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Dedicated to the memory of our teacher and friend
Arnold F. Nikiforov (18.11.1930 – 27.12.2005).

DUAL PROPERTIES OF ORTHOGONAL POLYNOMIALS OF DISCRETE VARIABLES ASSOCIATED WITH THE QUANTUM ALGEBRA $U_q(su(2))$

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Abstract

We show that for every set of discrete polynomials $y_n(x(s))$ on the lattice $x(s)$, defined on a finite interval (a, b), it is possible to construct two sets of dual polynomials $z_k(\xi(t))$ of degrees $k = s - a$ and $k = b - s - 1$. Here we do this for the classical and alternative Hahn and Racah polynomials as well as for their q-analogs. Also we establish the connection between classical and alternative families. This allows us to obtain new expressions for the Clerbsch–Gordan and Racah coefficients of the quantum algebra $U_q(su(2))$ in terms of various Hahn and Racah q-polynomials.

Keywords: discrete orthogonal polynomials, Hahn and Racah polynomials, $U_q(su(2))$ quantum algebra, q-Clebsch–Gordan and q-Racah coefficients.

1. Introduction

The symmetries of quantum states play an important role in explaining the degeneracy of energy levels [1]. The connection of the energy levels of the hydrogen atom with the irreducible representation of $O(4, 2)$ conformal symmetry was found in [2]. The q-deformed Heisenberg–Weyl symmetry was used to introduce the notion of quantum q-oscillators [3,4]. The physical meaning of q-deformations was clarified in [5] where it was shown that classical q-oscillators and their quantum partners are standard nonlinear oscillators vibrating with a frequency depending on the amplitude. Thus, the symmetry groups and the q-deformed symmetry groups are important ingredients in the description of states in quantum optics and quantum mechanics. A general consideration of constructing the irreducible representations of Lie groups and their connection with the formalism of classical mechanics was presented in [6] in the context of symmetry applications in quantum mechanics and quantum optics.

The matrix elements of the operator of irreducible representations of the Lie group $SU(2)$ and q-deformed $SU_q(2)$-quantum group are expressed in terms of some special functions. The special functions

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are related to Clebsch–Gordan coefficients and their generalizations. The aim of this work is to study some polynomials which naturally appear in the physical problems associated with the q-symmetries.

In the present paper, we study further the properties of polynomial solutions $y_n(x(s))$ of the second-order linear difference equation (SODE) of hypergeometric-type on the lattice $x(s)$ [7,8]

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla y(s)}{\nabla x(s)}+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0,$$

where $\Delta f(s)=f(s+1)-f(s-1)$ and $\nabla f(s)=f(s)-f(s-1)$, or written in equivalent form

$$A(s)y(s+1)+B(s)y(s)+C(s)y(s-1)+\lambda_n y(s)=0,\qquad B(s)=-A(s)-C(s)\,, \tag{1}$$

where

$$\phi(s)=\sigma(s)+\tau(s)\Delta x(s-1/2)$$

and

$$A(s)=\frac{\phi(s)}{\Delta x(s)\Delta x(s-1/2)},\qquad C(s)=\frac{\sigma(s)}{\nabla x(s)\Delta x(s-1/2)}. \tag{2}$$

The polynomial solutions of the above difference equations are called hypergeometric polynomials on the nonuniform lattice $x(s)$. They can be expressed using the Rodrigues-type formula [8]

$$P_n(x(s))_q=\frac{B_n}{\rho(s)}\nabla^{(n)}\rho_n(s),\qquad \nabla^{(n)}:=\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}, \tag{3}$$

where ρ is the solution of the Pearson-type equation $\Delta\sigma(s)\rho(s)=\tau(s)\rho(s)\Delta x(s-1/2)$,

$$\rho_n(s)=\rho(s+n)\prod_{k=1}^{n}\sigma(s+k),\qquad x_k(s)=x(s+k/2),$$

and B_n is a normalizing factor.

It is well known [7] that for any family $y_n(x(s))$ of orthogonal polynomials of discrete variable $x(s)$ on a finite interval $a\le s\le b-1$, there is a corresponding dual family $z_k(\xi(t))$ defined on a new discrete variable $\xi(t)$, $a'\le t\le b'-1$. In fact, if the boundary conditions

$$x^k(s-1/2)\sigma(s)\rho(s)\Big|_{s=a,b}=0$$

hold, then the polynomials $P_n(s)$ satisfy the orthogonality relation (for more details, see [7,8])

$$\sum_{s=a}^{b-1}y_n(x(s))y_m(x(s))\rho(s)\Delta(x(s-1/2))=\delta_{nm}d_n^2. \tag{4}$$

There is also another orthogonality relation that can be written as follows:

$$\sum_{n=0}^{b-a-1}y_n(x(s))y_n(x(s'))d_n^2=\delta_{ss'}\frac{1}{\rho(x(s))}\,. \tag{5}$$

This relation can be understood as a dual orthogonality relation (see [8], pp. 38, 39), i.e., the relation for the dual set $(z_k(\xi(t_n)))_k$

$$z_k(\xi(t_n)) \sim y_n(x(s)), \qquad k = 0, 1, \ldots, b - a - 1\,, \tag{6}$$

defined in some lattice $\xi(t_n)$, and orthogonal with respect to a weight function

$$\rho'(\xi(t_n)) = \frac{d_n^2}{\Delta x(t_n - 1/2)}$$

defined on the interval (a', b') with the norm

$$d_k'^2 = \frac{1}{\rho(x(s))\Delta(x(s - 1/2))}\,.$$

We will write (5) in the following way:

$$\sum_{t_n=a'}^{b'-1} z_k(t_n) z_\ell(t_n) \rho'(t_n) \Delta\xi(t_n - 1/2) = \delta_{k\ell}(d_k')^2, \tag{7}$$

where $\rho'(t)$ and $d_k'^2$ are the corresponding weight function and norm, respectively.

Notice that the dual polynomials are the solution of a second-order difference equation that corresponds to the three-term recurrence relation (TTRR) of the family $y_n(x(s))$

$$x(s)y_n(x(s)) = \alpha_n y_{n+1}(x(s)) + \beta_n y_n(x(s)) + \gamma_n y_{n-1}(x(s))\,. \tag{8}$$

Let us point out here that in [7,8] only one kind of dual polynomials was considered, indeed the ones with degree $k = s - a$ for which the following connection formula:

$$z_k(\xi(t)) = D_{kn} y_n(x(s)) \tag{9}$$

was established. Nevertheless, for a finite interval (a, b), there exists another possibility corresponding to the dual polynomials associated with the family $y_n(x(s))$ but with degree $k = b - s - 1$, i.e.,

$$z_k'(\xi(t_n)) = D_{kn}' y_n(x(s))\,. \tag{10}$$

In the following, for the sake of simplicity, we will write t instead of t_n.

This second kind of dual polynomials has not been considered (as far as we know), so the aim of the present work is to complete this point. Some results concerning the dual family of polynomials can be found in [9–11] for a finite support, and in [12] (and references therein) for an infinite (but countable) support.

We will focus our attention in the study of the dual sets to the Hahn and Racah polynomials and their q-analogs. For each family, we will construct two dual sets of polynomials.

We start rewriting (1) in the form

$$y_n(x(s)) = \left(-\frac{\lambda_n}{A(s-1)} - \frac{B(s-1)}{A(s-1)}\right) y_n(x(s-1)) - \frac{C(s-1)}{A(s-1)} y_n(x(s-2))\,. \tag{11}$$

Hence, by induction it is easy to see that the polynomial $y_n(x(s))$, which is a polynomial of degree n in $x(s)$, is also a polynomial of degree $k = s - a$ in λ_n. Rewriting (1) in the following way:

$$y_n(x(s)) = \left(-\frac{\lambda_n}{C(s+1)} - \frac{B(s+1)}{A(s+1)}\right) y_n(x(s+1)) - \frac{A(s+1)}{C(s+1)} y_n(x(s+2))\,, \tag{12}$$

we conclude that $y_n(x(s))$ is also a polynomial of λ_n but with degree $k = b - s - 1$.

Example: To illustrate this, let us consider the classical Hahn $h_n^{\alpha\beta}(s,a,b)$ and Racah $u_n^{\alpha\beta}(s,a,b)$ polynomials. For these families, we have [7,8]

$$\lambda_n = n(\alpha+\beta+n+1) = t(t+1) - \frac{1}{4}(\alpha+\beta)(\alpha+\beta+2)\,, \tag{13}$$

where

$$t = \frac{1}{2}(\alpha+\beta) + n.$$

For the corresponding alternative Hahn and Racah polynomials $\widetilde{h}_n^{\alpha\beta}(s,a,b)$ and $\widetilde{u}_n^{\alpha\beta}(s,a,b)$, we have

$$\lambda_n = n(2b-2a+\alpha+\beta-n-1) = -\widetilde{t}(\widetilde{t}+1) + \left(b-a+\frac{1}{2}(\alpha+\beta)-1\right)\left(b-a+\frac{1}{2}(\alpha+\beta)\right),$$

where

$$\widetilde{t} = b-a+\frac{1}{2}(\alpha+\beta)-n-1.$$

Therefore Eq. (9) leads to a polynomial of degree $k = s - a$ in the variables t or $\widetilde{t}$, whereas (10) leads to a polynomial of degree $k = b - s - 1$ in the same variables. Notice that $\widetilde{t}$ transforms into t if we replace n by $b - a - n - 1$, i.e., $\widetilde{t} \to t, \quad \text{if} \quad n \to b-a-n-1\,.$

Before continuing our analysis, let us mention that for a finite interval (a, b) the two different families of polynomials can always be constructed – the standard one and an alternative one. This is shown for the Hahn case in [8] (see Sec. 2.4.2.1, p. 32). We will consider here both cases in detail. □

Let continue with our analysis.

By iterating formulas (11) and (12), we obtain that the coefficient of the power $[t(t+1)]^k$ of the polynomial $y_n(x(s))$ is

$$\begin{aligned} A_{kn} &= (-1)^{s-a} y_n(x(a)) \prod_{l=1}^{s-a} \frac{1}{A(s-l)}\,, \qquad k = s-a,\\ A'_{kn} &= (-1)^{b-s-1} y_n(x(b-1)) \prod_{l=1}^{b-s-1} \frac{1}{C(s+l)}\,, \quad k = b-s-1. \end{aligned} \tag{14}$$

Analogously, iterating (11) and (12) for the alternative polynomials $\widetilde{y}_n(x(s))$ we find

$$\begin{aligned} \widetilde{A}_{kn} &= \widetilde{y}_n(x(a)) \prod_{l=1}^{s-a} \frac{1}{\widetilde{A}(s-l)}\,, \qquad k = s-a,\\ \widetilde{A}'_{kn} &= \widetilde{y}_n(x(b-1)) \prod_{l=1}^{b-s-1} \frac{1}{\widetilde{C}(s+l)}\,, \quad k = b-s-1. \end{aligned} \tag{15}$$

Therefore, the proportionality factors between the families y_n, $\widetilde{y}_n$ and their dual ones are

$$D_{kn}=\frac{a_k}{A_{kn}},\qquad D'_{kn}=\frac{a_k}{A'_{kn}},\qquad \widetilde{D}_{kn}=\frac{\widetilde{a}_k}{\widetilde{A}_{kn}},\qquad \text{and}\qquad \widetilde{D}'_{kn}=\frac{\widetilde{a}_k}{\widetilde{A}'_{kn}},\tag{16}$$

respectively. Here a_k and $\widetilde{a}_k$ are the leading coefficients of the corresponding dual polynomials $z_k(\xi(t))$ and $\widetilde{z}_k(\xi(t))$, i.e., the coefficient of $[t(t+1)]^k$ and $[\widetilde{t}(\widetilde{t}+1)]^k$, respectively. These coefficients and other needed characteristics of the classical polynomials can be found in [7,8].

