# On a modification of the Jacobi linear functional: asymptotic properties and zeros of OP *

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**Abstract.** In this contribution we study the asymptotic behaviour of polynomials orthogonal with respect to the linear functional $\mathcal{U}$ (J. Arvesú, 1999)

$$\mathcal{U} = \mathcal{J}_{\alpha,\beta} + A_1\delta(x-1) + B_1\delta(x+1) - A_2\delta'(x-1) - B_2\delta'(x+1),$$

where $\mathcal{J}_{\alpha,\beta}$ is the Jacobi linear functional, i.e.

$$\langle \mathcal{J}_{\alpha,\beta}, p\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1, \quad p \in \mathbb{P},$$

where $\mathbb{P}$ is the linear space of polynomials with real coefficients.

The asymptotic properties are analyzed in $(-1,1)$ (inner asymptotics) and $\mathbb{C} \setminus [-1,1]$ (outer asymptotics) with respect to the behaviour of Jacobi polynomials.

In a second step, we use the above results in order to obtain the location of zeros of such orthogonal polynomials (J. Arvesú, 1999).

Notice that the linear functional $\mathcal{U}$ is a generalization of one studied in (T. H. Koornwinder, 1984) when $A_2 = B_2 = 0$. From the point of view of rational approximation (A. A. Gonchar, 1975) the corresponding Markov function is a perturbation of the Jacobi-Markov function by a rational function with two double poles at $\pm 1$. The denominators of the $[n-1/n]$ Padé approximants are our orthogonal polynomials.

**Keywords:** semiclassical orthogonal polynomials, asymptotics, zeros

## 1. Introduction.

In this work we will study a generalization of the Jacobi polynomials introduced in (T. H. Koornwinder, 1984). Such polynomials are orthogonal with respect to the Jacobi measure *"perturbed"* by the addition of two delta Dirac measures as well as their derivatives at the points $x = \pm 1$.

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Such a kind of modification of a linear functional appear when an extension of the Gauss-Lobatto quadrature formulas is considered. In fact, in (C. Bernardi and Y. Maday, 1991) such quadrature formulas are used in a spectral method for solving a mono-dimensional fourth order differential problem. Here, the boundary conditions are values of the solution and its first derivative in the ends of the interval (-1,1).

The aforementioned modifications were firstly studied in (H. L. Krall, 1940) when he considered the polynomial solution of certain fourth order linear differential equations. There H. L. Krall obtained, apart the classical orthogonal polynomials (Hermite, Jacobi, Laguerre and Bessel), three new families of orthogonal polynomials with respect to positive measures with an absolutely continuous part plus some mass points. More precisely, the so-called classical-type orthogonal polynomials appear. Another approach to this subject was presented in (A. M. Krall, 1981).

The analysis of the asymptotic properties of polynomials orthogonal with respect to a perturbation of a measure via the addition of mass points was introduced by Nevai (P. Nevai, 1979). In particular, he proved how the location of the mass points with respect to the support of the measure has an influence in the asymptotic behaviour of perturbed polynomials.

The algebraic properties for such polynomials have attracted the interest of many researchers. A general approach when a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one delta Dirac measure was started by Chihara (T. S. Chihara, 1985) in the positive definite case and Marcellán and Maroni (F. Marcellán and P. Maroni, 1992) for quasi-definite linear functionals. From the point of view of differential equations see (F. Marcellán and A. Ronveaux, 1989). For two point masses there exist very few examples in the literature (see (N. Draïdi, 1990; R. Koekoek, 1990; T. H. Koornwinder, 1984; K. H. Kwon and S.B. Park, 1997)) but the difficulties increase as shows (N. Draïdi and P. Maroni, 1988).

A special emphasis was given to the modifications of classical linear functionals (Hermite, Laguerre, Jacobi and Bessel) in the framework of the so-called semiclassical orthogonal polynomials. In (T. H. Koornwinder, 1984) the Jacobi case with two masses at points $x = \pm 1$ was considered. The hypergeometric representation of the resulting polynomials as well as the existence of a second order differential equation that such polynomials satisfy have been established. Also the particular cases of the Krall-type polynomials (A. M. Krall, 1981; H. L. Krall, 1940) have been obtained from this general case as special cases or limit cases. In (J. Koekoek and R.Koekoek, 1991; R. Koekoek, 1988; R. Koekoek, 1990) the Laguerre case was considered in details.

The perturbation of a linear functional via the addition of the derivatives of a delta Dirac measure was started in (S. L. Belmehdi and F. Marcellán, 1992). In particular, necessary and sufficient conditions for the existence of a sequence of polynomials orthogonal with respect to such a linear functional are obtained. Furthermore, an extensive study for the new orthogonal polynomials was performed when the initial functional is semiclassical. This problem can be considered as a limit case of two masses located in two close points. The study of such a kind of modifications of a linear functional has known an increasing interest during the past years since their applications in approximation theory (see (A. A. Gonchar, 1975) for the bounded case and (G. López, 1989) for the unbounded one).

First, in (R. Álvarez-Nodarse and F. Marcellán, 1995; R. Álvarez-Nodarse and F. Marcellán, 1996) the perturbation of the Laguerre linear functional when we add the linear functional $M_0\delta(x)+M_1\delta'(x)$ is analyzed. More precisely they studied the behavior of the polynomials and their zeros as well as the hypergeometric character of them.

More recently, Arvesú et al. (J. Arvesú, R. Álvarez-Nodarse, F. Marcellán, and K.H. Kwon, 1998) have analyzed a generalization of the Bessel polynomials, which appears when one perturbs the Bessel linear functional by the addition of the linear functional $M_0\delta(x) + M_1\delta'(x)$. In particular, the hypergeometric character of these polynomials and the behavior of their zeros were studied. In the present work we will deal with the Jacobi case.

The plan of the paper is the following. In Section 2 we give some results concerning the Jacobi polynomials. Using these results in Section 3 we obtain a general formula for the generalized Jacobi polynomials in terms of the classical ones and their first and second derivatives. This allows us to find a symmetry property in the same sense as in (T. H. Koornwinder, 1984). Finally, in Section 4 we study the asymptotic properties which are useful to investigate in Section 5 the location of zeros of such polynomials.

## 2. Some Preliminary Results.

In this section we have enclosed some formulas for the Jacobi polynomials which are useful in the analysis of polynomials orthogonal with respect to the linear functional (16) from below. All the formulas as well as some special properties for the Jacobi polynomials can be found in the literature of special functions, see for instance the classical monograph *Orthogonal Polynomials* by Szegö (G. Szegö, 1975, Chapter 5). In this work we will use monic polynomials, i.e., polynomials with leading coefficient equal to 1, ($P_n(x) = x^n + b_n x^{n-1}+$ lower degree terms).

The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ are the polynomial solution of the second order linear differential equation of hypergeometric type

$$\sigma(x)\, y''(x) + \tau(x)\, y'(x) + \lambda_n\, y(x) = 0, \tag{1}$$

where

$$\sigma(x) = (1-x^2), \quad \tau(x) = \beta - \alpha - (\alpha+\beta+2)x, \quad \lambda_n = n(n+\alpha+\beta+1),$$

respectively.

