$ are defined by the three-term recurrence relation (2.1) with coefficients [39]

$$a_n = 0, \quad b_n^2 = \frac{n}{2}. \tag{3.8}$$

Theorem 1 allows us to find

$$\mu_m'^{(n)} = \begin{cases} \frac{2}{n} \sum_{p=1}^{k} \left(\sum_{(m)} F(0, r_1, 0, r_2, ..., 0, r_p) \right) \sum_{i=1}^{n-p} \prod_{k=1}^{p} \left[\frac{i+k-1}{2} \right]^{r_k}, & m = 2k \\ 0 \quad , & m = 2k-1 \end{cases},$$

for the m-th order spectral moment of the Hermite polynomial of degree n. Then, it is straightforward to obtain the values

$$\mu_1'^{(n)} = 0, \quad \mu_2'^{(n)} = \frac{n-1}{2}, \quad \mu_3'^{(n)} = 0, \quad \mu_4'^{(n)} = \frac{(n-1)(2n-3)}{4},$$

$$\mu_5 = 0, \quad \mu_6 = \frac{5n^3 - 20n^2 + 32n - 15}{8}.$$

for the six moments of lowest order.

3.2.2 Laguerre Polynomials.

The Laguerre polynomials $L_n^\alpha(x)$ satisfy a three-term recurrence relation of the form (2.1) with coefficients [39]

$$a_{n+1} = 2n + \alpha - 1, \quad b_n^2 = n(n+\alpha). \tag{3.9}$$

Theorem 1 gives

$$\mu_m'^{(n)} = \frac{1}{n} \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} [2(i+k-2) + \alpha + 1]^{r_k'} \times$$

$$\times \prod_{k=1}^{j} [(i+k-1)(i+k-1+\alpha)]^{r_k},$$

for the m-th order spectral moment of the Laguerre polynomial of degree n. This general expression supplies the values

$$\mu_1'^{(n)} = n + \alpha, \quad \mu_2'^{(n)} = (n+\alpha)\ (2n + \alpha - 1),$$

$$\mu_3'^{(n)} = (n+\alpha)\left(5n^2 + n(5\alpha - 6) + \alpha^2 - 3\alpha + 2\right),$$

$$\mu_4'^{(n)} = (n+\alpha)\left(14n^3 + n^2(21\alpha - 29) + n(\alpha - 2)(9\alpha - 11) + (\alpha - 3)(\alpha - 2)(\alpha - 1)\right).$$

for the four moments of lowest order.

3.2.3 Jacobi Polynomials.

The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ satisfy a three-term recurrence relation of the form (2.1) with coefficients [39]

$$\begin{aligned} a_{n+1} &= \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n-2+\alpha+\beta)}, \\ b_n^2 &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}. \end{aligned} \tag{3.10}$$

Theorem 1 allows us to find

$$\mu_m'^{(n)} = \frac{1}{n}\sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} \left[\frac{\beta^2-\alpha^2}{[2(i+k-2)+\alpha+\beta][2(i+k-1)+\alpha+\beta]}\right]^{r_k'} \times$$

$$\times \prod_{k=1}^{j} \left[\frac{4(i+k-1)(i+k-1+\alpha)(i+k-1+\beta)(i+k-1+\alpha+\beta)}{[2(i+k-1)+\alpha+\beta]^2[2(i+k-1)+\alpha+\beta-1][2(i+k-1)+\alpha+\beta+1]}\right]^{r_k},$$

for the m-th order spectral moment of the Jacobi polynomial of degree n. Then the following values for the three moments of lowest order inmediately follow

$$\mu_1'^{(n)} = \frac{\beta-\alpha}{2n+\alpha+\beta},$$

$$\mu_2'^{(n)} = \frac{4n^3+4n^2(\alpha+\beta-1)+2n((\alpha-2)\alpha+(\beta-2)\beta)+(\alpha+\beta)\left(\alpha^2+(\beta-1)\beta-\alpha(1+2\beta)\right)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)^2},$$

$$\mu_3 = -\frac{1}{(-2+2n+\alpha+\beta)(2n+\alpha+\beta-1)(2n+\alpha+\beta)^3}\times$$
$$\times\Big[(\alpha-\beta)\left(16n^4+4n^2(\alpha+\beta-2)(4\alpha+4\beta-1)+4n^3(7\alpha+7\beta-6)+\right.$$
$$\left.+(\alpha+\beta)^2\left(2+(\alpha-3)\alpha-3\beta-2\alpha\beta+\beta^2\right)+2n(\alpha+\beta)\left(4+\alpha(2\alpha-9)-9\beta+2\alpha\beta+2\beta^2\right)\right)\Big].$$