Comparing the orthogonality relations for the starting polynomials $y_n(x(s))$ and their dual ones, we conclude that the coefficient D_{kn} can be obtained by the formula

$$D_{kn}^2=\frac{\rho(s)d_k^2}{\rho(t)d_n^2}\,,\tag{17}$$

where $\rho(s)$ and $\rho'(t)$ are the weight functions for the polynomials $y_n(x(s))$ and $z_k(\xi(t))$, respectively, and d_n^2 and $(d'_k)^2$, the corresponding norms.

The last formula can be used for computing the corresponding characteristics of the dual family especially if we combine it with formulas (9) and (16). Notice that for using formula (14) we need the values of the polynomial $y_n(x(s))$ at the end of the interval of orthogonality (a,b). These values can be obtained in a straightforward way by using the Rodrigues formula for the polynomials $y_n(x(s))$ [7,8].

The structure of the paper is as follows.

In Secs. 2 and 3 we discuss the dual properties of Hahn and Racah polynomials. In Secs. 4 and 5 similar problems are considered for the Hahn and Racah q-polynomials. Finally, Sec. 6 is devoted to the Clebsch–Gordan and Racah coefficients for the algebras $su(2)$ and $U_q(su(2))$ for which various expressions in terms of Hahn and Racah polynomials and q-polynomials are obtained. In Sec. 7 final remarks are presented.

2. Dual Properties of the Hahn Polynomials

The dual Hahn polynomials were studied in several papers (see [18] and references therein) and, in particular, in [8]. Here we will complete this study applying the results discussed above.

The Hahn polynomials are defined by the expression (see [8], [Eq. (2.7.19), p. 52])

$$h_n^{\alpha,\beta}(x,N)=\frac{(1-N)_n(\beta+1)_n}{n!}\,{}_3\mathrm{F}_2\left(\begin{array}{c}-x,\alpha+\beta+n+1,-n\\ 1-N,\beta+1\end{array}\Bigg|\,1\right),\tag{18}$$

and the alternative Hahn polynomials are (see [8], p. 53)

$$\widetilde{h}_n^{\alpha,\beta}(x,N)=\frac{(1-N)_n(1-\beta-N)_n}{n!}\,{}_3\mathrm{F}_2\left(\begin{array}{c}-x,-2N-\alpha-\beta+n+1,-n\\ 1-N,1-\beta-N\end{array}\Bigg|\,1\right).\tag{19}$$

Their main characteristics can be found in [8] (Tables 2.1 and 2.2, pp. 42–43).

The dual Hanh polynomials are defined by

$$w_n^c(x(s),a,b)=\frac{(a-b+1)_n(a+c+1)_n}{n!}\,{}_3\mathrm{F}_2\left(\begin{array}{c}a-s,s+a+1,-n\\ a-b+1,a+c+1\end{array}\Bigg|\,1\right).\tag{20}$$

Notice that they are polynomials of degree n on the lattice $x(s)=s(s+1)$ (i.e., a quadratic lattice) with the leading coefficient $a_n=(n!)^{-1}$. Their main characteristics are given in [8] (Table 3.7, p. 109).

2.1. Polynomials of Degree $k = s$ Dual to the Classical Hahn Polynomials

The Hahn polynomials $h_n^{\alpha\beta}(s,N)$ are orthogonal in the finite interval $(a,b) = (0,N)$. They satisfy the difference equation (1), where

$$A(s) = \sigma(s) + \tau(s) = (N-s-1)(s+\beta+1)\,, \quad C(s) = \sigma(s) = s(N+\alpha-s)\,,$$

$$\lambda_n = n(\alpha+\beta+n+1) = t(t+1) - (\alpha+\beta)(\alpha+\beta+2)/4 \quad \text{with} \quad t = [(\alpha+\beta)/2] + n.$$

Since for the Hahn polynomials we have

$$h_n^{\alpha\beta}(0,N) = (-1)^n \frac{\Gamma(N)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(N-n)\Gamma(\beta+1)}$$

and

$$\prod_{i=1}^{s} \frac{1}{(N-s-i)(s+\beta+i)} = \frac{\Gamma(N-a)\Gamma(s+\beta+1)}{\Gamma(N-s)\Gamma(\beta+1)}\,,$$

so

$$A_k = (-1)^{k+n} \frac{\Gamma(N-k)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(N-n)\Gamma(k+\beta+1)}\,.$$

In this case, the dual to $h_n^{\alpha\beta}(s,N)$ polynomial of degree $k = s$ is the classical dual Hahn polynomial defined by (20)

$$w_k^c(t) := w_k^c(t,a,b), \quad \text{where} \quad a = \frac{\alpha+\beta}{2}\,, \quad b = N + \frac{\alpha+\beta}{2}\,, \quad c = \frac{\beta-\alpha}{2}\,. \tag{21}$$

The leading coefficient of the polynomials $w_k^c(t)$ (i.e., the coefficient of the power $[t(t+1)]^k$) is equal to $(k!)^{-1}$. In this case, the proportionality coefficient between the Hahn polynomials $h_n^{\alpha\beta}(s,N)$ and the dual ones $w_k^c(t)$ is

$$D_{kn} = \frac{a_k}{A_{kn}} = (-1)^{k+n} \frac{\Gamma(n+1)\Gamma(N-n)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(N-k)\Gamma(n+\beta+1)}\,.$$

Then, we have the following connection formula:

$$w_k^c(t,a,b) = (-1)^{n+k} \frac{\Gamma(n+1)\Gamma(N-n)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(N-k)\Gamma(n+\beta+1)} h_n^{\alpha\beta}(s,N)\,, \tag{22}$$

where $k = s$, $t = [(\alpha+\beta)/2] + n$, and a, b, and c are given by (21).

2.2. Polynomials of Degree $k = N - s - 1$ Dual to the Classical Hahn Polynomials

Next, taking into account that

$$h_n^{\alpha\beta}(N-1,N) = \frac{\Gamma(N)\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(N-n)\Gamma(\alpha+1)}$$

and

$$\prod_{i=1}^{N-s-1} \frac{1}{(s+i)\Gamma(N+\alpha-s-i)} = \frac{\Gamma(s+1)\Gamma(\alpha+1)}{\Gamma(N+\alpha-s)\Gamma(N)}\,,$$

we obtain

$$A'_{kn} = (-1)^k \frac{\Gamma(N-k)\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(N-n)\Gamma(k+\alpha+1)}\,.$$

In this case, the polynomials of degree $k = N-s-1$, dual to $h_n^{\alpha\beta}(s,N)$, are given by the formula

$$w_k^{c'}(t) := w_k^{c'}(t,a,b), \quad \text{where} \quad a = \frac{\alpha+\beta}{2}\,, \quad b = N + \frac{\alpha+\beta}{2}\,, \quad c' = \frac{\alpha-\beta}{2}\,. \tag{23}$$

Furthermore, the coefficient a_k of $[t(t+1)]^k$ is equal to $(k!)^{-1}$ and the proportionality coefficient is given by

$$D'_{kn} = \frac{a_k}{A'_k} = (-1)^k \frac{\Gamma(n+1)\Gamma(N-n)\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(N-k)\Gamma(n+\alpha+1)}\,.$$

Therefore, the following connection formula holds:

$$w_k^{c'}(t,a,b) = (-1)^{N-s-1} \frac{\Gamma(n+1)\Gamma(N-n)\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(N-k)\Gamma(n+\alpha+1)} h_n^{\alpha\beta}(s,N)\,, \tag{24}$$

where $k = N-s-1$, $t = [(\alpha+\beta)/2] + n$, and a, b, and c are given by (23).

2.3. Polynomials of Degree $k = s$ Dual to the Alternative Hahn Polynomials

The alternative Hahn polynomials $\widetilde{h}_n^{\alpha\beta}(s,N)$ are orthogonal in the finite interval $(a,b) = (0,N)$. They satisfy the difference equation (1) with

$$A(s) = \widetilde{\sigma}(s) + \widetilde{\tau}(s) = (N-s-1)(N+\beta-s-1)\,, \qquad C(s) = \widetilde{\sigma}(s) = s(s+\alpha)\,,$$

$$\lambda_n = n(2N+\alpha+\beta-n-1) = -\widetilde{t}(\widetilde{t}+1) + \left(N + \tfrac{\alpha+\beta}{2} - 1\right)(N+\alpha+\beta),$$

where $\widetilde{t} = N + [(\alpha+\beta)/2] - n - 1$. Since

$$\widetilde{h}_n^{\alpha\beta}(0,N) = \frac{\Gamma(N)\Gamma(N+\beta)}{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\beta-n)}$$

and

$$\prod_{i=1}^{s} \frac{1}{(s+i)(s+\alpha+i)} = \frac{\Gamma(N)\Gamma(N+\beta)}{\Gamma(s+1)\Gamma(s+\alpha+1)}\,,$$

we find

$$\widetilde{A}_{kn} = \frac{\Gamma(N-s)\Gamma(N+\beta-s)}{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\beta-n)}\,.$$

The corresponding dual polynomials to $\widetilde{h}_n^{\alpha\beta}(s,N)$ of degree $k = s$ are then given by

$$w_k^{c'}(\widetilde{t}) := w_k^{(\alpha,\beta)}(\widetilde{t},a,b), \quad \text{with} \quad a = \frac{\alpha+\beta}{2}\,, \quad b = N + \frac{\alpha+\beta}{2}\,, \quad c' = \frac{\alpha-\beta}{2}\,, \tag{25}$$

where $w_k^{c'}(\widetilde{t})$ denotes the same family as before but with different parameters and variable. Thus the leading coefficient is $a_k = (k!)^{-1}$. In this case, the proportionality coefficient reads

$$\widetilde{D}_{kn} = \frac{a_k}{\widetilde{A}_{kn}} = \frac{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\beta-n)}{\Gamma(s+1)\Gamma(N-s)\Gamma(N+\beta-s)}\,.$$

Therefore, the connection between the two families is as follows:

$$w_k^{c'}(\widetilde{t},a,b)=\frac{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\beta-n)}{\Gamma(s+1)\Gamma(N-s)\Gamma(N+\beta-s)}\widetilde{h}_n^{\alpha\beta}(s,N)\,, \tag{26}$$

where $k=s$, $\widetilde{t}=N+[(\alpha+\beta)/2]-n-1$, and a, b, and c' are given by (25).