They are orthogonal with respect to the linear functional $\mathcal{J}_{\alpha,\beta}$ on the linear space $\mathbb{P}$ of polynomials with real coefficients defined by

$$\langle \mathcal{J}_{\alpha,\beta}, P\rangle \;= \int_{-1}^{1} P(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1, \quad P \in \mathbb{P}. \tag{2}$$

The orthogonality relation is

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}dx = \delta_{nm}||P_n^{\alpha,\beta}||^2, \tag{3}$$

$$||P_n^{\alpha,\beta}||^2 = \frac{2^{\alpha+\beta+2n+1} n!\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+1)_n^2}.$$

They satisfy the differentiation formula

$$(P_n^{\alpha,\beta}(x))^{(\nu)} = \frac{n!}{(n-\nu)!} P_{n-\nu}^{\alpha+\nu,\beta+\nu}(x), \quad \nu = 0,1,\ldots, \tag{4}$$

where $n = 1,2,\ldots$, and $(P_n^{\alpha,\beta}(x))^{(\nu)}$ denotes the $\nu - th$ derivative of the function. Furthermore, the following symmetry property holds

$$P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x). \tag{5}$$

Now, from the *structure relation*

$$(1-x^2)P_n'(x) = \tilde{\alpha}_n P_{n+1}(x) + \tilde{\beta}_n P_n(x) + \tilde{\gamma}_n P_{n-1}(x), \qquad n \geq 0, \tag{6}$$

where

$$\begin{aligned}
\tilde{\alpha}_n &= -n, \\
\tilde{\beta}_n &= \frac{2(\alpha-\beta)n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\
\tilde{\gamma}_n &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},
\end{aligned} \tag{7}$$

and *the three-term recurrence relation*,

$$xP_n(x) = P_{n+1}(x) + \beta_n^{\alpha,\beta} P_n(x) + \gamma_n^{\alpha,\beta} P_{n-1}(x), \tag{8}$$

where

$$\begin{aligned} \beta_n^{\alpha,\beta} &= \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \gamma_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned} \tag{9}$$

we deduce

$$\begin{aligned} P_{n-1}^{\alpha,\beta}(x) = \frac{(1-x^2)(P_n^{\alpha,\beta})'(x)}{\hat{\gamma}_n^{\alpha,\beta}} - \frac{\left[\tilde{\beta}_n - n(x - \beta_n^{\alpha,\beta})\right] P_n^{\alpha,\beta}(x)}{\hat{\gamma}_n^{\alpha,\beta}}, \\ \hat{\gamma}_n^{\alpha,\beta} = (2n+\alpha+\beta+1)\gamma_n^{\alpha,\beta}. \end{aligned} \tag{10}$$

If we take derivates in the above formula we obtain

$$\begin{aligned} (P_{n-1}^{\alpha,\beta})'(x) \ &= -\frac{\left(2x + \left[\tilde{\beta}_n - n(x - \beta_n^{\alpha,\beta})\right]\right)(P_n^{\alpha,\beta})'(x)}{\hat{\gamma}_n^{\alpha,\beta}} \\ &+ \frac{nP_n^{\alpha,\beta}(x)}{\hat{\gamma}_n^{\alpha,\beta}} + \frac{(1-x^2)(P_n^{\alpha,\beta})''(x)}{\hat{\gamma}_n^{\alpha,\beta}}, \end{aligned} \tag{11}$$

where $\hat{\gamma}_n^{\alpha,\beta}$ and $\tilde{\beta}_n$ are given in (7) and (10), respectively.
The Jacobi polynomials have the following representation as hypergeometric series

$$P_n^{\alpha,\beta}(x) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n} {}_2F_1\left( \begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array} \middle| \frac{1-x}{2} \right), \tag{12}$$

where

$${}_pF_q\left( \begin{array}{c} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

and $(a)_k$ with $k = 1, 2, \ldots$ is the Pochhammer symbol or shifted factorial defined by

$$(a)_0 := 1, \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

A consequence of this representation is

$$P_n^{\alpha,\beta}(1) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}, \quad P_n^{\alpha,\beta}(-1) = \frac{(-1)^n 2^n(\beta+1)_n}{(n+\alpha+\beta+1)_n}. \tag{13}$$

Throughout this work we will use

$$\begin{aligned} K_n^{\alpha,\beta(p,q)}(x,y) &= \sum_{m=0}^{n} \frac{(P_m^{\alpha,\beta})^{(p)}(x)(P_m^{\alpha,\beta})^{(q)}(y)}{||P_m||^2} \\ &= \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_n^{\alpha,\beta(0,0)}(x,y), \end{aligned} \tag{14}$$

in order to denote the kernels of the Jacobi polynomials, as well as their derivatives with respect to $x$ and $y$, respectively. For $p = q = 0$ and $n = 1, 2, \ldots$ the well known Christoffel-Darboux formula

$$\sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x)P_m^{\alpha,\beta}(y)}{||P_m||^2} = \frac{P_n^{\alpha,\beta}(x)P_{n-1}^{\alpha,\beta}(y) - P_{n-1}^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)}{(x-y)||P_{n-1}||^2}, \tag{15}$$

holds.

## 3. The Definition and Orthogonal Relation.

Consider the linear functional $\mathcal{U}$ on $\mathbb{P}$, defined as

$$\langle \mathcal{U}, P\rangle = \langle \mathcal{J}_{\alpha,\beta}, P\rangle + A_1 P(1) + B_1 P(-1) + A_2 P'(1) + B_2 P'(-1), \tag{16}$$

where $\mathcal{J}_{\alpha,\beta}$ is the Jacobi linear functional (2).

For large $n$ we will determine the monic polynomial $\widetilde{P}_n(x)$ which is orthogonal with respect to the functional (16). The reason for this assumption is to guarantee the existence of the polynomials for all values of the masses $A_1$, $B_1$, $A_2$ and $B_2$. To obtain this we write the Fourier expansion of the generalized Jacobi polynomials in terms of the Jacobi polynomials

$$\widetilde{P}_n(x) := P_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(x) = P_n^{\alpha,\beta}(x) + \sum_{k=0}^{n-1} a_{n,k} P_k^{\alpha,\beta}(x), \tag{17}$$

where $P_n^{\alpha,\beta}(x)$ denotes the Jacobi monic polynomial of degree $n$. To find the coefficients $a_{n,k}$ we can use the orthogonality of the polynomials $\widetilde{P}_n(x)$ with respect to $\mathcal{U}$, i.e.,

$$\left\langle \mathcal{U}, \widetilde{P}_n(x) P_k^{\alpha,\beta}(x) \right\rangle = 0, \quad 0 \le k < n.$$

Putting (17) in (16) we find:

$$\left\langle \mathcal{U}, \widetilde{P}_n(x) P_k^{\alpha,\beta}(x) \right\rangle = \left\langle \mathcal{J}_{\alpha,\beta}, \widetilde{P}_n(x) P_k^{\alpha,\beta}(x) \right\rangle$$
$$+A_1 \widetilde{P}_n(1) P_k^{\alpha,\beta}(1) + B_1 \widetilde{P}_n(-1) P_k^{\alpha,\beta}(-1) \tag{18}$$
$$+ A_2 \left( \widetilde{P}_n(x) P_k^{\alpha,\beta}(x) \right)' \Big|_{x=1} + B_2 \left( \widetilde{P}_n(x) P_k^{\alpha,\beta}(x) \right)' \Big|_{x=-1},$$

where $\widetilde{P}_n'(x)$ and $\left( P_n^{\alpha,\beta}(x) \right)'$ denote the first derivatives of the generalized and the Jacobi polynomials, respectively. If we use the decomposition (17) and taking into account the orthogonality of the Jacobi polynomials with respect to the linear functional $\mathcal{J}_{\alpha,\beta}$ we find the following expression for the coefficients $a_{n,k}$

$$a_{n,k} = -\frac{A_1 \widetilde{P}_n(1) P_k^{\alpha,\beta}(1) + B_1 \left( \widetilde{P}_n \right)' (-1) P_k^{\alpha,\beta}(-1)}{||P_k||^2}$$
$$-\frac{A_2 \left[ \left( \widetilde{P}_n \right)' (1) P_k^{\alpha,\beta}(1) + \widetilde{P}_n(1) \left( P_k^{\alpha,\beta} \right)' (1) \right]}{||P_k||^2} \tag{19}$$
$$-\frac{B_2 \left[ \left( \widetilde{P}_n \right)' (-1) P_k^{\alpha,\beta}(-1) + \widetilde{P}_n(-1) \left( P_k^{\alpha,\beta} \right)' (-1) \right]}{||P_k||^2}.$$