In the special case of Legendre polynomials (i.e., when $\alpha=\beta=0$) we have

$$\mu_m'^{(n)} = \begin{cases} \dfrac{2}{n}\displaystyle\sum_{p=1}^{k}\left(\sum_{(m)} F(0,r_1,0,r_2,...,r_p)\right)\sum_{i=1}^{n-p}\prod_{k=1}^{p}\left[\frac{(i+k-1)^2}{4(i+k-1)^2-1}\right]^{r_k}, & m=2k \\ 0\quad, & m=2k-1 \end{cases},$$

for the m-th order spectral moment of the n-th degree polynomial, and

$$\mu_1'^{(n)}=0,\quad \mu_2'^{(n)}=\frac{n-1}{2n-1},\quad \mu_3'^{(n)}=0,\quad \mu_4'^{(n)}=\frac{(n-1)\left(3n^2-7n+3\right)}{(2n-1)^2(2n-3)},$$

$$\mu_5=0,\quad \mu_6=\frac{15+37n-74n^2-162n^3+170n^4+160n^5-180n^6+40n^7}{(-5+2n)\,(-3+2n)\,(-1+2n)^2\,(1+2n)^3}.$$

for the six moments of lowest order.

3.2.4 Bessel polynomials.

The Bessel polynomials $B_n^\alpha(x)$ satisfy a three-term recurrence relation of the form (2.1) with coefficients [28]

$$a_n = -\frac{2\alpha}{(2n+\alpha)(2n+\alpha-2)},\quad b_n^2 = -\frac{4n(n+\alpha)}{(2n+\alpha+1)(2n+\alpha)^2(2n+\alpha-1)}. \tag{3.11}$$

Then, Theorem 1 allows us to find the values

$$\mu_m'^{(n)} = \frac{1}{n}\sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} \left[\frac{-2\alpha}{[2(i+k-2)+\alpha][2(i+k-2)+\alpha]}\right]^{r_k'} \times$$

$$\times \prod_{k=1}^{j} \left[\frac{-4(i+k-1)(i+k-1+\alpha)}{[2(i+k-1)+\alpha]^2[2(i+k-1)+\alpha-1][2(i+k-1)+\alpha+1]}\right]^{r_k},$$

for the m-th order spectral moment of the Bessel polynomial of degree n. Then the following values for the three moments of lowest order inmediately follow

$$\mu_1'^{(n)} = -\frac{2}{2n+\alpha}, \quad \mu_2'^{(n)} = \frac{4(n+\alpha)}{(2n+\alpha-1)(2n+\alpha)^2},$$

$$\mu_3'^{(n)} = \frac{-8\alpha(n+\alpha)}{(2n+\alpha-1)(2n+\alpha-2)(2n+\alpha)^3}.$$

4 The asymptotic values of the spectral moments of the classical polynomials.

Here we will compute the asymptotic values of the moments of the distribution of the zeros of the classical polynomials by means of the general Theorem 2. For specific cases there exist other procedures which provide these asymptotic moments such as, e.g. the Nevai-Dehesa theorem [36] or any of its generalizations [47]. Recently, general potential theoretic considerations [42, 47] are being succesfully used [20, 31, 32, 33, 41] to determine the asymptotic distribution of the discrete and continuous classical orthogonal polynomials. Contrary to these approaches, the method used in our work does not use the orthogonality condition of the involved polynomials. This is an observation especially relevant when comparing our Kravchuk and Hahn results with the corresponding values based on potential theoretic techniques.

