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# A characterization of the classical orthogonal discrete and $q$ -polynomials

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## Abstract

In this paper we present a new characterization for the classical discrete and  $q$ -classical (discrete) polynomials (in the Hahn's sense).

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## 0. Introduction

The classical orthogonal polynomials are very interesting mathematical objects that have attracted the attention not only of mathematicians since their appearance at the end of the XVIII century connected with some physical problems. They are used in several branches of mathematical and physical sciences and they have a lot of useful properties: they satisfy a three-term recurrence relation (TTRR), they are the solution of a second order linear differential (or difference) equation, their derivatives (or finite differences) also constitute an orthogonal family, their generating functions can be given explicitly, among others (for a recent review see e.g. [1]). Among such properties, a fundamental role is played by the so-called characterization theorems, i.e., such properties that completely define and characterize the classical polynomials. Obviously not every property characterize the classical polynomials and as an example we can use the TTRR. It is well-known that, under certain conditions—by the so-called Favard Theorem (for a review see [7])—, the TTRR characterizes the orthogonal polynomials (OP) but there exist families of OP that satisfy a TTRR but not a linear differential equation with polynomial coefficients, or a Rodrigues-type formula, etc. In this paper we will complete the works [3,10] proving a new characterization for the classical discrete [3,6] and the  $q$ -classical [4,10] polynomials. For the continuous case see [8,9].

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**1. Preliminaries**

Let  $\mathbb{P}$  be the linear space of polynomial functions in  $\mathbb{C}$  with complex coefficients and let  $\mathbb{P}^*$  be its algebraic dual space, i.e.,  $\mathbb{P}^*$  is the linear space of all linear functionals  $\mathbf{u} : \mathbb{P} \rightarrow \mathbb{C}$ . In general, we will represent the action of a functional over a polynomial by

$$\langle \mathbf{u}, \pi \rangle, \quad \mathbf{u} \in \mathbb{P}^*, \quad \pi \in \mathbb{P}.$$

Therefore a functional is completely determined by a sequence of complex numbers  $\langle \mathbf{u}, x^n \rangle = u_n, n \geq 0$ , the so-called moments of the functional.

**Definition 1.1.** Let  $(P_n)_{n \geq 0}$  be a basis sequence of  $\mathbb{P}$  such that  $\deg P_n = n$ . We say that  $(P_n)_{n \geq 0}$  is an orthogonal polynomial sequence (OPS) with respect to a functional  $\mathbf{u} \in \mathbb{P}^*$ , if there exist non-zero numbers  $k_n, n \geq 0$ , such that

$$\langle \mathbf{u}, P_m P_n \rangle = k_n \delta_{mn}, \quad n, m \geq 0,$$

holds, where  $\delta_{mn}$  is the Kronecker delta. If the leading coefficient of  $P_n$  is equal to 1 for all  $n$ , i.e.,  $P_n(x) = x^n + \dots$ , we say that the sequence  $(P_n)_{n \geq 0}$  is a monic orthogonal polynomial sequence (MOPS) and denote it by  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$ .

It is very well-known that such an OPS exists if and only if the linear functional  $\mathbf{u}$  is quasi-definite.

Next, we introduce the forward difference operator defined on  $\mathbb{P}$  by

$$\Delta : \mathbb{P} \mapsto \mathbb{P}, \quad \Delta y(x) = y(x + 1) - y(x).$$

For the  $\Delta$  operator we have the property

$$\Delta[\pi(x)\rho(x)] = \pi(x)\Delta\rho(x) + \rho(x + 1)\Delta\pi(x). \tag{1.1}$$

We will also use the Jackson  $q$ -derivative operator  $\mathcal{D}_q$  on  $\mathbb{P}$  defined by

$$\mathcal{D}_q : \mathbb{P} \mapsto \mathbb{P}, \quad \mathcal{D}_q \pi = \frac{\pi(qx) - \pi(x)}{(q - 1)x}, \quad |q| \neq 0, 1.$$

Note that in this case we have

$$\mathcal{D}_q(\pi(x)\rho(x)) = \rho(x)\mathcal{D}_q\pi(x) + \pi(qx)\mathcal{D}_q\rho(x) = \rho(qx)\mathcal{D}_q\pi(x) + \pi(x)\mathcal{D}_q\rho(x). \tag{1.2}$$

Throughout the paper let  $[n]_q, n \in \mathbb{N}$ , denote the basic  $q$ -number