Finally (17) becomes

$$\widetilde{P}_n(x) = P_n^{\alpha,\beta}(x) - A_1 \widetilde{P}_n(1) K_{n-1}^{\alpha,\beta(0,0)}(x,1)$$
$$-B_1 \widetilde{P}_n(-1) K_{n-1}^{\alpha,\beta(0,0)}(x,-1) - A_2 \left( \widetilde{P}_n \right)' (1) K_{n-1}^{\alpha,\beta(0,0)}(x,1)$$
$$-B_2 \left( \widetilde{P}_n \right)' (-1) K_{n-1}^{\alpha,\beta(0,0)}(x,-1) - A_2 \widetilde{P}_n(1) K_{n-1}^{\alpha,\beta(0,1)}(x,1) \tag{20}$$
$$-B_2 \widetilde{P}_n(-1) K_{n-1}^{\alpha,\beta(0,1)}(x,-1).$$

In order to find $\widetilde{P}_n(1)$, $\widetilde{P}_n(-1)$, $\left( \widetilde{P}_n \right)' (1)$ and $\left( \widetilde{P}_n \right)' (-1)$ we can take derivatives in (20) and evaluate the resulting equation, as well as (20), at $x = 1$ and $x = -1$. This leads us to a linear system of equations

$$\mathcal{K} \cdot \tilde{\mathcal{P}}_n = \mathcal{P}_n, \tag{21}$$

being

$$\mathcal{K} := \mathcal{I}_4 + \mathcal{K}_4(n),$$

where $\mathcal{I}_4$ is the identity matrix and $\mathcal{K}_4(n) = (\mathcal{M}_1|\mathcal{M}_2|\mathcal{M}_3|\mathcal{M}_4)$ with the following column vectors

$$\mathcal{M}_1 = A_1 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) \end{pmatrix} + A_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,1)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,1) \\ K_{n-1}^{\alpha,\beta(1,1)}(1,1) \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) \end{pmatrix},$$

$$\mathcal{M}_2 = B_1 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,0)}(-1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,-1) \end{pmatrix} + B_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,1)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,-1) \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) \\ K_{n-1}^{\alpha,\beta(1,1)}(-1,-1) \end{pmatrix},$$

$$\mathcal{M}_3 = A_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) \end{pmatrix}, \quad \mathcal{M}_4 = B_2 \begin{pmatrix} K_{n-1}^{\alpha,\beta(0,0)}(1,-1) \\ K_{n-1}^{\alpha,\beta(0,0)}(-1,-1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,1) \\ K_{n-1}^{\alpha,\beta(0,1)}(-1,-1) \end{pmatrix}.$$

$\tilde{\mathcal{P}}_n$ and $\mathcal{P}_n$ are the column vectors

$$\tilde{\mathcal{P}}_n = \begin{pmatrix} \widetilde{P}_n(1) \\ \widetilde{P}_n(-1) \\ \left(\widetilde{P}_n\right)'(1) \\ \left(\widetilde{P}_n\right)'(-1) \end{pmatrix}, \quad \mathcal{P}_n = \begin{pmatrix} P_n^{\alpha,\beta}(1) \\ P_n^{\alpha,\beta}(-1) \\ (P_n^{\alpha,\beta})'(1) \\ (P_n^{\alpha,\beta})'(-1) \end{pmatrix},$$

respectively. Let us denote $\mathcal{K}_j(\mathcal{P}_n)$ the matrix obtained substituting the $j$ column in $\mathcal{K}$ by $\mathcal{P}_n$. Then, by the Cramer's rule, the system (21) has a unique solution if and only if the determinant of $\mathcal{K}$ is different from zero. Moreover, the solution is

$$\begin{aligned} \widetilde{P}_n(1) &= \frac{\det \mathcal{K}_1(\mathcal{P}_n)}{\det \mathcal{K}}, & \widetilde{P}_n(-1) &= \frac{\det \mathcal{K}_2(\mathcal{P}_n)}{\det \mathcal{K}}, \\ \left(\widetilde{P}_n\right)'(1) &= \frac{\det \mathcal{K}_3(\mathcal{P}_n)}{\det \mathcal{K}}, & \left(\widetilde{P}_n\right)'(-1) &= \frac{\det \mathcal{K}_4(\mathcal{P}_n)}{\det \mathcal{K}}. \end{aligned} \tag{22}$$

Then, the existence of the generalized polynomials is guaranteed if and only if $\det \mathcal{K}$ does not vanish for every $n \geq 0$. Later on, from the asymptotic formulas for the kernels (33)-(35) we will conclude that for $n$ large enough the monic polynomial $\widetilde{P}_n(x)$ exists.

PROPOSITION 1. *The following symmetry properties for the generalized Jacobi polynomials and their first derivatives hold*

$$P_n^{\alpha,\beta,A_1,B_1,A_2,B_2}(-x) = (-1)^n P_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2}(x), \tag{23}$$

$$(P_n^{\alpha,\beta,A_1,B_1,A_2,B_2})'(-1) = (-1)^{n+1}(P_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2})'(1). \tag{24}$$

Later on, will be useful to have an explicit representation of the generalized Jacobi polynomials in terms of the classical ones. To do that we rewrite the equation (20) in the form

$$\begin{aligned} \widetilde{P}_n(x) &= (1+n\zeta_n+n\eta_n)P_n^{\alpha,\beta}(x) \\ &+ [\zeta_n(1-x)-\eta_n(1+x)+(\beta+1)\chi_n+(\alpha+1)\omega_n]\left(P_n^{\alpha,\beta}(x)\right)' \\ &+ [\chi_n(1+x)-\omega_n(1-x)]\left(P_n^{\alpha,\beta}(x)\right)'', \end{aligned} \tag{25}$$

where

$$\begin{aligned} \zeta_n &= \left[B_1 - \frac{n(n+\alpha+\beta+1)B_2}{2(\beta+1)}\right] C_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2} \\ &\quad -B_2 D_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2}, \\ \eta_n &= \left[A_1 + \frac{n(n+\alpha+\beta+1)A_2}{2(\alpha+1)}\right] C_n^{\alpha,\beta,A_1,B_1,A_2,B_2} \\ &\quad +A_2 D_n^{\alpha,\beta,A_1,B_1,A_2,B_2}, \end{aligned} \tag{26}$$

$$\begin{aligned} \chi_n &= \frac{A_2 C_n^{\alpha,\beta,A_1,B_1,A_2,B_2}}{(\alpha+1)}, \\ \omega_n &= \frac{B_2 C_n^{\beta,\alpha,B_1,A_1,-B_2,-A_2}}{(\beta+1)}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} C_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \frac{\widetilde{P}_n(1)P_n^{\alpha,\beta}(1)}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}, \\ D_n^{\alpha,\beta,A_1,B_1,A_2,B_2} &= \frac{\left(\widetilde{P}_n\right)'(1)P_n^{\alpha,\beta}(1)}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}. \end{aligned} \tag{28}$$

Notice that $\zeta_n, \eta_n, \chi_n$, and $\omega_n$ depend on $n, \alpha, \beta, A_1, B_1, A_2$, and $B_2$. Now, using (4)-(8) we can rewrite (25) as follows

$$\begin{aligned} \widetilde{P}_n(x) &= A_n P_n^{\alpha,\beta}(x) + n\left[B_n P_{n-1}^{\alpha+1,\beta+1}(x) + C_n P_n^{\alpha+1,\beta+1}(x)\right] \\ &+n\left[D_n P_{n-2}^{\alpha+1,\beta+1}(x) + (n-1)E_n P_{n-2}^{\alpha+2,\beta+2}(x)\right] \\ &+n(n-1)\left[F_n P_{n-1}^{\alpha+2,\beta+2}(x) + G_n P_{n-3}^{\alpha+2,\beta+2}(x)\right], \end{aligned} \tag{29}$$

where

$$B_n = \zeta_n - \eta_n + C_n\beta_{n-1}^{\alpha+1,\beta+1} + (\beta+1)\chi_n + (\alpha+1)\omega_n,$$

$$A_n = 1 - nC_n, \quad C_n = -(\zeta_n + \eta_n), \quad D_n = C_n\gamma_{n-1}^{\alpha+1,\beta+1}, \tag{30}$$

$$E_n = \chi_n - \omega_n + F_n\beta_{n-2}^{\alpha+2,\beta+2},\ F_n = \chi_n + \omega_n,\ G_n = F_n\gamma_{n-2}^{\alpha+2,\beta+2}.$$

We would like to remark here that the generalized Jacobi polynomials satisfy a second order linear differential equation (SODE). To deduce it one can rewrite the representation formula (25) in terms of the polynomials and their first derivatives, and using that the Jacobi polynomials satisfy a SODE.