2.4. Polynomials of Degree $k=N-s-1$ Dual to the Alternative Hahn Polynomials

In this case, using

$$\widetilde{h}_n^{\alpha\beta}(N-1,N)=(-1)^n\frac{\Gamma(N)\Gamma(N+\alpha)}{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\alpha-n)}$$

and

$$\prod_{i=1}^{N-s-1}\frac{1}{(s+i)(s+\alpha+i)}=\frac{\Gamma(s+1)\Gamma(s+\alpha+1)}{\Gamma(N+\alpha)\Gamma(N)}\,,$$

we obtain

$$\widetilde{A}'_{kn}=(-1)^n\frac{\Gamma(s+1)\Gamma(s+\alpha+1)}{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\alpha-n)}\,.$$

Therefore, the polynomials of degree $k=N-s-1$, dual to $\widetilde{h}_n^{\alpha\beta}(s,N)$, are the dual Hahn polynomials $w_k^c(\widetilde{t})$

$$w_k^c(\widetilde{t}):=w_k^{(\alpha,\beta)}(\widetilde{t},a,b)\quad\text{where}\quad a=\frac{\alpha+\beta}{2}\,,\quad b=N+\frac{\alpha+\beta}{2}\,,\quad c=\frac{\alpha-\beta}{2}\,, \tag{27}$$

with leading coefficient $\widetilde{a}'_k=(k!)^{-1}=[(N-s-1)!]^{-1}$. The proportionality coefficient reads

$$\widetilde{D}'_{kn}=\frac{\widetilde{a}'_k}{\widetilde{A}'_k}=(-1)^n\frac{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\alpha-n)}{\Gamma(s+1)\Gamma(N-s)\Gamma(s+\alpha+1)}\,.$$

Therefore, they are related to the alternative Hahn polynomials by the formula

$$w_k^c(\widetilde{t},a,b)=(-1)^n\frac{\Gamma(n+1)\Gamma(N-n)\Gamma(N+\alpha-n)}{\Gamma(s+1)\Gamma(N-s)\Gamma(s+\alpha+1)}\widetilde{h}_n^{\alpha\beta}(s,N)\,, \tag{28}$$

where $k=s$, $\widetilde{t}=N+[(\alpha+\beta)/2]-n-1$, and a, b, and c are given by (27).

2.5. Connection between the Two Hahn Families $h_n^{\alpha\beta}(s,N)$ and $\widetilde{h}_n^{\alpha\beta}(s,N)$

Comparing (22) and (26) we see that the left-hand side of (26) transforms into the left-hand side of (22), if one changes $n\to N-n-1$ (i.e., $\widetilde{t}\to t$) and $c'\to c$ (i.e., $\alpha\leftrightarrow\beta$). Then we obtain

$$h_n^{\alpha\beta}(s,N)=(-1)^{n+s}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(s+\beta+1)\Gamma(N+\alpha-s)}\widetilde{h}_{N-n-1}^{\beta\alpha}(s,N)\,. \tag{29}$$

The same formula is obtained if we use Eqs. (24) and (28).

Finally, comparing formulas (22) and (24), we obtain

$$w^{c'}_{N-s-1}(t,a,b) = (-1)^{N+n-1}\frac{\Gamma(N+\alpha-s)\Gamma(n+\beta+1)}{\Gamma(s+\beta+1)\Gamma(n+\alpha+1)} w^{c}_{s}(t,a,b)\,,$$

and taking into account the values for the parameters [see (21) and (23)]

$$\alpha = a-c, \qquad \beta = a+c, \qquad n = t-a, \qquad N = b-a,$$

we arrive at the identity

$$w^{(-c)}_{N-s-1}(t,a,b) = (-1)^{b-t-1}\frac{\Gamma(b-c-s)\Gamma(t+c+1)}{\Gamma(s+a+c+1)\Gamma(t-c+1)} w^{c}_{s}(t,a,b)\,.$$

3. Dual Properties of the Racah Polynomials

The Racah polynomials were studied in several papers (see [18] and references therein) and, in particular, in [8]. Here we will study their dual properties. They are defined by [8]

$$\begin{aligned} u^{\alpha,\beta}_{n}(s,a,b) =& \frac{(a-b+1)_n(\beta+1)_n(a+b+\alpha+1)_n}{n!} \\ & \times {}_4\mathrm{F}_3\left(\begin{matrix} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{matrix} \,\middle|\, 1 \right) \end{aligned} \tag{30}$$

and satisfy the difference equation (1), where

$$\begin{aligned} &\sigma(s) = (s-a)(s+b)(s+a-\beta)(b+\alpha-s), \\ &\sigma(s)+\tau(s)\Delta x(s-1/2) = \sigma(-s-1) = (s+a+1)(b-s-1)(s-a+\beta+1)(b+\alpha+s+1). \end{aligned} \tag{31}$$

The alternative Racah polynomials are defined by [8]

$$\begin{aligned} \widetilde{u}^{\alpha,\beta}_{n}(s,a,b) \;=\; & \frac{(a-b+1)_n(2a-\beta+1)_n(a-b-\alpha+1)_n}{n!} \\ & \times {}_4\mathrm{F}_3\left(\begin{matrix} -n,\; 2a-2b-\alpha-\beta+n+1,\; a-s,\; a+s+1 \\ a-b+1,\; 2a-\beta+1,\; a-b-\alpha+1 \end{matrix} \,\middle|\, 1 \right) \end{aligned} \tag{32}$$

and satisfy the difference equation (1), where

$$\begin{aligned} \sigma(s) &= (s-a)(s+b)(s-a+\beta)(b+\alpha+s), \\ \sigma(s)+\tau(s)\Delta x(s-1/2) &= \sigma(-s-1) \\ &= (s+a+1)(b-s-1)(s+a-\beta+1)(b+\alpha-s-1). \end{aligned} \tag{33}$$

They are both polynomials of degree n on the lattice $x(s) = s(s+1)$ with leading coefficients

$$a_n = \frac{\Gamma(\alpha+\beta+2n+1)}{n!\Gamma(\alpha+\beta+n+1)} \quad \text{and} \quad \widetilde{a}_n = \frac{(-1)^n\Gamma(2b-2a+\alpha+\beta-n)}{n!\Gamma(2b-2a+\alpha+\beta-2n)},$$

respectively. Notice also that $\Delta x(s+m) = 2s+2m$.

The main characteristics of the Racah polynomials $u^{\alpha,\beta}_{n}(s,a,b)$ are given in [8] (Table 3.6, p. 108).

The main characteristics of the alternative Racah polynomials can be obtained changing $\alpha \to -2b-\alpha$ and $\beta \to 2a-\beta$.

3.1. Polynomials of Degree $k = s - a$ Dual to the Racah Polynomials

Let us construct the polynomials of degree $s-a$ dual with respect to the Racah polynomials $u_n^{\alpha\beta}(s,a,b)$. From (13) follows that the new variable is $t = [(\alpha+\beta)/2] + n$ and then the new interval of orthogonality is (a', b'), where $a' = (\alpha+\beta)/2$ and $b' = b - a + (\alpha+\beta)/2$. But a priori it is not clear what the values of the new parameters α' and β' are, nor what kind of polynomials the corresponding dual ones will be. Combining the two methods discussed in the first section, one concludes that in this case the dual family is ($k = s - a$)

$$u_k^{\alpha'\beta'}(t,a',b') \quad \text{with} \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta, \tag{34}$$

where $u_k^{\alpha'\beta'}(t,a',b')$ are the Racah polynomials (30). Iterating (11) for the polynomials $u_n(s) = u_n^{\alpha\beta}(s,a,b)$ we obtain

$$\begin{aligned} u_n(s) &= [t(t+1)]^k u_n(a) \prod_{l=1}^{s-a} \frac{1}{A(s-l)} + \cdots \\ &= \frac{(-1)^{s-a+n}\Gamma(b-s)\Gamma(\beta+n+1)\Gamma(b+a+\alpha+n+1)\Gamma(2s+1)}{\Gamma(s+a+1)\Gamma(s-a+\beta+1)\Gamma(b+\alpha+s+1)\Gamma(n+1)\Gamma(b-a-n)}[t(t+1)]^k + \cdots \end{aligned}$$

Comparing this relation with the leading coefficient of the polynomial $u_k^{\alpha'\beta'}(t,a',b')$ [see, e.g., [8] (Table 3.6, p. 108)]

$$a_k = \frac{\Gamma(\alpha'+\beta'+2k+1)}{\Gamma(k+1)\Gamma(\alpha'+\beta'+k+1)} = \frac{\Gamma(2s+1)}{\Gamma(s-a+1)\Gamma(s+a+1)},$$

we obtain ($k = s - a$)

$$u_k^{\alpha'\beta'}(t,a',b') = (-1)^{s-a+n}\frac{\Gamma(b-a-n)\Gamma(s-a+\beta+1)\Gamma(b+\alpha+s+1)\Gamma(n+1)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(n+\beta+1)\Gamma(b+a+\alpha+n+1)}u_n^{\alpha\beta}(s,a,b), \tag{35}$$

where the parameters are given by (34).

3.2. Polynomials of Degree $k = b - s - 1$ Dual to the Racah Polynomials

In this case, the dual set is given by $\widetilde{u}_k^{\alpha'\beta'}(t,a',b')$, where $\widetilde{u}_n$ are the alternative polynomials (32) with the parameters defined as in the previous section, i.e.,

$$t = \frac{\alpha+\beta}{2} + n, \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \tag{36}$$

In this case, we have

$$A'_{kn} = (-1)^k u_n^{\alpha\beta}(b-1,a,b) \prod_{l=1}^{b-s-1} \frac{1}{C(s+l)}.$$

Using

$$u_n^{\alpha\beta}(b-1,a,b) = \frac{\Gamma(b-a)\Gamma(\alpha+n+1)\Gamma(b+a-\beta)}{\Gamma(n+1)\Gamma(b-a-n)\Gamma(\alpha+1)\Gamma(b+a-\beta-n)},$$

along with (2) and (31), we obtain

$$A'_{kn}=(-1)^{b-s-1}\frac{\Gamma(s-a+1)\Gamma(s+b+1)\Gamma(s+a-\beta+1)\Gamma(\alpha+n+1)}{\Gamma(2s+2)\Gamma(b+\alpha-s)\Gamma(n+1)\Gamma(b-a-n)\Gamma(b+a-\beta-n)}\,.$$

Now taking into account that the leading coefficient for the polynomial $u_k^{\alpha'\beta'}(t,a',b')$

$$a'_k=(-1)^k\frac{\Gamma(2b'-2a'+\alpha'+\beta'-k)}{\Gamma(k+1)\Gamma(2b'-2a'+\alpha'+\beta'-2k)}=(-1)^{b-s-1}\frac{\Gamma(b+s+1)}{\Gamma(b-s)\Gamma(2s+2)}\,,$$

with $2b'-2a'+\alpha'+\beta'-k=b+s+1$, we obtain for the proportionality coefficient the value

$$D'_{kn}=\frac{\Gamma(b+\alpha-s)\Gamma(n+1)\Gamma(b-a-n)\Gamma(b+a-\beta-n)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(s+a-\beta+1)\Gamma(n+\alpha+1)}\,,$$

i.e., $(k=b-s-1)$

$$\widetilde{u}_k^{\alpha'\beta'}(t,a',b')=\frac{\Gamma(b+\alpha-s)\Gamma(n+1)\Gamma(b-a-n)\Gamma(b+a-\beta-n)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(s+a-\beta+1)\Gamma(n+\alpha+1)}u_n^{\alpha\beta}(s,a,b), \tag{37}$$

where the parameters are given by (36).