4.1 Classical discrete polynomials.

4.1.1 Charlier Polynomials.

The Charlier polynomials satisfy a three-term recurrence relation of type (2.1) with coefficients given by (3.1). Notice that these coefficients are of the form (2.2) with the parameters $\theta = 1$, $\beta = 0$, $\alpha = 1$ and $\gamma = 0$, as well as $(c_0, e_0) = (1,1)$. Then, Theorem 2 shows that, since $\theta > \frac{1}{2}(\alpha - \gamma)$, the Charlier polynomials belong to the class 7a, so that its asymptotical distribution of zeros $\rho^{**}(x) = \lim_{n\to\infty} \rho\left(\frac{x}{n}\right)$ has the moments

$$\mu_m''' = \frac{1}{m+1}, \quad m = 0, 1, 2, ... \tag{4.1}$$

This indicates that the contracted density of zeros of Charlier polynomials with large degree is uniform [30, Vol 2, p. 276],

$$\rho\left(\frac{x}{n}\right) = 1, \quad 0 \le \frac{x}{n} \le 1, \tag{4.2}$$

which is in agreement with the Nevai–Dehesa result [36] and the recent work of Kuijlaars–Van Assche [32].

4.1.2 Meixner Polynomials.

The coefficients (a_n, b_n) of the three-term recurrence relation of the Meixner polynomials $m_n^{\gamma,\mu}(x)$ are shown in (3.2). They have the form (2.2) with parameters $\theta = 1$, $\beta = 0$, $\alpha = 2$ and $\gamma = 0$. Then, these polynomials belong to the class 7b as described in Theorem 2. So, its asymptotical distribution of zeros $\rho^{**}(x) = \lim_{n\to\infty} \rho\left(\frac{x}{n}\right)$ has the moments

$$\mu_m''' = \frac{1}{m+1} \sum_{i=0}^{[\frac{m}{2}]} \frac{(1+\mu)^{m-2i}\mu^i}{(1-\mu)^m} \begin{pmatrix} 2i \\ i \end{pmatrix} \begin{pmatrix} m \\ 2i \end{pmatrix}, \quad m = 0, 1, 2, ... \tag{4.3}$$

where we have taken into account that $(c_0, d_0) = (1+\mu, 1-\mu)$ and $(e_0, f_0) = (\mu, (1-\mu)^2)$. This result has been previously obtained by other means [36] and coincide with the recent Kuijlaars & Van Assche's work [32].

4.1.3 Kravchuk Polynomials.

Here we will determine the asymptotic distribution of zeros of the Kravchuk polynomials $\{k_n(x,p,N);\ n \geq 1\}$ in the following two cases (i) for $n \to \infty$ and N fixed, and (ii) for $(n,N) \to \infty$, but so that $n/N = t \in (0,1)$. This will be done by calculating all the moments of the associated asymptotic density of zeros in an explicit and closed form which depends on p in the first case and on (p,t) in the second one.

The first case when $n \to \infty$ and N is fixed, is particularly important because then the polynomials are not orthogonal since the order n may be greater than N. Indeed, it is well known that the discrete orthogonality is preserved only for finite sequences, with $n < N$ [10, 40]. The recursion coefficients (a_n, b_n) of the polynomials $k_n(x,p,N)$, have in this case the form (2.2) with parameters $\theta = 1$, $\beta = 0$, $\alpha = 2$ and $\gamma = 0$, and $(c_0, e_0) = (1-2p, p(p-1))$, Theorem 2 shows that the Kravchuk polynomials $k_n(x,p,N)$, N fixed, belong to the class 7b. So, its asymptotical distribution of zeros $\rho^{**}(x) = \lim_{n\to\infty} \rho\left(\frac{x}{n}\right)$ has the moments

$$\mu'''_m = \frac{1}{m+1} \sum_{i=0}^{[\frac{m}{2}]} (1-2p)^{m-2i} [p(p-1)]^i \binom{2i}{i} \binom{m}{2i}, \quad m = 0, 1, 2, \ldots \tag{4.4}$$

This result have not been found by those approaches where the orthogonality condition plays a crucial role [20, 21, 31, 32, 33].