## 4. Some Asymptotic Formulas.

In this section we will study some asymptotic formulas for the generalized Jacobi polynomials. More precisely, the relative asymptotics $\widetilde{P}_n(x)/P_n^{\alpha,\beta}(x)$, outside the interval [-1,1] and the difference between the new polynomials and the classical ones, inside [-1,1]. First of all, we need to obtain some asymptotic formulas concerning to the classical polynomials and their kernels. In order to do that we use the asymptotic formula for the Gamma function see ((F. W. J. Olver, 1974, formula 8.16, page 88) and (G. Szegö, 1975))

$$\Gamma(ax+b) \sim \sqrt{2\pi}e^{-ax}(a\,x)^{a\,x+b-\frac{1}{2}}, \quad x >> 1, \qquad a,b,x \in \mathbb{R}. \tag{31}$$

Taking into account (3), (4) and (13), we find the following asymptotic formulas for $k \in \mathbb{N}$

$$(P_n^{\alpha,\beta})^{(k)}(1) \sim \frac{\sqrt{\pi}n^{\alpha+2k+\frac{1}{2}}}{\Gamma(\alpha+k+1)2^{n+\alpha+\beta+k}},\ ||P_{n-1}^{\alpha,\beta}||^2 \sim \frac{\pi}{2^{2n+\alpha+\beta-2}}, \tag{32}$$

and

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,0)}(1,1) &\sim \frac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)2^{\alpha+\beta+1}}, \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) &\sim \frac{(-1)^{n+1}n^{\alpha+\beta+1}}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}}, \end{aligned} \tag{33}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(1,1) &\sim \frac{n^{2\alpha+4}}{\Gamma(\alpha+1)\Gamma(\alpha+3)2^{\alpha+\beta+2}}, \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) &\sim \frac{(-1)^{n}n^{\alpha+\beta+3}}{\Gamma(\beta+2)\Gamma(\alpha+1)2^{\alpha+\beta+2}}, \end{aligned} \tag{34}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(1,1)}(1,1) &\sim \frac{(\alpha+2)n^{2\alpha+6}}{\Gamma(\alpha+2)\Gamma(\alpha+4)2^{\alpha+\beta+3}}, \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) &\sim \frac{(-1)^{n}n^{\alpha+\beta+5}}{\Gamma(\alpha+2)\Gamma(\beta+2)2^{\alpha+\beta+3}}, \end{aligned} \tag{35}$$

where $x_n \sim y_n$ means that $\lim_{n\to\infty} x_n/y_n = 1$. To obtain the other kernels, as well as their estimates, we can use the symmetry properties (62) and (33)-(35).

¿From these asymptotic formulas, (33)-(35), and doing some straightforward calculations we find that for $n$ large enough

$$\det \mathcal{K} \sim O(n^{16+4\alpha+4\beta}).$$

Then, the existence of $\widetilde{P}_n(x)$ for $n$ large enough is guaranteed for any choice of non zero masses $A_1$, $B_1$, $A_2$ and $B_2$. Now using the symmetry property (5) and the asymptotic formulas (32)-(35), we can compute the asymptotic behavior of the generalized Jacobi polynomials, as well as their first derivatives at the points $\pm 1$, i.e.,

$$\begin{aligned} &\widetilde{P}_n(1) \sim \frac{\sqrt{\pi}\Gamma(\alpha+4)}{2^{n-2}A_2 n^{\frac{7}{2}+\alpha}}, \quad \widetilde{P}_n(-1) \sim (-1)^{n+1}\frac{\sqrt{\pi}\Gamma(\beta+4)}{2^{n-2}B_2 n^{\frac{7}{2}+\beta}}, \\ &\left(\widetilde{P}_n\right)'(1) \sim -\frac{\sqrt{\pi}\Gamma(\alpha+3)}{2^{n-1}A_2 n^{\frac{3}{2}+\alpha}}, \; \left(\widetilde{P}_n\right)'(-1) \sim \frac{(-1)^{n+1}\sqrt{\pi}\Gamma(\beta+3)}{2^{n-1}B_2 n^{\frac{3}{2}+\beta}}. \end{aligned} \tag{36}$$

¿From (32)-(36) we can give the estimates for the constants defined by (26)-(28)

$$
\begin{gathered}
C^{\alpha,\beta,A_1,B_1,A_2,B_2} \sim 2\frac{\Gamma(\alpha+4)}{A_2\Gamma(\alpha+1)n^4}, \\
D^{\alpha,\beta,A_1,B_1,A_2,B_2} \sim -\frac{\Gamma(\alpha+3)}{A_2\Gamma(\alpha+1)n^2}, \\
\zeta_n \sim 2\frac{(\beta+2)}{n^2}, \quad \eta_n \sim 2\frac{(\alpha+2)}{n^2}, \quad \chi_n \sim 2\frac{(\alpha+2)(\alpha+3)}{n^4}, \\
\omega_n \sim -2\frac{(\beta+2)(\beta+3)}{n^4}.
\end{gathered} \tag{37}
$$

Finally, from (25), taking derivatives two times and using (32) and (37) we find

$$
\left(\widetilde{P}_n\right)''(1) \sim -\frac{\sqrt{\pi} n^{\alpha+\frac{9}{2}}(\alpha+2)(\alpha+5)}{\Gamma(\alpha+5)2^{n+\alpha+\beta+2}}. \tag{38}
$$

To obtain the relative asymptotics $\widetilde{P}_n(z)/P_n^{\alpha,\beta}(z)$, outside the interval [-1,1] we need to do some manipulations. First, we multiply (25) by $\sigma(z)$, and using the SODE (1) we find the following equivalent representation formula

$$
\sigma(z)\widetilde{P}_n(z) = a(z;n)P_n^{\alpha,\beta}(z) + b(z;n)\left(P_n^{\alpha,\beta}(z)\right)', \tag{39}
$$

where $a(z;n), b(z;n)$ are polynomials of uniformly bounded degree in $z$ with coefficients depending on $n$ given by

$$
\begin{aligned}
a(z;n) &= (1+n\zeta_n+n\eta_n)\sigma(z) - \lambda_n\left[\chi_n(1+z) - \omega_n(1-z)\right], \\
b(z;n) &= \left[\zeta_n(1-z) - \eta_n(1+z) + (\beta+1)\chi_n + (\alpha+1)\omega_n\right]\sigma(z) \\
&\quad -\tau(z)\left[\chi_n(1+z) - \omega_n(1-z)\right].
\end{aligned} \tag{40}
$$