3.3. Polynomials of Degree $k=s-a$ Dual to the Alternative Racah Polynomials

Now we find the polynomials of degree $s-a$ dual to the alternative Racah polynomials $\widetilde{u}_n^{\alpha\beta}(s,a,b)$. In this case, the dual family is $u_k^{\alpha'\beta'}(\widetilde{t},a',b')$, where

$$\widetilde{t}=b-a-1-n+\frac{\alpha+\beta}{2}\,,\quad a'=\frac{\alpha+\beta}{2}\,,\quad b'=b-a+\frac{\alpha+\beta}{2}\,,\quad \alpha'=2a-\beta,\quad \beta'=\beta. \tag{38}$$

Notice that

$$\lambda_n=n(2b-2a+\alpha+\beta-n-1)=-\widetilde{t}(\widetilde{t}+1)-\left(b-a+\frac{\alpha+\beta}{2}-1\right)\left(b-a+\frac{\alpha+\beta}{2}\right).$$

Then, the coefficient of $[\widetilde{t}(\widetilde{t}+1)]^k$ for the polynomial $\widetilde{u}_n^{\alpha\beta}(s,a,b)$ reads

$$\widetilde{A}_{kn}=\widetilde{u}_n^{\alpha\beta}(a,a,b)\prod_{l=1}^{s-a}\frac{(2s+1+2l)(2s+2-2l)}{\sigma(-s-l-l)}\,.$$

Using (2), (33), and

$$\widetilde{u}_n^{\alpha\beta}(a,a,b)=\frac{\Gamma(b-a)\Gamma(2a-\beta+n+1)\Gamma(b-a+\alpha)}{\Gamma(n+1)\Gamma(b-a-n)\Gamma(2a-\beta+1)\Gamma(b-a+\alpha-n)}\,,$$

we obtain

$$\widetilde{A}_{kn}=\frac{\Gamma(2s+1)\Gamma(b-s)\Gamma(b+\alpha-s)\Gamma(2a-\beta+n+1)}{\Gamma(s+a+1)\Gamma(s+a-\beta+1)\Gamma(n+1)\Gamma(b-a-n)\Gamma(b-a+\alpha-n)}\,.$$

Combining this with the value of the leading coefficient [see [8] (Table 3.6)]

$$\widetilde{a}_k = \frac{\Gamma(\alpha' + \beta' + 2k + 1)}{\Gamma(k+1)\Gamma(\alpha' + \beta' + k + 1)} = \frac{\Gamma(2s+1)}{\Gamma(s-a+1)\Gamma(s+a+1)}\,,$$

we finally obtain

$$\widetilde{D}_{nk} = \frac{\widetilde{a}_k}{\widetilde{A}_{nk}} = \frac{\Gamma(s+a-\beta+1)\Gamma(n+1)\Gamma(b-a-n)\Gamma(b-a+\alpha-n)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(b+\alpha-s)\Gamma(2a-\beta+n+1)}\,,$$

i.e., $(k = s - a)$

$$u_k^{\alpha'\beta'}(\widetilde{t}, a', b') = \frac{\Gamma(s+a-\beta+1)\Gamma(n+1)\Gamma(b-a-n)\Gamma(b-a+\alpha-n)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(b+\alpha-s)\Gamma(2a-\beta+n+1)}\,\widetilde{u}_n^{\alpha\beta}(s,a,b)\,, \tag{39}$$

where the set of parameters is given by (38).

3.4. Polynomials of Degree $k = b - s - 1$ Dual to the Alternative Racah Polynomials

In this case, the dual polynomials are $\widetilde{u}_{k=b-s-1}^{\alpha'\beta'}(\widetilde{t}, a', b')$ and the set of parameters is the same as in the previous case (38). Since

$$\widetilde{u}_n^{\alpha\beta}(b-1,a,b) = (-1)^n \frac{\Gamma(b-a)\Gamma(2b+\alpha)\Gamma(b-a+\beta)}{\Gamma(n+1)\Gamma(b-a-n)\Gamma(2b+\alpha-n)\Gamma(b-a+\beta-n)}\,,$$

using (2) and (33), and then (15), we arrive at

$$\widetilde{A}'_{kn} = (-1)^n \frac{\Gamma(s-a+1)\Gamma(s+b+1)\Gamma(s-a+\beta+1)\Gamma(b+\alpha+s+1)}{\Gamma(n+1)\Gamma(b-a-n)\Gamma(2b+\alpha-n)\Gamma(b-a+\beta-n)\Gamma(2s+2)}\,.$$

Combining the above formula with the value of the leading coefficient for the $\widetilde{u}_n^{\alpha\beta}$ polynomials

$$\widetilde{a}_k = (-1)^k \frac{\Gamma(2b' - 2a' + \alpha' + \beta' - k)}{\Gamma(k+1)\Gamma(2b' + 2a' + \alpha' + \beta' - 2k)} = (-1)^{b-s-1} \frac{\Gamma(b+s+1)}{\Gamma(b-s)\Gamma(2s+2)},$$

we obtain

$$\widetilde{D}'_{kn} = (-1)^{b-s-n-1} \frac{\Gamma(n+1)\Gamma(b-a-n)\Gamma(b-a+\beta-n)\Gamma(2b+\alpha-n)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(s-a+\beta+1)\Gamma(b+\alpha+s+1)}\,,$$

i.e., $(k = b - s - 1)$

$$\widetilde{u}_k^{\alpha',\beta'}(\widetilde{t}, a', b') = (-1)^{b-s-n-1} \frac{\Gamma(n+1)\Gamma(b-a-n)\Gamma(b-a+\beta-n)\Gamma(2b+\alpha-n)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(s-a+\beta+1)\Gamma(b+\alpha+s+1)} \widetilde{u}_n^{\alpha\beta}(s,a,b), \tag{40}$$

where $\widetilde{t} = b - a - 1 - n + (\alpha+\beta)/2$, $a' = (\alpha+\beta)/2$, $b' = b - a + (\alpha+\beta)/2$, $\alpha' = 2a - \beta$, and $\beta' = \beta$.

3.5. Connection between the Two Families of Racah Polynomials $u_n^{\alpha\beta}(s,a,b)$ and $\widetilde{u}_n^{\alpha\beta}(s,a,b)$

If we change $n \to b-a-n-1$ and $\widetilde{t} \to t$ in (39), it becomes

$$u_{s-a}^{\alpha'\beta'}(t,a',b') = \frac{\Gamma(s+a-\beta+1)\Gamma(n+1)\Gamma(b-a-n)\Gamma(\alpha+n+1)}{\Gamma(s-a+1)\Gamma(b-s)\Gamma(b+\alpha-s)\Gamma(b+a-\beta-n)}\,\widetilde{u}_{b-a-n-1}^{\alpha\beta}(s,a,b)\,.$$

Comparing this with (35), we derive

$$\widetilde{u}_{b-a-n-1}^{\alpha\beta}(s,a,b) = (-1)^{s-a+n}\frac{\Gamma(s-a+\beta+1)\Gamma(b+\alpha-s)\Gamma(b+\alpha+s+1)\Gamma(b+a-\beta-n)}{\Gamma(s+a-\beta+1)\Gamma(\alpha+n+1)\Gamma(n+\beta+1)\Gamma(b+a+\alpha+n+1)}\,u_n^{\alpha\beta}(s,a,b)\,. \tag{41}$$

The same formula is obtained when we use (37) and (40).

4. The q-Hahn Polynomials

4.1. The q-Hahn Polynomials $h_n^{\alpha\beta}(x(s),N)_q$

In this section, we will consider the q-Hahn polynomials defined by the basic series [7, 13, 21]

$$h_n^{\alpha,\beta}(s,N)_q = \frac{(-1)^n(q^{2\beta+2};q^2)_n(q^{2-2N};q^2)_n}{q^{-2n(\alpha+N)}(q-q^{-1})^n(q^2;q^2)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-2n}, q^{-2s}, q^{2(n+\alpha+\beta+1)} \\ q^{2\beta+2}, q^{2-2N}\end{matrix}\;;q^2, q^{2(s-N-\alpha+1)}\right)$$

$$= \frac{(q^{2\beta+2};q^2)_n(q^{2(N+\alpha+\beta+1)};q^2)_n}{q^{n(2\beta+n+1)}(q-q^{-1})^n(q^2;q^2)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-2n}, q^{2s+2\beta+2}, q^{2(n+\alpha+\beta+1)} \\ q^{2\beta+2}, q^{2(N+\alpha+\beta+1)}\end{matrix}\;;q^2, q^2\right).$$

Notice that they are polynomials of degree n on the lattice $x(s) = q^{2s}$. Furthermore, they satisfy the difference equation (1) or (4) on the lattice $x(s) = q^{2s}$ with

$$\begin{aligned} A(s) &= q^{2\alpha+\beta+N+1-2s}[N-s-1]_q[\beta+s+1]_q\,, \\ C(s) &= q^{\alpha+N+1-2s}[s]_q[\alpha+N-s]_q\,, \\ \lambda_n &= q^{\alpha+\beta+2}[n]_q[n+\alpha+\beta+1]_q \\ &= q^{\alpha+\beta+2}\left[\left(n+\frac{\alpha+\beta}{2}\right)_q\left(n+\frac{\alpha+\beta}{2}+1\right)_q - \left(\frac{\alpha+\beta}{2}\right)_q\left(\frac{\alpha+\beta}{2}+\right)_q\right]. \end{aligned} \tag{42}$$

Here and in the following, we will denote by $[n]$ the symmetric q-number

$$[n] = \frac{q^n - q^{-n}}{q-q^{-1}}, \quad \text{and} \quad [n]! = [n][n-1]\cdots[1], \quad n \in \mathbb{N}.$$

In general, we will use the following notation by Nikiforov and Uvarov (see Eq. (3.2.24) in [8]):

The $\widetilde{\Gamma}_q$ function constitutes the q-analog of the Γ function and is related to the classical q-Gamma function Γ_q by the formula

$$\widetilde{\Gamma}_q(s) = q^{-(s-1)(s-2)/2}\Gamma_{q^2}(s) = q^{-(s-1)(s-2)/2}(1-q^2)^{1-s}\frac{(q^2;q^2)_\infty}{(q^{2s};q^2)_\infty}, \qquad 0<q<1.$$

For simplicity, we will also use the symbol

$$[x]! := \widetilde{\Gamma}_q(x+1), \qquad x \in \mathbb{R}.$$

In the following, we will use the simpler notation $[x] := [x]_q$ except where indicated.

All the characteristics of the q-Hahn polynomials can be found in Table 1.