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \dots + q^{n-1}, \quad n > 0, \quad [0]_q := 0.$$

All the above operators are linear and

$$\Delta x^n = nx^{n-1} + \dots, \quad \mathcal{D}_q x^n = [n]_q x^{n-1}, \quad n > 0, \quad \Delta 1 = \mathcal{D}_q 1 = 0.$$

**Definition 1.2.** Let  $\mathbf{u} \in \mathbb{P}^*$  and  $\pi \in \mathbb{P}$ . We define the action of the  $\Delta$ -difference operator  $\Delta$  on  $\mathbb{P}^*$  by  $\Delta : \mathbb{P}^* \rightarrow \mathbb{P}^*$ ,  $\langle \Delta \mathbf{u}, \pi \rangle = -\langle \mathbf{u}, \Delta \pi \rangle$ . We define the action of the  $q$ -derivative  $\mathcal{D}_q$  on  $\mathbb{P}^*$  by  $\mathcal{D}_q : \mathbb{P}^* \rightarrow \mathbb{P}^*$ ,  $\langle \mathcal{D}_q \mathbf{u}, \pi \rangle = -\langle \mathbf{u}, \mathcal{D}_q \pi \rangle$ .

**Definition 1.3.** Let  $\mathbf{u} \in \mathbb{P}^*$  and  $\pi \in \mathbb{P}$ . We define a polynomial modification of a functional  $\mathbf{u}$ , the functional  $\pi \mathbf{u}$ , by  $\langle \pi \mathbf{u}, \rho \rangle = \langle \mathbf{u}, \pi \rho \rangle, \forall \rho \in \mathbb{P}$ .

From the above definition and identities (1.1) and (1.2) it follows that

$$\Delta(\pi(x)\mathbf{u}) = \pi(x - 1)\Delta\mathbf{u} + \Delta\pi(x - 1)\mathbf{u}, \tag{1.3}$$

$$\mathcal{D}_q(\pi(x)\mathbf{u}) = \pi(x/q)\mathcal{D}_q\mathbf{u} + \mathcal{D}_q\pi(x/q)\mathbf{u}, \tag{1.4}$$

for the discrete and the  $q$ -case, respectively.

Given a basis sequence of polynomials  $(B_n)_{n \geq 0}$  we define the so-called dual basis of  $(B_n)_{n \geq 0}$  as a sequence of linear functionals  $(\mathbf{b}_n)_{n \geq 0}$  such that

$$\langle \mathbf{b}_n, B_m \rangle = \delta_{nm}, \quad n, m \geq 0.$$

Furthermore, if  $(P_n)_{n \geq 0}$  is a MOPS associated to the quasi-definite functional  $\mathbf{u} \in \mathbb{P}^*$ , then their corresponding dual basis  $(\mathbf{p}_n) \subset \mathbb{P}^*$  is given by

$$\mathbf{p}_n = k_n^{-1} P_n \mathbf{u}, \quad k_n = \langle \mathbf{u}, P_n^2 \rangle, \quad n \geq 0. \tag{1.5}$$

**Definition 1.4.** Let  $\mathbf{u} \in \mathbb{P}^*$  be a quasi-definite functional and  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$ . We say that  $\mathbf{u}$  (respectively  $(P_n)_{n \geq 0}$ ) is a  $\Delta$ -classical functional (respectively a  $\Delta$ -classical MOPS), if and only if the sequence  $(\Delta P_{n+1})_{n \geq 0}$  is also orthogonal. We say that  $\mathbf{u}$  (respectively  $(P_n)_{n \geq 0}$ ) is a  $q$ -classical functional (respectively a  $q$ -classical MOPS), if and only if the sequence  $(\mathcal{D}_q P_{n+1})_{n \geq 0}$  is also orthogonal.

In the following  $(Q_n)_{n \geq 0}$  will denote either the sequence of monic  $\Delta$ -differences or  $q$ -derivatives of  $(P_n)_{n \geq 0}$ , i.e., either  $Q_n = 1/(n+1)\Delta P_{n+1}$ , or  $Q_n = 1/([n+1]_q)\mathcal{D}_q P_{n+1}$ , respectively, for  $n \geq 0$ . We have the following proposition.