Second, we will rewrite (39) in the form

$$
\widetilde{P}_n(z) = \tilde{a}(z;n)P_n^{\alpha,\beta}(z) + \tilde{b}(z;n)\left(P_n^{\alpha,\beta}(z)\right)', \tag{41}
$$

where

$$
\begin{aligned}
\tilde{a}(z;n) &= (1+n\zeta_n+n\eta_n) - \lambda_n\left[\frac{\chi_n}{(1-z)} - \frac{\omega_n}{(1+z)}\right] \\
&\sim 1 + 2\frac{(\alpha+\beta+4)}{n} - \frac{2}{n^2}\left[\frac{(\alpha+2)(\alpha+3)}{(1-z)} + \frac{(\beta+2)(\beta+3)}{(1+z)}\right],
\end{aligned} \tag{42}
$$

$$\tilde{b}(z;n) = [\zeta_n(1-z) - \eta_n(1+z) + (\beta+1)\chi_n + (\alpha+1)\omega_n]$$

$$-\tau(x)\left[\frac{\chi_n}{(1-z)} - \frac{\omega_n}{(1+z)}\right]$$

$$\sim \frac{2}{n^2}\left[(\beta+2)(1-z) - (\alpha+2)(1+z)\right] \tag{43}$$

$$+\frac{2}{n^4}[(\beta+1)(\alpha+2)(\alpha+3) - (\alpha+1)(\beta+2)(\beta+3)]$$

$$-2\frac{[\beta-\alpha-(\alpha+\beta+2)z]}{n^4}\left[\frac{(\alpha+2)(\alpha+3)}{(1-z)} + \frac{(\beta+2)(\beta+3)}{(1+z)}\right].$$

Then, from (41) and using (42)-(44), as well as,

$$\frac{1}{n}\frac{(P_n^{\alpha,\beta})'(z)}{P_n^{\alpha,\beta}(z)} = \frac{1}{\sqrt{z^2-1}} + o(1)\,, \tag{44}$$

we obtain the following estimate for the ratio

$$\begin{aligned}\frac{\widetilde{P}_n(z)}{P_n^{\alpha,\beta}(z)} &= 1 + \frac{2(\beta+2)}{n}\left[1-\sqrt{\frac{z-1}{z+1}}\right] \\ &+\frac{2(\alpha+2)}{n}\left[1-\sqrt{\frac{z+1}{z-1}}\right] + o\left(\frac{1}{n}\right),\end{aligned} \tag{45}$$

where $z \in \mathbb{C}\backslash[-1,1]$.

In order to obtain the asymptotic behavior of the difference between the new polynomials and the classical ones, when $z$ belongs to $[-1,1]$, we use the Darboux formula for the asymptotics of the Jacobi polynomials on the interval $\theta \in [\varepsilon, \pi-\varepsilon], 0 < \varepsilon << 1$ (G. Szegö, 1975, equation 8.21.10 page 196)

$$\begin{aligned} a_n P_n^{\alpha,\beta}(\cos\theta) &= \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{\sqrt{n\pi}} \\ \times\cos&\left[n\theta + \frac{1}{2}(\alpha+\beta+1)\theta - \frac{1}{2}(\alpha+\frac{1}{2})\pi\right] + O\left(\frac{1}{n^{\frac{3}{2}}}\right),\end{aligned} \tag{46}$$

with $a_n = \dfrac{(n+\alpha+\beta+1)_n}{2^n\, n!} \sim \dfrac{2^{n+\alpha+\beta}}{\sqrt{n\pi}}$.

The expression (29), as well as the following asymptotic estimates for the coefficients

$$A_n \sim 1 + 2\frac{(\alpha+\beta+4)}{n}, \quad B_n \sim 2\frac{(\beta-\alpha)}{n^2},$$
$$C_n \sim -2\frac{(\alpha+\beta+4)}{n^2}, \quad D_n \sim -\frac{(\alpha+\beta+4)}{2n^2}, \tag{47}$$
$$E_n = O\left(\frac{1}{n^6}\right), \quad F_n = O\left(\frac{1}{n^4}\right), \quad G_n = O\left(\frac{1}{n^4}\right),$$

which follow from (37).
Then, using (29) and (46)-(47) we deduce

$$2^{n+\alpha+\beta}\left[\widetilde{P}_n(x) - P_n^{\alpha,\beta}(x)\right] \sim \frac{\left[\sin\left(\frac{\theta}{2}\right)\right]^{-\alpha-\frac{3}{2}}\left[\cos\left(\frac{\theta}{2}\right)\right]^{-\beta-\frac{3}{2}}}{n}$$
$$\times\left\{(\alpha+\beta+4)\sin\theta\cos\left[n\theta + \frac{1}{2}(\alpha+\beta+1)\theta - \frac{1}{2}(\alpha+\frac{1}{2})\pi\right]\right.$$
$$\left.-2(\alpha+2)\cos\left[n\theta + \frac{1}{2}(\alpha+\beta+3)\theta - \frac{1}{2}(\alpha+\frac{3}{2})\pi\right]\right\} + O\left(\frac{1}{n^2}\right).$$

## 5. Zeros.

Here we will study the properties of the zeros of the generalized Jacobi polynomials, for non zero values of the masses.

THEOREM 1. *For $n$ large enough, the orthogonal polynomial $\widetilde{P}_n(x)$ has at least $n-4$ different, real and simple zeros in $(-1,1)$.*

*Proof.* Let $x_1, x_2, x_3, ..., x_k$ be the different real zeros of odd multiplicity of $\widetilde{P}_n(x)$ in $(-1,1)$. Hence the sign of the product $\widetilde{P}_n(x)q(x)$ does not change $\forall x \in (-1,1)$, where

$$q(x) = (x-x_1)(x-x_2)...(x-x_k).$$

Now define

$$h(x) = \left(1-x^2\right)^2 q(x).$$

Thus

$$\left\langle \mathcal{U}, \widetilde{P}_n(x)\, h(x)\right\rangle = \left\langle \mathcal{J}_{\alpha,\beta}, \widetilde{P}_n(x)\, h(x)\right\rangle > 0,$$

so $\deg h(x) \geq n$, i.e., $k \geq n-4$. □

COROLLARY 1. *When the masses $A_2, B_2$ have the same sign, the generalized Jacobi polynomial for $n$ large enough has exactly $n-3$ different, real and simple zeros belonging to the interval $(-1,1)$.*

*Proof.* Let us consider the case of even $n$ and $A_2, B_2 > 0$. The other cases can be proved in a similar way.
Since $\widetilde{P}_n(1) > 0$ and $\left(\widetilde{P}_n\right)'(1) < 0$ (see (36)), then for some positive $x > 1$, the polynomial $\widetilde{P}_n(x)$ has a minimum. This implies that on the right of $x = 1$ it has two zeros, which can be complex conjugates, real and simple or with multiplicity 2. Again from (36), since $\widetilde{P}_n(-1)$ and $\left(\widetilde{P}_n\right)'(-1)$ are negative the polynomial $\widetilde{P}_n(x)$ is a convex upward function for $x < -1$ and has a simple real zero; otherwise the number of zeros off [-1,1] was greater than 4, which yields a contradiction. □

COROLLARY 2. *For $n$ large enough if $A_2 > 0$, $B_2 < 0$ the generalized Jacobi polynomial has exactly 4 zeros off $[-1,1]$, of which two are located on the right of $x = 1$, and the other two are on the left of $x = -1$.*

PROPOSITION 2. *For $n$ large enough, if $A_2 < 0$ and $B_2 > 0$ the orthogonal polynomial $\widetilde{P}_n(x)$ has exactly $n$ different, real and simple zeros, $n-2$ of them belong to the interval $(-1,1)$, and the two remainder zeros are outside the interval being one positive and the other one negative.*

*Proof.* Let $x_1, x_2, x_3, ..., x_k$ be the different real zeros of odd multiplicity of $\widetilde{P}_n(x)$ on the interval $(-1,1)$ and

$$q(x) = (x - x_1)(x - x_2)...(x - x_k), \tag{48}$$

such that the sign of the product $\widetilde{P}_n(x)q(x)$ does not change $\forall x \in (-1,1)$. Now define $h(x)$

$$h(x) = (1 - x^2)q(x) = (1 - x^2)(x - x_1)(x - x_2)...(x - x_k). \tag{49}$$

Hence, for $n$ large enough

$$\begin{aligned} \left\langle \mathcal{U}, \widetilde{P}_n(x)\, h(x) \right\rangle &= \int_{-1}^{1} (1 - x^2)q(x)\widetilde{P}_n(x)\rho(x)dx \\ &+2\left[-A_2\widetilde{P}_n(1)q(1) + B_2\widetilde{P}_n(-1)q(-1)\right] < 0, \end{aligned} \tag{50}$$

which implies that $\deg h(x) \geq n$, i.e. $k \geq n-2$.
To prove that $\widetilde{P}_n(x)$ has one real simple negative zero and one real simple positive zero outside [-1, 1] we use the fact that for $n$ large

enough $\widetilde{P}_n(1) < 0$, $\left(\widetilde{P}_n\right)'(1) > 0$ (see formula (36)) and the polynomial $\widetilde{P}_n(x)$ is a continuous convex upward function for $x > 1$, then in some positive value $x > 1$ it changes its sign. Using the symmetry property (Proposition 1), Eq. (36), and a similar argument we can prove that the polynomial has one simple real negative zero off [-1, 1]. This implies that $k = n - 2$, hence the proposition holds. □

When $A_2 < 0$, $B_2 > 0$ we denote the zeros of $\widetilde{P}_n(x)$ as $x_{n,1} < -1 < x_{n,2} < ... < x_{n,n-1} < 1 < x_{n,n}$. Now we proceed to study the zeros $x_{n,1}$ and $x_{n,n}$ in more detail.