Using the Rodrigues formula for the q-polynomials (3), we obtain the following explicit formula:

$$h_n^{\alpha\beta}(x(s), N, q) = (-1)^n q^{n[2\alpha+\beta+N+(n+1)/2]} \frac{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)}{\widetilde{\Gamma}_q(\alpha+N-s)\widetilde{\Gamma}_q(\beta+s+1)}$$
$$\times \sum_{m=0}^{n} (-1)^m q^{-m(\alpha+\beta+n+1)} \frac{\widetilde{\Gamma}_q(\alpha+N-s+m)\widetilde{\Gamma}_q(\beta+s+n-m+1)}{[m]![n-m]!\widetilde{\Gamma}_q(s-m+1)\widetilde{\Gamma}_q(N-s-n+m)},$$

from which the following formulas follow (see also the hypergeometric representation):

$$h_n^{\alpha\beta}(x(0), N; q) = (-1)^n \frac{[N-1]!\widetilde{\Gamma}_q(\beta+n+1)}{[n]!\widetilde{\Gamma}_q(\beta+1)[N-n-1]!} q^{n[2\alpha+\beta+N+(n+1)/2]}, \tag{43}$$

$$h_n^{\alpha\beta}(x(N-1), N; q) = \frac{[N-1]!\widetilde{\Gamma}_q(\alpha+n+1)}{[n]!\widetilde{\Gamma}_q(\alpha+1)[N-n-1]!} q^{n[\alpha+N-(n+1)/2]}. \tag{44}$$

These polynomials transform into the Hahn polynomials (18) at $q \to 1$.

4.2. The Alternative q-Hahn Polynomials $\widetilde{h}_n^{\alpha\beta}(x(s), N)_q$

The alternative q-Hahn polynomials are defined by

$$\widetilde{h}_n^{\alpha,\beta}(s, N)_q = \frac{(-1)^n (q^{-2N-2\beta+2}; q^2)_n (q^{2-2N}; q^2)_n}{q^{2n\alpha}(q-q^{-1})^n (q^2; q^2)_n} \times {}_3\varphi_2 \left(\begin{matrix} q^{-2n}, q^{-2s}, q^{2(n-\alpha-\beta-2N+1)} \\ q^{-2\beta-2N+2}, q^{2-2N} \end{matrix} ; q^2, q^{2(s+\alpha+1)} \right)$$
$$= \frac{(q^{-2\beta-2N+2}; q^2)_n (q^{2(-N-\alpha-\beta+1)}; q^2)_n}{q^{n(-2\beta-2N+n+1)}(q-q^{-1})^n (q^2; q^2)_n} \times {}_3\varphi_2 \left(\begin{matrix} q^{-2n}, q^{2s-2N-2\beta+2}, q^{2(n-\alpha-\beta-2N+1)} \\ q^{-2N-2\beta+2}, q^{2(-N-\alpha-\beta+1)} \end{matrix} ; q^2, q^2 \right).$$

Notice that they are polynomials of degree n on the lattice $x(s) = q^{2s}$ and satisfy the difference equation (1) or (4) on $x(s)$ with

$$\begin{aligned} A_s &= q^{-2N-2\alpha-\beta-2s+1}[N-s-1][N+\beta-s-1], \\ C_s &= q^{-\alpha-2s+1}[s][s+\alpha], \\ \lambda_n &= q^{-2N-\alpha-\beta+2}[n][2N+\alpha+\beta-n-1] \\ &= q^{-2N-\alpha-\beta+2}(-[\widetilde{t}][\widetilde{t}+1] + \{N+[(\alpha+\beta)/2]-1\}[N+(\alpha+\beta)/2], \end{aligned} \tag{45}$$

TABLE 1. The Main Data of the q-Hahn Polynomials $h_n^{\alpha\beta}(x(s),N)_q$.

	$h_n^{\alpha\beta}(x(0),N;q), \quad x(s)=q^{2s}$
$\rho(x)$	$q^{[\alpha(\alpha+2N+2s-3)/2]+\beta(\beta+2s-1)/2}\dfrac{\widetilde{\Gamma}_q(\alpha+N-s)\widetilde{\Gamma}_q(\beta+s+1)}{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)}$
$\sigma(s)$	$q^{\alpha+N+2s}(q-q^{-1})^2[s][\alpha+N-s]$
$\tau(s)$	$(q-q^{-1})q^{\alpha+\beta+2}\left\{q^{\alpha+N}[\beta+1][N-1]-q^s[s][\alpha+\beta+2]\right\}$
$\phi(s)$	$(q-q^{-1})^2q^{2\alpha+b+N+2s+2}[N-s-1][\beta+s+1]$
λ_n	$q^{\alpha+\beta+2}[n][n+\alpha+\beta+1]$ $=q^{\alpha+\beta+2}([n+(\alpha+\beta)/2][n+(\alpha+\beta)/2+1]-[(\alpha+\beta)/2][(\alpha+\beta)/2+1])$
B_n	$(-1)^n\dfrac{1}{[n]!q^{2n}(q-q^{-1})}$
$\rho_n(s)$	$q^{s(\alpha+\beta+2n)+n(2\alpha+\beta+N+n+1)+[\alpha(\alpha+2N-3)+\beta(\beta-1)]/2}$ $\times(q-q^{-1})^{2n}\dfrac{\widetilde{\Gamma}_q(\alpha+N-s)\widetilde{\Gamma}_q(\beta+s+n+1)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N-s-n)}$
a_n	$\dfrac{q^{n(\alpha+\beta+1)}\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{(q-q^{-1})^n\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(\alpha+\beta+n+1)}$
d_n^2	$(q-q^{-1})\dfrac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(\alpha+\beta+N+n+1)}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(\alpha+\beta+n+1)[\alpha+\beta+2n+1]}$ $\times q^{\{\alpha[(\alpha+2N-3)+\beta(\beta-1)]/2\}+n(3\alpha+\beta+2N)+(\beta+1)(N-1)}$
α_n	$\dfrac{[n+1][\alpha+\beta+n+1](q-q^{-1})}{[\alpha+\beta+2n+1][\alpha+\beta+2n+2]q^{\alpha+\beta+1}}$
β_n	$\dfrac{q^{-(\alpha+\beta+2)}}{[\alpha+\beta+2n][\alpha+\beta+2n+2]}$ $\times(q^{2\alpha+N+1}([N-n][n][\alpha+\beta+2n+2]-[N-n-1][n+1][\alpha+\beta+2n])$ $+q^{\alpha+\beta+N+1}([\alpha+\beta+N+n+1][n+1][\alpha+\beta+2n]-[\alpha+\beta+N+n][n][\alpha+\beta+2n+2]))$
γ_n	$(q-q^{-1})q^{2\alpha+2N-1}\dfrac{[\alpha+n][\beta+n][\alpha+\beta+N+n][N-n]}{[\alpha+\beta+2n][\alpha+\beta+2N+1]}$

where $\widetilde{t} = N + [(\alpha+\beta)/2] - n - 1$. Their main characteristics are given in Table 2.

From the Rodrigues formula follows that

$$\widetilde{h}_n^{\alpha\beta}(x(s),N,q) = q^{-n[2N+2\alpha+\beta-(n+1)/2]}\,\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(\alpha+s+1)\widetilde{\Gamma}_q(N+\beta-s)$$
$$\times \sum_{m=0}^{n} \frac{(-1)^m q^{m(2N+2\alpha+\beta-n-1)}}{[m]![n-m]![s-m]!\widetilde{\Gamma}_q(\alpha+s-m+1)\widetilde{\Gamma}_q(N-n-s+m)\widetilde{\Gamma}_q(N+\beta-n-s+m)}\,.$$

So, for $s=0$ and $s=N-1$ we find, respectively,

$$\widetilde{h}_n^{\alpha\beta}(x(0),N;q) = q^{-n[2N+2\alpha+\beta-(n+1)/2]}\,\frac{[N-1]!\widetilde{\Gamma}_q(N+\beta)}{[n]![N-n-1]!\widetilde{\Gamma}_q(N+\beta-n)} \tag{46}$$

and

$$\widetilde{h}_n^{\alpha\beta}(x(N-1),N;q) = (-1)^n q^{-n[\alpha+(n+1)/2]}\,\frac{[N-1]!\widetilde{\Gamma}_q(N+\alpha)}{[n]![N-n-1]!\widetilde{\Gamma}_q(N+\alpha-n)}\,. \tag{47}$$

These polynomials transform into the alternative Hahn polynomials (19) at $q \to 1$.

4.3. The q-Dual Hahn Polynomials

The q-dual Hahn polynomials $W_n^c(x(s),a,b)_q$ are defined by the basic series [14, 15]

$$W_n^c(x(s),a,b)_q = \frac{(-1)^n (q^{2(a-b+1)};q^2)_n (q^{2(a+c+1)};q^2)_n}{q^{n(3a-b+c+1+n)}\varkappa_q^n (q^2;q^2)_n}$$
$$\times\, {}_3\varphi_2\left(\begin{matrix} q^{-2n}, q^{2a-2s}, q^{2a+2s+2} \\ q^{2(a-b+1)}, q^{2(a+c+1)} \end{matrix}\,\middle|\, q^2, q^2\right),$$

and they are polynomials of degree n on the q-quadratic lattice $x(s) = [s][s+1]$, where $a \le s \le b-1$, with the leading coefficient

$$a_n = q^{-3n(n-1)/2}\,\frac{1}{[n]!}\,. \tag{48}$$

All their characteristics can be found in Table 3.

These polynomials transform into the dual Hahn polynomials (20) at $q \to 1$. The main data of the polynomials can be found in [14].

4.4. Polynomials of Degree $k = s$ Dual to the q-Hahn Polynomials

Following the results for the classical case (non "q"), one can expect that the polynomials of degree $k = s$ dual to the q-Hahn polynomials $h_n^{\alpha\beta}(x(s),0,N;q)$ are the polynomials $W_k^c(t,a,b')_q$ with

$$t = n + \frac{\alpha+\beta}{2}\,, \qquad a = \frac{\alpha+\beta}{2}\,, \qquad b = N + \frac{\alpha+\beta}{2}\,, \quad \text{and} \quad c = \frac{\beta-\alpha}{2}\,.$$

Nevertheless, since for the q-dual Hahn polynomials $\lambda_n = q^{-n+1}[n] = \dfrac{q^{-2n}-1}{q^{-2}-1}$, the corresponding polynomials $h_n^{\alpha\beta}(x(s),N,q)$ should be defined in the lattice $x(s) = q^{-2s}$; therefore, the right choice will be $W_k^c(t,a,b)_{q^{-1}}$.