**Proposition 1.5** (Garcia et al. [3], Medem et al. [10]). Let  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$  and  $(Q_n)_{n \geq 0}$  be the sequence of monic  $\Delta$ -differences or  $q$ -derivatives. If  $(Q_n)_{n \geq 0} = \text{mops}(\mathbf{v})$ , then  $\mathbf{v} = \phi \mathbf{u}$  where  $\phi \in \mathbb{P}$ ,  $\deg \phi \leq 2$ .

**Theorem 1.6.** Let  $\mathbf{u} \in \mathbb{P}^*$  be a quasi-definite functional and  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$ . Then the following statements are equivalent [3]:

- (a)  $\mathbf{u}$  and  $(P_n)_{n \geq 0}$  are a  $\Delta$ -classical functional and a  $\Delta$ -classical MOPS, respectively.
- (b) There exist two polynomials  $\phi$  and  $\psi$ ,  $\deg \phi \leq 2$ ,  $\deg \psi = 1$ , such that

$$\Delta(\phi \mathbf{u}) = \psi \mathbf{u}. \tag{1.6}$$

- (c)  $(P_n)_{n \geq 0}$  satisfies the distributional Rodrigues formula, i.e., there exist a polynomial  $\phi \in \mathbb{P}$ ,  $\deg \pi \leq 2$ , and a sequence of complex numbers  $r_n$ ,  $r_n \neq 0$ ,  $n \geq 1$ , such that

$$P_n \mathbf{u} = r_n \Delta^n (\phi_{(n)} \mathbf{u}), \quad n \geq 1 \quad \text{where} \quad \phi_{(n)}(x) = \prod_{k=0}^{n-1} \phi(x+k). \tag{1.7}$$

Whereas for the  $q$ -classical polynomials the following statements are equivalent [10]:

- (i)  $\mathbf{u}$  and  $(P_n)_{n \geq 0}$  are a  $q$ -classical functional and a  $q$ -classical MOPS, respectively.
- (ii) There exist two polynomials  $\phi$  and  $\psi$ ,  $\deg \phi \leq 2$ ,  $\deg \psi = 1$ , such that

$$\mathcal{D}_q(\phi \mathbf{u}) = \psi \mathbf{u}. \tag{1.8}$$

- (iii)  $(P_n)_{n \geq 0}$  satisfies the distributional Rodrigues formula, i.e., there exist a polynomial  $\phi \in \mathbb{P}$ ,  $\deg \pi \leq 2$ , and a sequence of complex numbers  $r_n$ ,  $r_n \neq 0$ ,  $n \geq 1$ , such that

$$P_n \mathbf{u} = r_n \mathcal{D}_q^n (\phi_{(n)} \mathbf{u}), \quad n \geq 1 \quad \text{where} \quad \phi_{(n)}(x) = \prod_{i=0}^{n-1} \phi(q^i x). \tag{1.9}$$

Moreover, if

$$\phi(x) = Ax^2 + Bx + C, \quad \psi(x) = Mx + M_1, \quad M \neq 0, \tag{1.10}$$

then we have the regularity condition  $nA + M \neq 0$ ,  $n \geq 0$ , for the  $\Delta$ -case, and  $[n]_q A + M \neq 0$ ,  $n \geq 0$ , for the  $q$ -case, respectively.

Let us mention here that if (a), (b), and (c) (respectively, (i), (ii), and (iii)) of Theorem 1.6 hold, then  $\phi$  is the same as in Proposition 1.5.

Later, we will use the next technical result.

**Proposition 1.7.** *Let  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$  and let  $(Q_n)_{n \geq 0}$  be the sequence of their monic  $\Delta$ -differences or  $q$ -derivatives, respectively. If  $\mathbf{u}$  is either a  $\Delta$ -classical functional or a  $q$ -classical functional then,*

$$\Delta(Q_n \phi \mathbf{u}) = (M + nA)P_{n+1} \mathbf{u}, \quad n \geq 0, \tag{1.11}$$

$$\mathcal{D}_q(Q_n \phi \mathbf{u}) = q^{-n}(M + [n]_q A)P_{n+1} \mathbf{u}, \quad n \geq 0, \tag{1.12}$$

for the discrete and  $q$ -cases, respectively, where  $M, A$ , and  $\phi$ , are as in (1.10).