PROPOSITION 3. *For $n$ large enough $x_{n,n} - 1 = O\left(n^{-\alpha-4}\right)$ and $x_{n,1} + 1 = O\left(n^{-\beta-4}\right)$. More precisely*

$$\begin{aligned} 1 < x_{n,n} < 1 + \frac{\sqrt{2^{\alpha+\beta+5}\Gamma(\alpha+4)\Gamma(\alpha+5)}}{n^{\alpha+4}\sqrt{A_2(\alpha+2)(\alpha+5)}} + O\left(n^{-2\alpha-6}\right), \\ -1 > x_{n,1} > -1 - \frac{\sqrt{2^{\alpha+\beta+5}\Gamma(\beta+4)\Gamma(\beta+5)}}{n^{\beta+4}\sqrt{B_2(\beta+2)(\beta+5)}} + O\left(n^{-2\beta-6}\right). \end{aligned} \tag{51}$$

*Proof.* Using Taylor's Theorem we have for $x > 1$,

$$\begin{aligned} \widetilde{P}_n(x) &= \widetilde{P}_n(1) + \left(\widetilde{P}_n\right)'(1)(x-1) + \\ &+ \left(\widetilde{P}_n\right)''(1)\frac{(x-1)^2}{2} + \left(\widetilde{P}_n\right)'''(\xi)\frac{(x-1)^3}{6}, \end{aligned} \tag{52}$$

where $1 < \xi < x$. ¿From (36) for $n$ large enough, $\widetilde{P}_n(1)$ is negative while $\left(\widetilde{P}_n\right)'(1)$ is positive. Moreover, $\widetilde{P}_n(x)$ is a convex upward function for $x > 1$ and has its first saddle point (from the right) somewhere at $x < 1$. Then for all $x > 1$, $\left(\widetilde{P}_n\right)'''(x) \geq 0$. Hence

$$\begin{aligned} \widetilde{P}_n(x) &\geq \frac{\left(\widetilde{P}_n\right)''(1)}{2}x^2 + \left[\left(\widetilde{P}_n\right)'(1) - \left(\widetilde{P}_n\right)''(1)\right]x \\ &+ \frac{\left(\widetilde{P}_n\right)''(1)}{2} - \left(\widetilde{P}_n\right)'(1) + \widetilde{P}_n(1), \end{aligned}$$

and the zero $x_{n,n}$ is located between the zeros of the quadratic polynomial on the right hand side of the previous expression. If we denote

$$x_{1,2} = 1 - \frac{\left(\widetilde{P}_n\right)'(1)}{\left(\widetilde{P}_n\right)''(1)} \pm \sqrt{\left[\frac{\left(\widetilde{P}_n\right)'(1)}{\left(\widetilde{P}_n\right)''(1)}\right]^2 - 2\frac{\widetilde{P}_n(1)}{\left(\widetilde{P}_n\right)''(1)}} \tag{53}$$

and taking into account (36) and (38) we get

$$\begin{aligned} \frac{\left(\widetilde{P}_n\right)'(1)}{\left(\widetilde{P}_n\right)''(1)} &\sim \frac{2^{\alpha+\beta+3}\Gamma(\alpha+2)\Gamma(\alpha+5)}{A_2(\alpha+5)n^{2\alpha+6}}, \\ \frac{\widetilde{P}_n(1)}{\left(\widetilde{P}_n\right)''(1)} &\sim -\frac{2^{\alpha+\beta+4}\Gamma(\alpha+4)\Gamma(\alpha+5)}{A_2(\alpha+2)(\alpha+5)n^{2\alpha+8}}. \end{aligned} \tag{54}$$

Thus, when $n$ is large enough

$$x_2 \sim 1 + \frac{\sqrt{2^{\alpha+\beta+5}\Gamma(\alpha+4)\Gamma(\alpha+5)}}{n^{\alpha+4}\sqrt{A_2(\alpha+2)(\alpha+5)}} + O\left(n^{-2\alpha-6}\right). \tag{55}$$

In the same way, using the symmetry property (1) we find the speed of convergence for $x_{n,1}$, then (51) holds. □

PROPOSITION 4. *For $n$ large enough, the pair of complex or real zeros $z_{1,2}$ ($z'_{1,2}$) located on the right of $x = 1$ (on the left of $x = -1$) (see the Corollaries 1 and 2) are such that*

$$\begin{cases} 1 < \Re(z_{1,2}) < 1 + \dfrac{\Re(\delta)}{n^{\frac{\epsilon}{2}}}, \\ 0 \le \Im(z_{1,2}) < \widetilde{P}_n(1), \\ -1 - \dfrac{\Re(\delta)}{n^{\frac{\epsilon}{2}}} < \Re\left(z'_{1,2}\right) < -1, \\ 0 \le \Im\left(z'_{1,2}\right) < \widetilde{P}_n(-1) \end{cases} \quad 0 < \epsilon < 2, \quad \textit{and} \quad \delta \in \mathbb{C}. \tag{56}$$

*Proof.* Suppose that we have two zeros located on the right of $x = 1$, from (32) and (36) the generalized polynomial for $x = 1$ is positive and tends to zero, the first derivative is negative, whereas $P_n^{\alpha,\beta}(1)$ and $\left(P_n^{\alpha,\beta}\right)'(1)$ tend to $+\infty$.

Let $\{\tilde{x}_n\}_{n=1}^{\infty}$ be the sequence

$$\tilde{x}_n = 1 + \frac{\delta}{n^{2-\epsilon}},$$

where $\delta \in \mathbb{C}$ is an arbitrary constant, and $\epsilon \in (0, 2)$.
If we evaluate (45) in the set $\{\tilde{x}_n\}_{n=1}^{\infty}$ we have

$$\frac{\widetilde{P}_n(\tilde{x}_n)}{P_n^{\alpha,\beta}(\tilde{x}_n)} = 1 - \frac{2(\alpha+2)}{n^{\frac{\epsilon}{2}}}\left(\frac{2}{\delta}\right)^{\frac{1}{2}} + O\left(\frac{1}{n}\right). \tag{57}$$

So, from (57) we see that for the values $\tilde{x}_n$, $P_n^{\alpha,\beta}(x)$ and $\widetilde{P}_n(x)$ have the same asymptotic behavior, then the zeros (and also the minimum) of the generalized Jacobi polynomials are in $(1, \tilde{x}_n)$. Then, the result holds. □

The case when the zeros are located on the left of $x = -1$ follows in an analog way.

COROLLARY 3. *If n tends to infinity, all the zeros of the generalized Jacobi polynomials $\widetilde{P}_n(x)$ off $[-1,1]$ tend to $\pm 1$.*

Concerning the distribution of zeros for the generalized Jacobi polynomials inside [-1,1] the next result shows that it is an arcsin distribution.