In the second case, i.e., when the degree n of the polynomial and the number N of points of growth of its orthogonality measure approach to infinity so that $n/N = t \in (0,1)$, it is interesting to better consider the rescaled monic polynomials $N^{-n} k_n(Nx, p, N)$ which have the recurrence coefficients

$$a_n^* = \frac{a_n}{N} = p + \frac{(1-2p)(n-1)}{N}, \qquad b_n^{*\,2} = \frac{b_n^2}{N^2} = p(1-p)\left(\frac{n}{N}\right)\left(1 - \frac{n}{N} + \frac{1}{N}\right). \tag{4.5}$$

Then, according to theorem 1, the discrete density of zeros of the rescaled Kravchuk polynomials has the moments

$$\mu^{*\prime(n)} = \frac{1}{n} \sum_{(m)} F(r'_1, r_1, \ldots, r_j, r'_{j+1}) \sum_{i=1}^{n-s} \prod_{k=1}^{j+1} \left[p + \frac{(1-2p)(i+k-2)}{N}\right]^{r'_k} \times$$
$$\times \prod_{k=1}^{j} \left[\frac{p(1-p)(i+k-1)(N-i-k+2)}{N^2}\right]^{r_k}, \tag{4.6}$$

Then, the first few moments $\mu_n = \lim_{n\to\infty,\, n/N\to t} \mu^{*\prime(n)}$, of the asymptotic density of zeros of the rescaled Kravchuk polynomials are

$$\mu_1 = p + \frac{t}{2} - p\,t\,,$$

$$\mu_2 = -2\,p\,(-1+t)\,t + \frac{t^2}{3} + p^2\,\left(1 - 3\,t + 2\,t^2\right)\,,$$

$$\mu_3 = -3\,p\,(-1+t)\,t^2 + \frac{t^3}{4} + \frac{3\,p^2\,t\,(3 - 8\,t + 5\,t^2)}{2} + p^3\,\left(1 - 6\,t + 10\,t^2 - 5\,t^3\right)\,,$$

$$\mu_4 = -4\,p\,(-1+t)\,t^3 + \frac{t^4}{5} + 6\,p^2\,t^2\,\left(2 - 5\,t + 3\,t^2\right) - 4\,p^3\,t\,\left(-2 + 10\,t - 15\,t^2 + 7\,t^3\right) + $$
$$+p^4\,\left(1 - 10\,t + 30\,t^2 - 35\,t^3 + 14\,t^4\right)\,,$$

which coincide with the corresponding values of $\mu_m = \lim_{n\to\infty,\, n/N\to t} \frac{\mu_m'^{(n)}}{N^m}$, where $\mu_m'^{(n)}$ are given in (3.4). Moreover, they are consistent with the corresponding distribution of zeros recently found [21] on a potential theoretic basis.

In the special case, $p = t = \frac{1}{2}$, from (4.6) by some straightforward calculations we obtain that $\mu_m = \dfrac{1}{m+1}$, which corresponds to an uniform density of distribution of zeros; and this is in agreement with the recent results of Dragnev & Saff [20] and Kuijlaars & Rakhmanov [31] which use some technics extracted from potential theory.

4.1.4 Hahn Polynomials.

As in the previous case, we should consider two different cases (i) for $n \to \infty$ and N fixed, and (ii) for $(n, N) \to \infty$, but so that $n/N = t \in (0,1)$.

In the first case, i.e., $n \to \infty$ and N fixed, since the Hahn polynomials $h_n^{\alpha,\beta}(x, N)$ form a finite system of orthogonal polynomials defined for the degrees $n < N$ (N is the number of points in the discrete set [40]), we will study the asymptotic distribution of zeros of the Hahn polynomials which are not orthogonal like the the first N members of the sequence. In this case, using the fact that these polynomials obey a three-term recurrence relation with coefficients (a_n, b_n) behaving as (see (3.5))

$$a_n = \frac{(2N + \alpha - \beta - 2)n^2 + O(n)}{4n^2 + O(n)}, \quad b_n^2 = \frac{-n^6 + O(n^5)}{16n^4 + O(n^3)},$$