TABLE 2. The Main Data of the Alternative q-Hahn Polynomials $\widetilde{h}_n^{\alpha\beta}(x(s),N)_q$

	$\widetilde{h}_n^{\alpha\beta}(x(0),N;q), \quad x(s)=q^{2s}$
$\rho(s)$	$\dfrac{q^{-s(2N+\alpha+\beta)-[(N+\alpha)(N-\alpha-3)/2]+(N+\beta)(N+\beta+1)/2}}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(\alpha+s+1)\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(N+\beta-s)}$
$\sigma(s)$	$q^{2s-\alpha}[s][s+\alpha](q-q^{-1})^2$
$\tau(s)$	$(q-q^{-1})q^{2-2\alpha-\beta-2N}\{[N-1][N+\beta-1]-q^{\alpha+s}[2N+\alpha+\beta-2][s]\}$
$\phi(s)$	$(q-q^{-1})^2q^{-2N-2\alpha-\beta-2s+1}[N-s-1][N+\beta-s-1]$
λ_n	$q^{-2N-\alpha-\beta+2}[n][2N+\alpha+\beta-n-1]$
B_n	$\dfrac{1}{[n]!q^n(q-q^{-1})^n}$
$\rho_n(s)$	$\dfrac{(q-q^{-1})^{2n}q^{-[(N+\alpha)(N-\alpha-3)/2]+[(N+\beta)(N+\beta+1)/2]-s(2N+\alpha+\beta-2N)-n(2N+2\alpha+\beta-n-1)}}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(\alpha+s+1)\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(N+\beta-s)}$
a_n	$(-1)^n\dfrac{q^{n(2N+\alpha+\beta-1)}\widetilde{\Gamma}_q(2N+\alpha+\beta-n)}{(q-q^{-1})^n\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(2N+\alpha+\beta-2n)}$
d_n^2	$\dfrac{(q-q^{-1})\widetilde{\Gamma}_q(2N+\alpha+\beta-n)}{[2N+\alpha+\beta-2n-1]\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N+\alpha-n)\widetilde{\Gamma}_q(N+\beta-n)}$ $\times\dfrac{q^{-[(N+\alpha)(N+\alpha-3)/2]+[(N+\beta)(N+\beta+1)/2]-n(2N+3\alpha+\beta)-(N-1)(N+\beta-1)}}{\widetilde{\Gamma}_q(N+\alpha+\beta-n)\widetilde{\Gamma}_q(N-n)}$
α_n	$-\dfrac{(q-q^{-1})q^{2N+\alpha+\beta-1}[n+1][2N+\alpha+\beta-n-1]}{[2N+\alpha+\beta-2n-1][2N+\alpha+\beta-2n-2]}$
β_n	$\dfrac{q^{2N+\alpha+\beta-2}}{[2N+\alpha+\beta-2n-2][2N+\alpha+\beta-2n]}$ $\times(q^{-N-2\alpha+1}([N-n-1][n+1][2N+\alpha+\beta-2n])$ $+q^{-N-\alpha-\beta+1}([N+\alpha+\beta-n-1][n+1][2N+\alpha+\beta-2n]$ $-[N+\alpha+\beta-n][n][2N+\alpha+\beta-2n-2]))$
γ_n	$-(q-q^{-1})q^{2N+\alpha+\beta-1}\dfrac{[N+\alpha-n][N+\beta-n][N-n]}{[2N+\alpha+\beta-2n-1][2N+\alpha+\beta-2n]}$

TABLE 3. The Main Data of the q-Dual Hahn Polynomials

$P_n(s)$	$W_n^c(x(s),a,b)_q\,,\quad x(s)=[s]_q[s+1]_q$
(a,b)	$[a,b-1]$
$\rho(s)$	$\dfrac{q^{-s(s+1)}\widetilde{\Gamma}_q[s+a+1]\widetilde{\Gamma}_q[s+c+1]}{\widetilde{\Gamma}_q[s-a+1]\widetilde{\Gamma}_q[s-c+1]\widetilde{\Gamma}_q[s+b+1]\widetilde{\Gamma}_q[b-s]}$ $-\frac{1}{2}\le a<b-1,\quad \lvert c\rvert<a+1$
$\sigma(s)$	$q^{s+c+a-b+2}[s-a]_q[s+b]_q[s-c]_q$
$\tau(s)$	$-x(s)+q^{a-b+c+1}[a+1]_q[b-c-1]_q+q^{c-b+1}[b]_q[c]_q$
λ_n	$q^{-(n-1)}[n]_q$
B_n	$\dfrac{(-1)^n}{[n]_q!}$
d_n^2	$q^{ac-ab-bc+a+c-b+1+2n(a+c-b)-n^2+5n}\dfrac{\widetilde{\Gamma}_q[a+c+n+1]_q}{[n]_q\widetilde{\Gamma}_q[b-c-n]_q\widetilde{\Gamma}_q[b-a-n]_q}$
$\rho_n(s)$	$\dfrac{q^{-s(s+1+n)-(n^2/2)+n(a+c-b+3/2)}\widetilde{\Gamma}_q[s+a+n+1]\widetilde{\Gamma}_q[s+c+n+1]}{\widetilde{\Gamma}_q[s-a+1]\widetilde{\Gamma}_q[s-c+1]\widetilde{\Gamma}_q[s+b+1]\widetilde{\Gamma}_q[b-s-n]}$
a_n	$\dfrac{q^{-3n(n-1)/2}}{[n]_q!}$
α_n	$q^{3n}[n+1]_q$
β_n	$q^{2n-b+c+1}[b-a-n+1]_q[a+c+n+1]_q$ $+q^{2n+2a+c-b+1}[n]_q[b-c-n]_q+[a]_q[a+1]_q$
γ_n	$q^{n+3+2(c+a-b)}[n+a+c]_q[b-a-n]_q[b-c-n]_q$

Then, in view of the same ideas as before (see the previous sections), from (43) and (42) we obtain

$$\prod_{l=1}^{s}\frac{-q^{\alpha+\beta+2}}{A(s-l)}=(-1)^s q^{s(\alpha+\beta+2)}\frac{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(\beta+1)}{\widetilde{\Gamma}_q(N)\widetilde{\Gamma}_q(\beta+s+1)}q^{-s(2\alpha+\beta+N+1)+s(s+1)}\,;$$

thus

$$A_{sn}=(-1)^{s+n}\frac{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(\beta+n+1)}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(\beta+s+1)}q^{n(2\alpha+\beta+N)+n(n+1)/2}q^{-s(N+\alpha-s)}\,.$$

Taking into account the value of the leading coefficient of the q-dual Hahn, we have $a_s=\frac{q^{3s(s-1)/2}}{[s]!}$ and, therefore,

$$D_{sn} = (-1)^{s+n} \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(\beta+s+1)}{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(\beta+n+1)} q^{-n[2\alpha+\beta+N+(n+1)/2]} q^{s[N+\alpha+(s-3)/2]} ,$$

i.e.,

$$\begin{aligned} W^c_{k=s}(t,a,b)_{q^{-1}} &= (-1)^{s+n} \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(\beta+s+1)}{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(\beta+n+1)} \\ &\quad \times q^{-n[2\alpha+\beta+N+(n+1)/2]} q^{s(N+\alpha+(s-3)/2]} h_n^{\alpha\beta}(s,N,q) , \end{aligned} \tag{49}$$

where

$$t = n + \frac{\alpha+\beta}{2} , \qquad a = \frac{\alpha+\beta}{2} , \qquad b = N + \frac{\alpha+\beta}{2} , \qquad c = \frac{\beta-\alpha}{2} .$$

Another way of obtaining the above result is to use the relation

$$D^2_{sn} = \frac{\rho(s)\Delta(x(s-1/2))d^2_s}{\rho(t)\Delta(\xi(t-1/2))d^2_n}$$

equivalent to (17), where $\rho(s)$ and d^2_n are the weight function and the norm, respectively, of the q-Hahn polynomials $h_n^{\alpha\beta}(s,N,q)$ and $\rho(t)$; and d^2_s are the ones of the q-dual Hahn polynomials $W^c_s(t,a,b)_{q^{-1}}$. Straightforward computation leads to the same formula (49).

4.5. Polynomials of Degree $k = N - s - 1$ Dual to the q-Hahn Polynomials

This case is quite similar to the previous one. We use (44) and (42), which leads to

$$A'_{kn} = (-1)^{N-s-1} \frac{q^{n(N+\alpha-(n+1)/2]} q^{(N-s-1)(\beta+s+1)} \widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(n+\alpha+1)}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\alpha-s)} .$$

Since, in this case, $a_k = \frac{q^{-3k(k-1)/2}}{[k]!}$ and we obtain

$$D'_{kn} = (-1)^{N-s-1} q^{-n(N+\alpha-(n-1)/2]} q^{-(N-s-1)\{[(3N+s)/2]+\beta-2\}}$$

$$\times \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\alpha-s)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(n+\alpha+1)}$$

and, therefore,

$$\begin{aligned} W^{(\alpha-\beta)/2}_{N-s-1}(t)_q &= (-1)^{N-s-1} \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\alpha-s)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(n+\alpha+1)} \\ &\quad \times q^{-n[N+\alpha-(n+1)/2]} q^{-(N-s-1)\{[(3N+s)/2]+\beta-2\}} h_n^{\alpha\beta}(s,N,q). \end{aligned} \tag{50}$$

4.6. Polynomials of Degree $k = s$ Dual to the Alternative q-Hahn Polynomials

In this case, following the same procedure as before and using (45), (46), and (48), we find

$$\widetilde{A}_{sn} = \frac{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(N+\beta-s)}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\beta-n)} q^{-n(2N+2\alpha+\beta-(n+1)/2]} q^{s(s+\alpha)} ,$$

$$\widetilde{D}_{sn} = \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\beta-n)}{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N+\beta-s)} q^{n[2N+2\alpha+\beta-(n+1)/2]} q^{-s[\alpha-(s-3)/2]} ,$$

and, therefore,

$$W_s^{\frac{1}{2}(\alpha-\beta)}(\widetilde{t})_{q^{-1}} = \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\beta-n)}{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N+\beta-s)} \times q^{n(2N+2\alpha+\beta-(n+1)/2]} q^{-s[\alpha-(s-3)/2]} \widetilde{h}_n^{\alpha\beta}(s,N,q),$$

where $\widetilde{t} = N + [(\alpha+\beta)/2] - n - 1$.

4.7. Polynomials of Degree $k = N - s - 1$ Dual to the Alternative q-Hahn Polynomials

Finally, for this case using (45), (47), and (48) we find

$$\widetilde{A}'_{kn} = (-1)^n \frac{q^{-n[\alpha+(n+1)/2]} q^{(N-s-1)(N+\beta-s-1)} \widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(s+\alpha+1)}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\beta-n)},$$

$$\widetilde{D}'_{kn} = (-1)^n \frac{q^{n[\alpha+(n+1)/2]} q^{-(N-s-1)\{[(N-s)/2]-\beta-2\}} \widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\alpha-n)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+\alpha+1)}$$

and

$$W_{N-s-1}^{(\beta-\alpha)/2}(\widetilde{t})_q = (-1)^n \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(N+\alpha-n)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+\alpha+1)} \times q^{n[\alpha+(n+1)/2]} q^{-(N-s-1)\{[(N-s)/2]-\beta-2\}} \widetilde{h}_n^{\alpha\beta}(s,N,q), \tag{52}$$

where $\widetilde{t} = N + [(\alpha+\beta)/2] - n - 1$.