THEOREM 2. *Let $\nu_n$ be the discrete unit measure defined on the Borel sets in $\mathbb{C}$ having mass $\frac{1}{n}$ at each zero of $\widetilde{P}_n(x)$. Then*

$$\nu_n \xrightarrow{*} \frac{1}{\pi\sqrt{1-x^2}}, \tag{58}$$

*in the weak star topology.*

*Proof.* From (29) and (47), we get

$$\begin{aligned} &||\widetilde{P}_n(x)||_{[-1,1]} \le |A_n|\,||P_n^{\alpha,\beta}(x)||_{[-1,1]} + n\,|B_n|\,||P_{n-1}^{\alpha+1,\beta+1}(x)||_{[-1,1]} \\ &+n\,|C_n|\,||P_n^{\alpha+1,\beta+1}(x)||_{[-1,1]} + n\,|D_n|\,||P_{n-2}^{\alpha+1,\beta+1}(x)||_{[-1,1]} \\ &+n(n-1)\left[|E_n|\,||P_{n-2}^{\alpha+2,\beta+2}(x)||_{[-1,1]} + |F_n|\,||P_{n-1}^{\alpha+2,\beta+2}(x)||_{[-1,1]}\right] \\ &+n(n-1)\,|G_n|\,||P_{n-3}^{\alpha+2,\beta+2}(x)||_{[-1,1]}, \end{aligned} \tag{59}$$

where $||\cdot||_{[-1,1]}$ denotes the sup-norm in the interval [-1, 1].
Because of $\overline{\lim_{n\to\infty}}\,||P_n^{\alpha,\beta}(x)||_{[-1,1]}^{\frac{1}{n}} = \dfrac{1}{2}$ (see (G. Szegö, 1975)), we deduce

$$\overline{\lim_{n\to\infty}}\,||\widetilde{P}_n(x)||_{[-1,1]}^{\frac{1}{n}} \le \frac{1}{2}. \tag{60}$$

Thus, from Theorem 2.1 in (H. P. Blatt, E. B. Saff, and M. Simkani, 1988)

$$\nu_n \xrightarrow{*} \frac{1}{\pi\sqrt{1-x^2}}. \tag{61}$$

□

In the next figures we show some numerical examples when the zeros are located outside the interval $[-1,1]$.

## Appendix

¿From (14), (15), and (5) we get

$$K_n^{\alpha,\beta(0,0)}(x,y) = K_n^{\beta,\alpha(0,0)}(-x,-y),$$
$$K_n^{\alpha,\beta(0,1)}(x,y) = -K_n^{\beta,\alpha(0,1)}(-x,-y), \tag{62}$$
$$K_n^{\alpha,\beta(1,1)}(x,y) = K_n^{\beta,\alpha(1,1)}(-x,-y).$$

Here we will compute the different Jacobi kernels, in order to express the generalized Jacobi polynomials as $x^n+\tilde{b}_n x^{n-1}+$ lower degree terms. First of all, we obtain the kernel $K_{n-1}^{\alpha,\beta(0,0)}(x,1)$. Evaluating (15) in $y=1$ and using (10) to eliminate $P_{n-1}^{\alpha,\beta}(1)$ and $P_{n-1}^{\alpha,\beta}(x)$, we obtain

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,0)}(x,1) &= \sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x) P_m^{\alpha,\beta}(1)}{||P_m||^2} \\ &= \frac{1}{x-1} \frac{P_n^{\alpha,\beta}(x) P_{n-1}^{\alpha,\beta}(1) - P_{n-1}^{\alpha,\beta}(x) P_n^{\alpha,\beta}(1)}{||P_{n-1}||^2} \\ &= \frac{P_n^{\alpha,\beta}(1)\left[(1+x)\left(P_n^{\alpha,\beta}\right)'(x) - nP_n^{\alpha,\beta}(x)\right]}{||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}}. \end{aligned} \tag{63}$$

Taking derivates with respect to $y$ formula (15) becomes

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(x,y) &= \sum_{m=0}^{n-1} \frac{P_m^{\alpha,\beta}(x)\left(P_m^{\alpha,\beta}\right)'(y)}{||P_m||^2} = \frac{K_{n-1}^{\alpha,\beta(0,0)}(x,y)}{(x-y)} \\ &+ \frac{P_n^{\alpha,\beta}(x)\left(P_{n-1}^{\alpha,\beta}\right)'(y) - P_{n-1}^{\alpha,\beta}(x)\left(P_n^{\alpha,\beta}\right)'(y)}{(x-y)||P_{n-1}||^2}. \end{aligned} \tag{64}$$

Evaluating (64) at $y=1$ one gets

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(x,1) &= \frac{K_{n-1}^{\alpha,\beta(0,0)}(x,1)}{x-1} + \frac{P_n^{\alpha,\beta}(x)\left(P_{n-1}^{\alpha,\beta}\right)'(1)}{||P_{n-1}||^2(x-1)} \\ &- \frac{\left(P_n^{\alpha,\beta}\right)'(1) P_{n-1}^{\alpha,\beta}(x)}{||P_{n-1}||^2(x-1)}. \end{aligned} \tag{65}$$

If we now use (11) evaluated in $x=1$ to eliminate $\left(P_{n-1}^{\alpha,\beta}\right)'(1)$ as well as (10) to substitute $P_{n-1}^{\alpha,\beta}(x)$ in the above formula we get

$$K_{n-1}^{\alpha,\beta(0,1)}(x,1) = -\frac{\left(P_n^{\alpha,\beta}\right)'(1)\left[nP_n^{\alpha,\beta}(x)-(1+x)\left(P_n^{\alpha,\beta}\right)'(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}$$
$$+\frac{\left[(1+x)P_n^{\alpha,\beta}(1)(P_n^{\alpha,\beta})'(x)-2(P_n^{\alpha,\beta})'(1)P_n^{\alpha,\beta}(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}(x-1)}.$$

Now we will use the differentiation formula (4) for $\nu=1$ to replace $\left(P_n^{\alpha,\beta}\right)'(1)$ by $P_n^{\alpha,\beta}(1)$. This yields

$$K_{n-1}^{\alpha,\beta(0,1)}(x,1) = \frac{\left(P_n^{\alpha,\beta}\right)'(1)\left[(1+x)\left(P_n^{\alpha,\beta}\right)'(x)-nP_n^{\alpha,\beta}(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}$$
$$+\frac{P_n^{\alpha,\beta}(1)\left[(1+x)\left(P_n^{\alpha,\beta}\right)'(x)-\frac{n(n+\alpha+\beta+1)}{\alpha+1}P_n^{\alpha,\beta}(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}(x-1)}. \tag{66}$$

Finally, using the second order linear differential equation (1) we obtain

$$K_{n-1}^{\alpha,\beta(0,1)}(x,1) = \frac{\left(P_n^{\alpha,\beta}\right)'(1)\left[(1+x)\left(P_n^{\alpha,\beta}\right)'(x)-nP_n^{\alpha,\beta}(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}$$
$$-\frac{P_n^{\alpha,\beta}(1)\left[(1+\beta)\left(P_n^{\alpha,\beta}\right)'(x)+(x+1)\left(P_n^{\alpha,\beta}\right)''(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}(\alpha+1)}. \tag{67}$$

Handling as above, it is not difficult to deduce a similar expression for the kernels $K_{n-1}^{\alpha,\beta(0,0)}(x,-1)$ and $K_{n-1}^{\alpha,\beta(0,1)}(x,-1)$. Applying formulas