it is easy to check that the Hahn polynomials $h_n^{\alpha,\beta}(x, N)$, N fixed, belong to the class 7c of Theorem 2. So, the moments of its corresponding asymptotical density of zeros $\rho^*(x) = \lim_{n\to\infty} \rho\left(\frac{x}{n}\right)$ are given by

$$\begin{cases} \mu''_{2m} = \dfrac{1}{2m+1}\left(\dfrac{-1}{16}\right)^m \begin{pmatrix} 2m \\ m \end{pmatrix}, \\ \mu''_{2m+1} = 0, \end{cases} \qquad m = 0, 1, 2, \ldots \tag{4.7}$$

since $(e_0, f_0) = (-1, 16)$. Notice that these moments (i) do not depend on the parameters α and β which characterize the polynomial $h_n^{\alpha,\beta}(x, N)$, and (ii) do not correspond to the asymptotic distribution of zeros of the orthogonal Hahn polynomials $h_n^{\alpha,\beta}(x, N)$ $(n < N)$ which we discuss in the following.

Now, let us consider the second case. Since the asymptotic density of the contracted zeros of any family of this class (Hahn class) does not depend on α and β, we shall consider here for simplicity the study of the asymptotic distribution of the afore mentioned Chebyshev family. Precisely, we will calculate the moments of the asymptotic density of zeros of the rescaled Chebyshev polynomials $T_n(x,N)$ defined by (3.6) along the rays $n/N = t \in (0,1)$ $((n,N)\to\infty)$. In this case, the recurrence coefficients behave according to (3.7) as

$$a_n = 0, \qquad b_n^2 = \frac{\left(\frac{n}{N-1}\right)^2 \left[1-\left(\frac{n}{N}\right)^2\right]}{4\left(\frac{n}{N}\right)^2 - \frac{1}{N^2}},$$

so that the first few asymptotic moments are

$$\mu_1 = 0, \quad \mu_2 = \frac{1}{2} - \frac{t^2}{6}, \quad \mu_3 = 0, \quad \mu_4 = \frac{3}{8} - \frac{t^2}{4} + \frac{3\,t^4}{40}, \quad \mu_5 = 0,$$

$$\mu_6 = \frac{5}{16} - \frac{5\,t^2}{16} + \frac{3\,t^4}{16} - \frac{5\,t^6}{112}.$$

A simple calculation shows that the above moments correspond to the ones given by the asymptotic density of zero distribution ρ for the Chebyshev polynomials $T_n(x,N)$

$$\rho(x) = \begin{cases} \dfrac{1}{\pi t}\arctan\left(\dfrac{t}{\sqrt{r^2-x^2}}\right) & x\in[-r,r], \\ \dfrac{1}{2t} & |x|\in[r,1], \end{cases} \qquad r = \sqrt{1-t^2},$$

obtained by Rakhmanov in [41, Eq. (1.3) page 114] (see also [31]) by potential theoretic considerations.

4.2 Classical continuous polynomials.

Finally, for completeness, we include here the asymptotic values of the spectral moments of the classical continuous orthogonal polynomials as briefly calculated with our method. These values and the associated asymptotic distribution of zeros were previously obtained in the literature [16, 24, 38, 36, 46, 47, 48] by other means.

4.2.1 Hermite Polynomials.

The coefficients of the three-term recurrence relation of these polynomials, which are given by Eq. (3.8), have the form (2.2) with the parameters $\theta = \beta = 0$, $\alpha = 1$ and $\gamma = 0$, as well as $(e_0, f_0) = (\frac{1}{2}, 1)$. Then, Hermite polynomials belong to both classes 3 and 7c of Theorem 2; so that, its asymptotical distribution of zeros $\rho^{**}(x) = \lim_{n\to\infty}\rho\left(\frac{x}{n}\right)$ has the moments

$$\begin{cases} \mu''_{2m} = \dfrac{1}{m+1}\left(\dfrac{1}{\sqrt{2}}\right)^{2m}\begin{pmatrix} 2m \\ m\end{pmatrix}, \\ \mu''_{2m+1} = 0, \end{cases} \qquad m = 0,1,2,\ldots, \tag{4.8}$$