**Proof.** Using (1.5), Lemmas 1.7 and 1.8 in [3] for the discrete case, and Corollary 2.3 in [10] for the  $q$ -case, and Proposition 1.5 we find

$$\Delta(Q_n \phi \mathbf{u}) = -(n + 1) \frac{k'_n}{k_{n+1}} P_{n+1} \mathbf{u}, \quad \mathcal{D}_q(Q_n \phi \mathbf{u}) = -[n + 1]_q \frac{k'_n}{k_{n+1}} P_{n+1} \mathbf{u},$$

respectively, where  $k_{n+1} = \langle \mathbf{u}, P_{n+1}^2 \rangle$ ,  $k'_n = \langle \phi \mathbf{u}, Q_n^2 \rangle$ . Next we compute  $k'_n/k_{n+1}$ . For the discrete case we have

$$\begin{aligned} k'_n = \langle \phi \mathbf{u}, Q_n^2 \rangle &= \frac{1}{n + 1} \langle \phi \mathbf{u}, (x + 1)^n \Delta P_{n+1} \rangle \\ &= \frac{1}{n + 1} \langle \phi \mathbf{u}, \Delta(x^n P_{n+1}) - \Delta(x^n) P_{n+1} \rangle \\ &= -\frac{1}{n + 1} \{ \langle \Delta(\phi \mathbf{u}), x^n P_{n+1} \rangle + \langle \phi \mathbf{u}, \Delta(x^n) P_{n+1} \rangle \} \\ &= -\frac{1}{n + 1} \{ \langle \mathbf{u}, Mx^{n+1} P_{n+1} \rangle + \langle \mathbf{u}, nAx^{n+1} P_{n+1} \rangle \} \\ &= -\frac{M + nA}{n + 1} k_{n+1}, \end{aligned}$$

where we use Eq. (1.1) in the second equality, and (1.6), (1.10) in the fourth one. In the same way, but using (1.2) and (1.8), we find for the  $q$ -case

$$k'_n = \langle \phi \mathbf{u}, Q_n^2 \rangle = \frac{q^{-n}}{[n + 1]_q} \langle \phi \mathbf{u}, (qx)^n \mathcal{D}_q P_{n+1} \rangle = -q^{-n} \frac{M + [n]_q A}{[n + 1]_q} k_{n+1}. \quad \square$$

## 2. Main result

In this section, we prove the characterization theorem in both situations.

### 2.1. Classical discrete polynomials

**Theorem 2.1.** *Let  $\mathbf{u} \in \mathbb{P}^*$  be a quasi-definite functional and  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$ . Then  $(P_n)_{n \geq 0}$  is a  $\Delta$ -classical MOPS if and only if for every  $n \geq 1$ ,*

$$P_n \mathbf{u} = \Delta(\alpha_{n-1} \phi \mathbf{u}), \tag{2.1}$$

where  $\alpha_{n-1}$  is a polynomial of degree  $n - 1$  and  $\phi$  is a polynomial of degree less or equal to 2.

**Proof.** ( $\Leftarrow$ ) Taking  $n = 1$  we get

$$P_1 \mathbf{u} = \Delta(\alpha_0 \phi \mathbf{u}) = \alpha_0 \Delta(\phi \mathbf{u}),$$

thus by Theorem 1.6,  $\mathbf{u}$  is a  $\Delta$ -classical functional with  $\psi = P_1/\alpha_0$ .

( $\Rightarrow$ ) If  $\mathbf{u}$  is  $\Delta$ -classical then, by Theorem 1.6, Eq. (1.7),

$$P_n \mathbf{u} = r_n \Delta^n(\phi_{(n)} \mathbf{u}), \quad n \geq 1 \quad \text{where } \phi_{(n)}(x) = \prod_{k=0}^{n-1} \phi(x+k).$$

Therefore, it is enough to show that  $r_n \Delta^n(\phi_{(n)} \mathbf{u}) = \Delta(\alpha_{n-1} \phi \mathbf{u})$  where  $\alpha_{n-1}$  is a  $(n - 1)$ -degree polynomial. For doing that, let  $n$  be a fixed positive integer. We prove that

$$r_n \Delta^n(\phi_{(n)} \mathbf{u}) = \Delta^{n-k}(\alpha_k \phi_{(n-k)} \mathbf{u}), \quad k = 0, 1, \dots, n - 1, \tag{2.2}$$

where  $\alpha_k$  is a polynomial of degree  $k$ . Obviously the formula is correct for  $k = 0$  by taking  $\alpha_0(x) = r_n$ . Let suppose that it is true for  $k = 0, 1, 2, \dots, p, p < n - 1$ :

$$r_n \Delta^n(\phi_{(n)} \mathbf{u}) = \Delta^{n-p}(\alpha_p \phi_{(n-p)} \mathbf{u}) = \Delta^{n-(p+1)} \Delta(\alpha_p \phi_{(n-p)} \mathbf{u}).$$