(10)-(15) and (64) and evaluating them at $y=-1$, we get

$$K_{n-1}^{\alpha,\beta(0,0)}(x,-1)=\frac{P_n^{\alpha,\beta}(-1)\left[(x-1)\left(P_n^{\alpha,\beta}\right)'(x)-nP_n^{\alpha,\beta}(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}},$$

$$\begin{aligned}K_{n-1}^{\alpha,\beta(0,1)}(x,-1)&=\frac{\left(P_n^{\alpha,\beta}\right)'(-1)}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}\\&\times\left[(x-1)\left(P_n^{\alpha,\beta}\right)'(x)-nP_n^{\alpha,\beta}(x)\right]\\&+\frac{P_n^{\alpha,\beta}(-1)\left[(1-x)\left(P_n^{\alpha,\beta}\right)''(x)-(\alpha+1)\left(P_n^{\alpha,\beta}\right)'(x)\right]}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}(\beta+1)},\end{aligned}\tag{68}$$

respectively. Now we obtain the kernel $K_{n-1}^{\alpha,\beta(1,1)}(x,1)$ in terms of the polynomials $P_n^{\alpha,\beta}(x)$ and their first, second and third derivatives. From (67), after some straightforward calculations we find

$$\begin{aligned}K_{n-1}^{\alpha,\beta(1,1)}(x,1)&=\frac{1}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}\left\{\left(P_n^{\alpha,\beta}\right)'(1)\right.\\&\times\left[(1+x)\left(P_n^{\alpha,\beta}\right)''(x)-(n-1)\left(P_n^{\alpha,\beta}\right)'(x)\right]-\frac{P_n^{\alpha,\beta}(1)}{(\alpha+1)}\\&\left.\times\left[(2+\beta)\left(P_n^{\alpha,\beta}\right)''(x)+(x+1)\left(P_n^{\alpha,\beta}\right)'''(x)\right]\right\}.\end{aligned}\tag{69}$$

$$\begin{aligned}K_{n-1}^{\alpha,\beta(1,1)}(x,-1)&=\frac{1}{||P_{n-1}||^2\hat{\gamma}_n^{\alpha,\beta}}\left\{\left(P_n^{\alpha,\beta}\right)'(-1)\right.\\&\times\left[(x-1)\left(P_n^{\alpha,\beta}\right)''(x)-(n-1)\left(P_n^{\alpha,\beta}\right)'(x)\right]+\frac{P_n^{\alpha,\beta}(-1)}{(\beta+1)}\\&\times\left[(x-1)\left(P_n^{\alpha,\beta}\right)'''(x)-(2+\alpha)\left(P_n^{\alpha,\beta}\right)''(x)\right].\end{aligned}\tag{70}$$

As simple consequences of (63) and (67)-(69), we get

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,0)}(1,1) &= \frac{\left(P_n^{\alpha,\beta}(1)\right)^2 n(n+\beta)}{||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}(\alpha+1)}, \\ K_{n-1}^{\alpha,\beta(0,0)}(1,-1) &= -\frac{nP_n^{\alpha,\beta}(-1)P_n^{\alpha,\beta}(1)}{||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}}. \end{aligned} \tag{71}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(0,1)}(1,1) &= \frac{\left(P_n^{\alpha,\beta}\right)'(1)P_n^{\alpha,\beta}(1)(n+\beta)(n-1)}{||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}(\alpha+2)}, \\ K_{n-1}^{\alpha,\beta(0,1)}(1,-1) &= -\frac{\left(P_n^{\alpha,\beta}\right)'(-1)P_n^{\alpha,\beta}(1)(n-1)}{||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}}, \end{aligned} \tag{72}$$

$$\begin{aligned} K_{n-1}^{\alpha,\beta(1,1)}(1,1) &= \frac{P_n^{\alpha,\beta}(1)\left(P_n^{\alpha,\beta}\right)'(1)(n-1)(n+\beta)}{2||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}(\alpha+1)(\alpha+2)(\alpha+3)} \\ &\times \left[n(\alpha+2)(n+\alpha+\beta)-(\alpha+1)(\alpha+\beta+2)\right], \\ K_{n-1}^{\alpha,\beta(1,1)}(1,-1) &= -\frac{\left(P_n^{\alpha,\beta}\right)'(-1)P_n^{\alpha,\beta}(1)(n-1)}{2||P_{n-1}||^2 \hat{\gamma}_n^{\alpha,\beta}(\alpha+1)} \\ &\times \left[n(n+\alpha+\beta)-\alpha-\beta-2\right]. \end{aligned} \tag{73}$$

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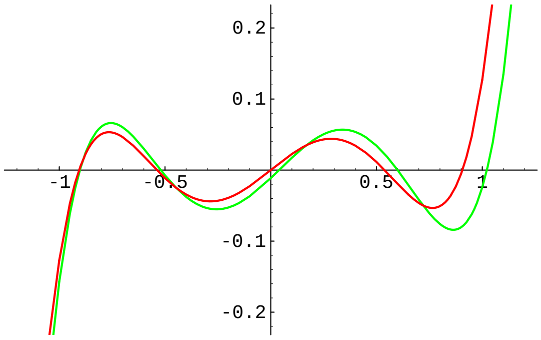
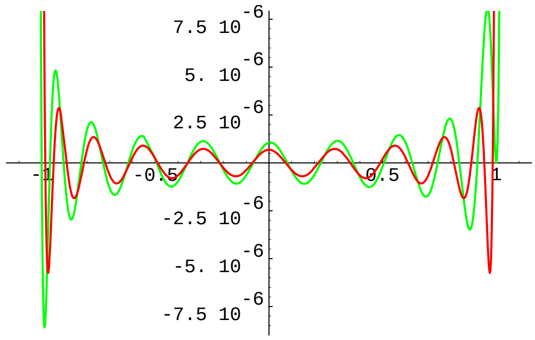
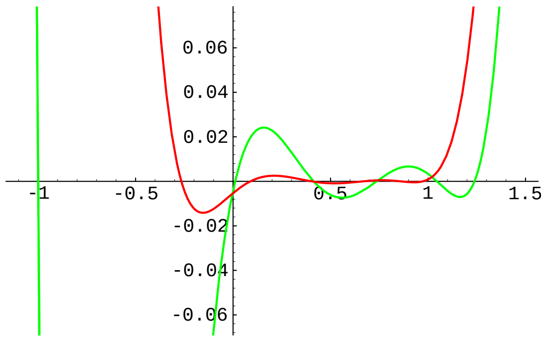



Comparison between classical (bold line) and generalized polynomials for $n = 5$, $A_1 = -0.5$, $B_1 = 0$, $A_2 = 0$, $B_2 = 0$, $\alpha = \beta = 0$, and $x \in [-1.2, 1.2]$.

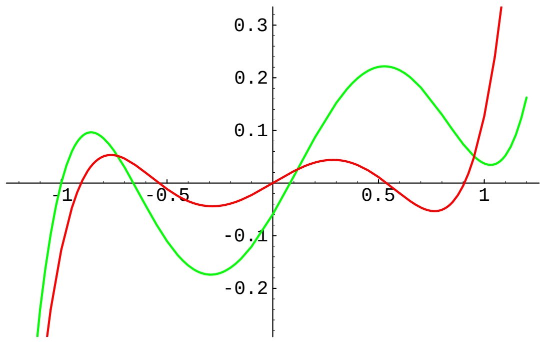



Comparison between classical (bold line) and generalized polynomials for $n = 20$, $A_1 = 0$, $B_1 = 0$, $A_2 = -4$, $B_2 = 2$, $\alpha = \beta = 1$, and $x \in [-1.1, 1.1]$.

*Figure 1.* Some numerical tests performed by using the symbolic computer algebra package *Mathematica* to show the number of zeros for both families of orthogonal polynomials. These graphics are in accordance with the previous results.




Comparison between classical (bold line) and generalized polynomials for $n = 6$, $A_1 = B_1 = 0$, $A_2 = 10$, $B_2 = -10^3$, $\alpha = 0$, $\beta = 10$, and $x \in [-1.1, 1.5]$.




Comparison between classical (bold line) and generalized polynomials for $n = 5$, $A_1 = B_1 = 0$, $A_2 = -0.5$, $B_2 = -10^6$, $\alpha = \beta = 0$, and $x \in [-1.2, 1.2]$.

*Figure 2.* Some numerical tests performed by using the symbolic computer algebra package *Mathematica* to show the number of zeros for both families of orthogonal polynomials. These graphics are in accordance with the previous results.