4.8. Connection between q-Hanh Polynomials

Making the change $\widetilde{t} = N + [(\alpha+\beta)/2] - n - 1 \to t = [(\alpha+\beta)/2] + n$ and $-c \to c$ (or equivalently, $n \to N - n - 1$ and $\alpha, \beta \to \beta, \alpha$) in (51) one obtains

$$W_s^{(\beta-\alpha)/2}(\widetilde{t}, a, b)_{q^{-1}} = \frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(N-n)\widetilde{\Gamma}_q(\alpha+n+1)}{\widetilde{\Gamma}_q(N-s)\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(N+\alpha-s)} \times q^{(N-n-1)\{[(3N+n)/2]+\alpha+2\beta\}} q^{-s[\beta-(s-3)/2]} \widetilde{h}_{N-n-1}^{\beta\alpha}(s,N,q)) . \tag{53}$$

If we compare the above expression with (49), we find

$$h_n^{\alpha\beta}(s,N;q) = (-1)^{s+n} q^{n(\alpha-\beta)+(N-1)[(3N/2)+\alpha+2\beta]-s(N+\alpha+\beta)} \times \frac{\widetilde{\Gamma}_q(n+\alpha+1)\widetilde{\Gamma}_q(n+\beta+1)}{\widetilde{\Gamma}_q(N+\alpha-s)\widetilde{\Gamma}_q(\beta+s+1)} \widetilde{h}_{N-n-1}^{\beta\alpha}(s,N;q) . \tag{54}$$

The last formula can be rewritten in the following way:

$$h_n^{\alpha\beta}(s,N;q) = (-1)^{s+n}\sqrt{\frac{\widetilde{\rho}(s)d_n^2}{\rho(s)\widetilde{d}^2_{N-n-1}}}\,\widetilde{h}^{\beta\alpha}_{N-n-1}(s,N;q)\,, \tag{55}$$

where $\rho(s)$ and d_n^2 are the weight function and the norm of the polynomials $h_n^{\alpha\beta}(s,N;q)$, and $\widetilde{\rho}(s)$ and $\widetilde{d}^2_{N-n-1}$ are the weight function and the norm of the polynomials $\widetilde{h}^{\beta\alpha}_{N-n-1}(s,N;q)$.

Now using (49) and (50) we find

$$\begin{aligned} W_s^{(\beta-\alpha)/2}(t,a,b)_{q^{-1}} &= (-1)^{N-n-1}\frac{\widetilde{\Gamma}_q(n+\alpha+1)\widetilde{\Gamma}_q(\beta+s+1)}{\widetilde{\Gamma}_q(N+\alpha-s)\widetilde{\Gamma}_q(\beta+n+1)} \\ &\times\ q^{-n(\alpha+\beta+n+1)-s(N-\alpha+\beta-s-1)+(N-1)[(3N/2+\beta-2]}W^{(\alpha-\beta)/2}_{N-s-1}(t,a,b)_q\,. \end{aligned} \tag{56}$$

If we now substitute $\alpha=a-c$, $\beta=a+c$, $n=t-a$, and $N=b-a$, we obtain

$$\begin{aligned} W_s^{c}(t,a,b)_{q^{-1}} &= (-1)^{b-t-1}\frac{\widetilde{\Gamma}_q(t-c+1)\widetilde{\Gamma}_q(a+c+s+1)}{\widetilde{\Gamma}_q(t+c+1)\widetilde{\Gamma}_q(b-c-s)} \\ &\times q^{(b-a-1)\{[(3b-a)/2]+c-2\}-s(b-a+2c-s-1)-(t-a)(t+a+1)}\,W^{-c}_{N-s-1}(t,a,b)_q\,. \end{aligned} \tag{57}$$

5. The q-Racah Polynomials

The q-Racah polynomials are defined by

$$\begin{aligned} u_n^{\alpha,\beta}(x(s),a,b)_q =&\frac{q^{-n(2a+\alpha+\beta+n+1)}(q^{2(a-b+1)};q^2)_n(q^{2\beta+2};q^2)_n(q^{2(a+b+\alpha+1)};q)_n}{(q-q^{-1})^{2n}(q^2;q^2)_n} \\ &\times {}_4\varphi_3\left(\begin{matrix} q^{-2n},q^{2(\alpha+\beta+n+1)},q^{2a-2s},q^{2(a+s+1)} \\ q^{2(a-b+1)},q^{2\beta+2},q^{2(a+b+\alpha+1)}\end{matrix}\,\middle|\, q^2\,,\,q^2\right). \end{aligned} \tag{58}$$

They are polynomials of degree n on the q-quadratic lattice $x(s)=[s][s+1]$ with the leading coefficient

$$a_n = \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]!\widetilde{\Gamma}_q(\alpha+\beta+n+1)},$$

A detailed study of this family was done in [16] (see also [13, 17]). Their main characteristics are given in Table 1 of [16].

The alternative q-Racah polynomials are defined by

$$\begin{aligned} \widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =&\frac{q^{-n(4a-2b-\alpha-\beta+n+1)}(q^{2(a-b+1)};q^2)_n(q^{2(2a-\beta+1)};q^2)_n(q^{2(a-b-\alpha+1)};q^2)_n}{(q-q^{-1})^{2n}(q^2;q^2)_n} \\ &\times {}_4\varphi_3\left(\begin{matrix} q^{-2n},q^{2(2a-2b-\alpha-\beta+n+1)},q^{2a-2s},q^{2(a+s+1)} \\ q^{2(a-b+1)},q^{2(2a-\beta+1)},q^{2(a-b-\alpha+1)}\end{matrix}\,\middle|\, q^2\,,\,q^2\right). \end{aligned} \tag{59}$$

They are polynomials of degree n on the q-quadratic lattice $x(s) = [s][s+1]$ with the leading coefficient

$$a_n = \frac{(-1)^n \widetilde{\Gamma}_q[2b - 2a + \alpha + \beta - n]_q}{[n]_q! \widetilde{\Gamma}_q[2b - 2a + \alpha + \beta - 2n]_q}.$$

Moreover [16],

$$\begin{aligned} \widetilde{u}_n^{\alpha,\beta}(x(a), a, b)_q &= \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2a - \beta + n + 1)\widetilde{\Gamma}_q(b - a + \alpha)}{[n]!\widetilde{\Gamma}_q(b - a - n)\widetilde{\Gamma}_q(2a - \beta + 1)\widetilde{\Gamma}_q(b - a + \alpha - n)}, \\ \widetilde{u}_n^{\alpha,\beta}(x(b-1), a, b)_q &= \frac{(-1)^n\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2b + \alpha)\widetilde{\Gamma}_q(b - a + \beta)}{[n]!\widetilde{\Gamma}_q(b - a - n)\widetilde{\Gamma}_q(2b + \alpha - n)\widetilde{\Gamma}_q(b - a + \beta - n)}. \end{aligned} \tag{60}$$

A detailed study of this family was done in [16] and their main characteristics are given in Table 2 of [16].

Let us now study the duality properties of the q-Racah polynomials. First of all, notice that all the characteristics of these polynomials transform into the corresponding ones by replacing the q-numbers $[m]$ with the standard ones m and the q-Gamma functions $\widetilde{\Gamma}_q(x)$, with the classical ones $\Gamma(x)$. Therefore, it is reasonable to expect that all the results in Sec. 3 can be extended to this case just replacing the standard numbers and functions by their symmetric q-analogs. We will show only the details for the first case, since the other three are equivalent and we will include only the final result.

5.1. Polynomials of Degree $k = s - a$ Dual to the $q-$Racah Polynomials

Let us consider in detail the first case.

Let us construct the polynomials of degree $s - a$ dual with respect to the q-Racah polynomials $u_n^{\alpha\beta}(s, a, b)_q$. First of all, notice that for these polynomials

$$\lambda_n = [n][n + \alpha + \beta + 1] = \Big([t][t+1] - [(\alpha+\beta)/2]\{[(\alpha+\beta)/2] + 1\}\Big),$$

where $t = n + (\alpha+\beta)/2$. Following the same ideas as for the non q-case, we see that the dual polynomials to $u_n^{\alpha\beta}(s, a, b)_q$ should be ($k = s - a$)

$$u_k^{\alpha'\beta'}(t, a', b'), \text{ where } a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \tag{61}$$

Iterating (11) for the polynomials $u_n(s) = u_n^{\alpha\beta}(s, a, b)_q$ we find

$$\begin{aligned} u_n(s) &= \{[t][t+1]\}^k u_n^{\alpha\beta}(a, a, b)_q \prod_{l=1}^{s-a} \frac{1}{A(s-l)} + \cdots \\ &= \frac{(-1)^{s-a+n}\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b+a+\alpha+n+1)\Gamma(2s+1)}{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b+\alpha+s+1)\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)}\{[t][t+1]\}^k + \cdots \end{aligned} \tag{62}$$

Comparing this with the leading coefficient of the polynomial $u_k^{\alpha'\beta'}(t, a', b')_q$

$$a_k = \frac{\widetilde{\Gamma}_q(\alpha' + \beta' + 2k + 1)}{\widetilde{\Gamma}_q(k+1)\widetilde{\Gamma}_q(\alpha' + \beta' + k + 1)} = \frac{\widetilde{\Gamma}_q(2s+1)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+a+1)}, \tag{63}$$

we obtain $(k = s - a)$

$$u_k^{\alpha',\beta'}(t,a',b')_q = (-1)^{s-a+n}\frac{\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b+\alpha+s+1)\widetilde{\Gamma}_q(n+1)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(n+\beta+1)\widetilde{\Gamma}_q(b+a+\alpha+n+1)}u_n^{\alpha\beta}(s,a,b)_q. \tag{64}$$

5.2. Polynomials of Degree $k = b - s - 1$ Dual to the q-Racah Polynomials

In this case, the dual set is given by $\widetilde{u}_k^{\alpha'\beta'}(t,a',b')_q$, where $\widetilde{u}_n$ are the alternative polynomials with the parameters defined as follows:

$$t = \frac{\alpha+\beta}{2} + n, \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta, \tag{65}$$

so, we have $(k = b - s - 1)$

$$\widetilde{u}_k^{\alpha'\beta'}(t,a',b')_q = \frac{\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(b+a-\beta-n)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(n+\alpha+1)}u_n^{\alpha\beta}(s,a,b)_q, \tag{66}$$

where the parameters are given by (65).

5.3. Polynomials of Degree $k = s - a$ Dual to the Alternative q-Racah Polynomials

In this case, the dual family is $u_k^{\alpha'\beta'}(\widetilde{t},a',b')_q$, where

$$\widetilde{t} = b - a - 1 - n + \frac{\alpha+\beta}{2}, \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \tag{67}$$

The relation between the two families is $(k = s - a)$

$$u_k^{\alpha'\beta'}(\widetilde{t},a',b')_q = \frac{\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(b-a+\alpha-n)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(2a-\beta+n+1)}\,\widetilde{u}_n^{\alpha\beta}(s,a,b)_q\,, \tag{68}$$

where the set of parameters is given by (67).