which describe a special case of Beta distribution [30, Vol. 2, p. 210]: the semicircular density distribution. Therefore, the contracted density of zeros of Hermite polynomials is

$$\rho\left(\frac{x}{\sqrt{2n}}\right) = \frac{1}{\pi\sqrt{n}}\sqrt{1-\left(\frac{x}{\sqrt{2n}}\right)^2}, \quad -1 \le \frac{x}{\sqrt{2n}} \le 1, \tag{4.9}$$

as already found in the literature by other means [7, 16, 36, 24, 47].

4.2.2 Laguerre Polynomials.

For this case, the recurrence coefficients given by (3.9) are of the form (2.2) with parameters $\theta = 1$, $\beta = 0$, $\alpha = 2$ and $\gamma = 0$, as well as $(c_0, d_0) = (2,1)$ and $(e_0, f_0) = (1,1)$. Then, the Laguerre polynomials $L_n^\alpha(x)$ belong to the class 7b described in Theorem 2; so that, its asymptotical distribution of zeros has the moments

$$\mu'''_m = \frac{1}{m+1} \sum_{i=0}^{[\frac{m}{2}]} 2^{m-2i} \begin{pmatrix} 2i \\ i \end{pmatrix} \begin{pmatrix} m \\ 2i \end{pmatrix} = \frac{1}{m+1} \begin{pmatrix} 2m \\ m \end{pmatrix}, \quad m = 0,1,2,\dots, \tag{4.10}$$

which characterize another special case of Beta distribution [30, Vol. 2, p. 210]. This result has been previously obtained in the literature [16, 36]. This indicates that the contracted density of zeros of the Laguerre polynomials with large degree n is given by

$$\rho\left(\frac{x}{n}\right) = \frac{1}{2\pi}\left(\frac{x}{n}\right)^{-\frac{1}{2}}\left(4 - \frac{x}{n}\right)^{\frac{1}{2}}, \quad 0 \le \frac{x}{n} \le 4\,.$$

Similar analytical expressions, which coincide with this one for very large values of n, have been derived in the WKB framework [50], by use of random matrix methods [7] and also in [24].

4.2.3 Jacobi Polynomials.

From (3.10) one notices that the recurrence coefficients of these polinomials behave as

$$a_n = \frac{\beta^2 - \alpha^2}{4n^4 + O(n)}, \quad b_n^2 = \frac{4n^4 + O(n^3)}{16n^4 + O(n^3)}.$$

These expressions are of the form (2.2) with parameters $\theta = 0$, $\beta = 2$ and $\alpha = \gamma = 4$, as well as $(e_0, f_0) = (4,16)$. Then, Jacobi polynomials belong to class 2 as decribed in Theorem 2; so that, the moments of the asymptotic density of zeros are

$$\begin{cases} \mu'_{2m} = \left(\frac{1}{2}\right)^{2m} \begin{pmatrix} 2m \\ m \end{pmatrix}, \\ \mu'_{2m+1} = 0, \end{cases} \quad m = 0,1,2,\dots \tag{4.11}$$

This corresponds to the so-called *arc-sin* density [22]

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad -1 \le x \le 1. \tag{4.12}$$

4.2.4 Bessel polynomials.

From (3.11) one notices that the recurrence coefficients of these polinomials behave as

$$a_n = -\frac{2\alpha}{4n^4 + O(n)}, \quad b_n^2 = -\frac{4n^2 + O(n)}{16n^4 + O(n^3)}.$$

These expressions are of the form (2.2) with parameters $\theta = 0$, $\beta = 2$, $\alpha = 2$ and $\gamma = 4$. Then, Bessel polynomials belong to class 1 as decribed in Theorem 2; so that, the moments of the asymptotic density of zeros are

$$\begin{cases} \mu'_0 = 1, \\ \mu'_m = 0, \end{cases} \quad m = 1,2,\dots \tag{4.13}$$

which correspond to a *delta-Dirac* density [23].

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