Let us show that

$$\Delta(\alpha_p \phi_{(n-p)} \mathbf{u}) = \alpha_{p+1} \phi_{(n-p-1)} \mathbf{u}, \quad \text{deg } \alpha_{p+1} = p + 1.$$

For doing that, notice that  $\phi_{(n-p)}(x) = \phi(x)\phi_{(n-p-1)}(x + 1)$ , and therefore using (1.3)

$$\begin{aligned} \Delta(\alpha_p(x)\phi_{(n-p)}(x)\mathbf{u}) &= \Delta(\alpha_p(x)\phi_{(n-p-1)}(x+1)\phi(x)\mathbf{u}) \\ &= \alpha_p(x-1)\phi_{(n-p-1)}(x)\Delta(\phi(x)\mathbf{u}) + \Delta(\alpha_p(x-1)\phi_{(n-p-1)}(x))\phi(x)\mathbf{u} \\ &= \{\alpha_p(x-1)\phi_{(n-p-1)}(x)\psi(x) + \Delta(\alpha_p(x-1)\phi_{(n-p-1)}(x))\phi(x)\}\mathbf{u}. \end{aligned}$$

To complete the proof it suffices to show that

$$\begin{aligned} \Delta(x) &:= \alpha_p(x-1)\phi_{(n-p-1)}(x)\psi(x) + \Delta(\alpha_p(x-1)\phi_{(n-p-1)}(x))\phi(x) \\ &= \{\alpha_p(x-1)\psi(x) + \phi(x)\Delta\alpha_p(x-1)\}\phi_{(n-p-1)}(x) \\ &\quad + \alpha_p(x)\phi(x)\Delta\phi_{(n-p-1)}(x) \\ &= \alpha_{p+1}(x)\phi_{(n-p-1)}(x), \end{aligned}$$

being  $\alpha_{p+1}$  a polynomial of degree  $p + 1$ . Using that

$$\Delta\phi_{(n-p-1)}(x) = \frac{\phi(x+n-p-1) - \phi(x)}{\phi(x)}\phi_{(n-p-1)}(x),$$

we finally obtain

$$\Delta(x) = \{\alpha_p(x-1)\psi(x) + \alpha_p(x)\phi(x+n-p-1) - \alpha_p(x-1)\phi(x)\}\phi_{(n-p-1)}(x).$$

To prove that the above expression is a polynomial of degree  $p + 1$  we substitute (1.10),  $\alpha_p(x) = a_p x^p + \dots$ , and equate the coefficients of  $x^{p+1}$ . This gives  $a_p\{M + A(2n - p - 2)\} \neq 0$ , due to the regularity condition (see Theorem 1.6) and the fact that  $a_p \neq 0$  since the polynomial  $\alpha_p$  has degree equal to  $p$ . This prove (2.2). Now putting  $k = n - 1$ , the result follows.  $\square$

To conclude this section notice that comparing (2.1) and (1.11) we obtain that

$$\alpha_{n-1}(x) = \frac{Q_{n-1}(x)}{M + A(n - 1)} = \frac{\Delta P_n(x)}{n(M + A(n - 1))}.$$

Table 1  
The  $\Delta$ -classical polynomials

	Hahn	Meixner	Kravchuk	Charlier
$\phi$	$(x + \beta + 1)(N - x - 1)$	$\mu x + \mu v$	$-\frac{p}{1-p}(x - N)$	$a$
$\psi$	$-(\alpha + \beta + 2)x - (\beta + 1)(N - 1)$	$-(1 - \mu)x + \mu v$	$\frac{-x + Np}{1-p}$	$-x + a$
$\alpha_{n-1}$	$\frac{h_{n-1}^{(\alpha+1, \beta+1)}(x, N-1)}{n + \alpha + \beta + 1}$	$\frac{m_{n-1}^{(v+1, \mu)}(x)}{\mu - 1}$	$(p - 1)k_{n-1}^{(p)}(x, N - 1)$	$c_{n-1}^{(a)}(x)$

2.1.1. Examples

As examples we will take the monic polynomials of Hahn  $h_n^{(\alpha, \beta)}(x, N)$ , Meixner  $m_n^{(v, \mu)}(x)$ , Kravchuk  $k_n^{(p)}(x, N)$ , and Charlier  $c_n^{(a)}(x)$ , (for the main data see [3,5]). The results are given in Table 1.

2.2.  $q$ -Classical polynomials

**Theorem 2.2.** Let  $\mathbf{u} \in \mathbb{P}^*$  be a quasi-definite functional and  $(P_n)_{n \geq 0} = \text{mops}(\mathbf{u})$ . Then  $(P_n)_{n \geq 0}$  is a  $q$ -classical if and only if for every  $n \geq 1$ ,

$$P_n \mathbf{u} = \mathcal{D}_q(\alpha_{n-1} \phi \mathbf{u}), \tag{2.3}$$

where  $\alpha_{n-1}$  is a polynomial of degree  $n - 1$  and  $\phi$  is a polynomial of degree less or equal to 2.

**Proof.** ( $\Leftarrow$ ) As in Theorem 2.1, it is enough to take  $n = 1$ .

( $\Rightarrow$ ) Starting from Theorem 1.6, Eq. (1.9), it suffices to show that  $r_n \mathcal{D}_q^n(\phi_{(n)} \mathbf{u}) = \mathcal{D}_q(\alpha_{n-1} \phi \mathbf{u})$  holds, where  $\alpha_{n-1}$  is a  $(n - 1)$ -degree polynomial. This can be done following the same steps as in Theorem 2.1 taking into account that the analog of the formula 2.2 is

$$r_n \mathcal{D}_q^n(\phi_{(n)} \mathbf{u}) = \mathcal{D}_q^{n-k}(\alpha_k \phi_{(n-k)} \mathbf{u}), \quad k = 0, 1, \dots, n - 1.$$

Also, we use (1.4) instead of (1.3) and  $A(x)$  is now defined by

$$A(x) := \alpha_p(x/q) \phi_{(n-p-1)}(x) \psi(x) + \mathcal{D}(\alpha_p(x/q) \phi_{(n-p-1)}(x)) \phi(x).$$

Since

$$\mathcal{D}_q \phi_{(n-p-1)}(x) = \frac{\phi(q^{n-p-1}x) - \phi(x)}{(q-1)x} \frac{\phi_{(n-p-1)}(x)}{\phi(x)}$$

we can derive that

$$A(x) = \left\{ \alpha_p(x/q) \psi(x) + \frac{\alpha_p(x) \phi(q^{n-p-1}x) - \alpha_p(x/q) \phi(x)}{(q-1)x} \right\} \phi_{(n-p-1)}(x),$$

holds. The expression in the brackets is a polynomial of degree  $p + 1$  with the leading coefficient  $\alpha_p q^{-p} \{M + A[2n - p - 2]_q\}$ , which is different from zero because of the regularity condition (see Theorem 1.6) and the fact that the polynomial  $\alpha_p$  has degree  $p$ . So, the result follows.  $\square$

Observe that comparing (2.3) and (1.12) we obtain that

$$\alpha_{n-1}(x) = \frac{q^{n-1} Q_{n-1}(x)}{M + A[n-1]_q} = \frac{q^{n-1} \mathcal{D}_q P_n(x)}{[n]_q (M + A[n-1]_q)}.$$

Table 2  
The  $q$ -classical polynomials

	Big $q$ -Jacobi	Little $q$ -Jacobi
$\phi$	$aq(x-1)(bx-c)$	$ax(bqx-1)$
$\psi$	$\frac{1-abq^2}{(1-q)q}x + \frac{a(bq-1)+c(aq-1)}{1-q}$	$\frac{1}{(1-q)q} \{(1-abq^2)x + aq - 1\}$
$\alpha_{n-1}$	$\frac{(1-q)q}{1-abq^{n+1}} P_{n-1}(qx, qa, qb, qc; q)$	$\frac{(1-q)q^n}{1-abq^{n+1}} p_{n-1}(x, qa, qb q)$

### 2.2.1. Examples

In this case we have 12 families of classical  $q$ -polynomials, (see [2,10]). We will take two representatives examples corresponding to the big  $q$ -Jacobi polynomials  $P_n(x, a, b, c; q)$  and the little  $q$ -Jacobi polynomials  $p_n(x; a, b|q)$ . The main data of such polynomials can be found in [2,5]. The results are given in Table 2. The other ten cases can be obtained in an analogous way or by taking appropriate limits (see e.g. [2,5]).

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