5.4. Polynomials of Degree $k = b - s - 1$ Dual to the Alternative Racah Polynomials

In this case, the dual polynomials are $\widetilde{u}_{k=b-s-1}^{\alpha'\beta'}(\widetilde{t},a',b')_q$ and the set of parameters are given by (67), i.e., $(k = b - s - 1)$

$$\widetilde{u}_k^{\alpha',\beta'}(\widetilde{t},a',b')_q = (-1)^{b-s-n-1}\frac{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(b-a+\beta-n)\widetilde{\Gamma}_q(2b+\alpha-n)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b+\alpha+s+1)}\widetilde{u}_n^{\alpha\beta}(s,a,b)_1, \tag{69}$$

where

$$\widetilde{t} = b - a - 1 - n + \frac{\alpha+\beta}{2}, \quad a' = \frac{\alpha+\beta}{2}, \quad b' = b - a + \frac{\alpha+\beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta.$$

5.5. Connection between the Two Families of q-Racah Polynomials $u_n^{\alpha\beta}(s,a,b)_q$ and $\widetilde{u}_n^{\alpha\beta}(s,a,b)_q$

If we change $n \to b-a-n-1$ and $\widetilde{t} \to t$ in (68), it becomes

$$u_{s-a}^{\alpha'\beta'}(t,a',b')_q = \frac{\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(\alpha+n+1)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(b+a-\beta-n)}\,\widetilde{u}_{b-a-n-1}^{\alpha\beta}(s,a,b)_q\,. \tag{70}$$

Comparing this with (64) we deduce that

$$\begin{aligned}\widetilde{u}_{b-a-n-1}^{\alpha\beta}(s,a,b)_q &= (-1)^{s-a+n}\\ &\times\frac{\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(b+\alpha+s+1)\widetilde{\Gamma}_q(b+a-\beta-n)}{\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(n+\beta+1)\widetilde{\Gamma}_q(b+a+\alpha+n+1)}\,u_n^{\alpha\beta}(s,a,b)_q\,.\end{aligned} \tag{71}$$

The same formula is obtained when we use (66) and (69).

6. Connection with the $su(2)$ and $su_q(2)$ Algebras

Here we discuss the connection of the Hahn and Racah polynomials with the $su(2)$ and $su_q(2)$ algebras. A more detailed discussion can be found in [19, 20, 22] and references therein.

6.1. Clebsch–Gordan Coefficients for the $su(2)$ Algebra and Hahn Polynomials

It is well known [see, e.g., [8] (§5.2.2)] that the Hahn polynomials $h_n^{\alpha\beta}(s,N)$ are related to the Clebsch–Gordan coefficients (CGC) $(j_1m_1j_2m_2|jm)$ of the $su(2)$ algebra by the formula

$$(-1)^{j_1-m_1}(j_1m_1j_2m_2|jm) = \sqrt{\frac{\rho(s)}{d_n^2}}h_n^{\alpha\beta}(s,N), \tag{72}$$

where $n=j-m$, $s=j_2-m_2$, $N=j_1+j_2-m+1$, $\alpha=m-m'$, $\beta=m+m'$, $m'=j_1-j_2$, and ρ and d_n^2 are the weight function and the norm of the Hahn polynomials, respectively. Notice that, using the symmetry properties for the CCG, we can consider, without loss of generality, $m-m'\ge 0$ and $m+m'\ge 0$, i.e., $m-j_1+j_2\ge 0$ and $m+j_1-j_2\ge 0$, and $m\ge|j_1-j_2|$.

For finding the connection between the alternative Hahn polynomials $\widetilde{h}_n^{\alpha\beta}(s,N)$ and the CGC, we can use the connection between these two families (29) and substitute it in the last formula. This yields

$$(-1)^{j_1+j_2-j}(j_1m_1j_2m_2|jm) = \sqrt{\frac{\widetilde{\rho}(s)}{\widetilde{d}_{N-n-1}^2}}\widetilde{h}_{N-n-1}^{\alpha\beta}(s,N), \tag{73}$$

where $n=j-m$, $s=j_2-m_2$, $N=j_1+j_2-m+1$, $\alpha=m-m'$, $\beta=m+m'$, $m'=j_1-j_2$, and $\widetilde{\rho}$ and $\widetilde{d}_n^2$ are the weight function and the norm of the alternative Hahn polynomials, respectively.

To establish the connection of the CGC with the dual Hahn polynomials $w_k^{(c)}(t)$ we can use, e.g., (22). This leads to

$$(-1)^{j_1+j_2-j}(j_1m_1j_2m_2|jm) = \sqrt{\frac{\rho(t)(2t+1)}{d_k^2}}w_k^{(c)}(t,a,b), \tag{74}$$

where $c = j_1 - j_2$, $k = j_2 - m_2$, $t = j$, $a = m$, and $b = j_1 + j_2 + 1$ [compare it with [8](Eq. (5.2.14), p. 246)] and ρ and d_k^2 are the weight function and the norm of the dual Hahn polynomials.

Now, using the connection between the alternative Hahn polynomials and the dual Hahn polynomials [see (26) or (28)], we find another expression connecting CGC and dual Hahn polynomials

$$(j_1 m_1 j_2 m_2 | jm) = \sqrt{\frac{\rho(t)(2t+1)}{d_k^2}}\, w_k^{(c)}(t, a, b), \tag{75}$$

where now $c = j_1 - j_2$, $k = j_1 - m_1$, $t = j$, $a = m$, and $b = j_1 + j_2 + 1$.

6.2. 6j-Symbols for the $su(2)$ Algebra and Racah Polynomials

In [8] the formula

$$(-1)^{j_1+j+j_{23}} \sqrt{2j_{12}+1} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = \sqrt{\frac{\rho(s)}{d_n^2}}\, u_n^{\alpha\beta}(s, a, b) \tag{76}$$

connecting the 6j-symbols for the $su(2)$ algebra and Racah polynomials was proved, where $\rho(s)$ and d_n are the weight function and the norm of the Racah polynomials; $n = j_{12} - j_1 + j_2$, $x = s(s+1)$, $s = j_{23}$, $a = j_3 - j_2$, $b = j_2 + j_3 + 1$, $\alpha = j_1 - j_2 - j_3 + j$, and $\beta = j_1 - j_2 + j_3 - j$. Here it was supposed that $j_1 - j_2 \geq |j_3 - j|$ and $j_3 - j_2 \leq |j_1 - j|$.

Using the connection between the Racah polynomials $u_n^{\alpha\beta}(s, a, b)$ and the alternative Racah polynomials $\widetilde{u}_n^{\alpha\beta}(s, a, b)$ (41), we find another connection formula

$$(-1)^{j_{12}+j_3+j} \sqrt{2j_{12}+1} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = \sqrt{\frac{\widetilde{\rho}(s)}{\widetilde{d}_n^2}}\, \widetilde{u}_n^{\alpha\beta}(s, a, b)\,, \tag{77}$$

where $\widetilde{\rho}(s)$ and $\widetilde{d}_n$ are the weight function and the norm of the alternative Racah polynomials $\widetilde{u}_n^{\alpha\beta}(s, a, b)$ but now $n = j_1 + j_2 - j_{12}$. The other parameters s, a, b, α, and β are defined as before.

Finally, let us mention that the duality relation for the Racah polynomials is equivalent to the symmetry property of the 6j-symbols

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = \left\{ \begin{array}{ccc} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{array} \right\}. \tag{78}$$

6.3. Clebsch–Gordan Coefficients for the $su_q(2)$ Algebra and Hahn Polynomials

Following [21] we state that the relation between the CGC of the $su_q(2)$ algebra and the q-Hahn polynomials $h_n^{\alpha\beta}(s, N; q)$ is

$$(j_1 m_1 j_2 m_2 | jm)_{q^{-1}} = (-1)^{n+s} \sqrt{\frac{\rho(s)\Delta(x(s-1/2))}{d_n^2}}\, h_n^{\alpha\beta}(s, N; q)\,, \tag{79}$$

where $s = j_1 - m_1$, $N = j_1 + j_2 - m + 1$, $\alpha = m + j_1 - j_2$, $\beta = m - j_1 + j_2$, $n = j - m$, and ρ and d_n^2 are the weight function and the norm of the q-Hahn polynomials.

If we now use the identity [see, e.g., [13] (Eq. (8.198), p. 314)]

$$(-1)^n q^{n(\alpha+\beta+N)} h_n^{\beta\alpha}(N-s-1,N,q^{-1}) = h_n^{\alpha\beta}(s,N,q)\,, \tag{80}$$

we obtain an alternative setting

$$(j_1m_1j_2m_2|jm)_q = (-1)^{N-s-1}\sqrt{\frac{\rho(s)\Delta(x(s-1/2))}{d_n^2}}h_n^{\alpha\beta}(s,N;q)\,, \tag{81}$$

where now $s = j_2 - m_2$, $N = j_1 + j_2 - m + 1$, $\alpha = m - j_1 + j_2$, $\beta = m + j_1 - j_2$, and $n = j - m$.

For the q-dual Hahn polynomials, we have [14]

$$(-1)^{j_1+j_2-j}(j_1m_1j_2m_2|jm)_q = \sqrt{\frac{\rho(s)\Delta x(s-1/2)}{d_n^2}}W_n^c(x(s),a,b)_{q^{-1}}, \tag{82}$$

where $|j_1 - j_2| < m$, $n = j_2 - m_2$, $s = j$, $a = m$, $c = j_1 - j_2$, $b = j_1 + j_2 + 1$, and ρ and d_n^2 are the weight function and the norm of the q-dual Hahn polynomials.

Now using the relation between the q-dual Hahn polynomials (57), we find another equivalent relation

$$(j_1m_1j_2m_2|jm)_q = \sqrt{\frac{\rho(s)\Delta x(s-1/2)}{d_{N-n-1}^2}}W_{N-n-1}^{-c}(x(s),a,b)_q\,, \tag{83}$$

where n, s, a, b, and c have the same values as before.

6.4. $6j$-Symbols for the $su_q(2)$ Algebra and q-Racah Polynomials

Now for the q-case, the relation between the q-analog of $6j$-symbols and the q-Racah polynomials $u_n^{\alpha\beta}(s,a,b)_q$ is given as follows:

$$(-1)^{j_1+j+j_{23}}\sqrt{2j_{12}+1}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q = \sqrt{\frac{\rho(s)}{d_n^2}}u_n^{\alpha\beta}(s,a,b)_q\,, \tag{84}$$

where $\rho(s)$ and d_n are the weight function and the norm of the q-Racah polynomials $u_n^{\alpha\beta}(s,a,b)_q$ and $n = j_{12)} - j_1 + j_2$, $x(s) = [s][s+1]$, $s = j_{23}$, $a = j_3 - j_2$, $b = j_2 + j_3 + 1$, $\alpha = j_1 - j_2 - j_3 + j$, and $\beta = j_1 - j_2 + j_3 - j$, and it is supposed that $j_1 - j_2 \geq |j_3 - j|$ and $j_3 - j_2 \leq |j_1 - j|$.

If we use the connection between the Racah polynomials $u_n^{\alpha\beta}(s,a,b)_q$ and the alternative Racah polynomials $\widetilde{u}_n^{\alpha\beta}(s,a,b)_q$ (71), we find another formula

$$(-1)^{j_{12}+j_3+j}\sqrt{2j_{12}+1}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q = \sqrt{\frac{\widetilde{\rho}(s)}{\widetilde{d}_n^2}}\widetilde{u}_n^{\alpha\beta}(s,a,b)_q\,, \tag{85}$$

where now $\widetilde{\rho}(s)$ and $\widetilde{d}_n$ are the weight function and the norm of the q-Racah polynomials $\widetilde{u}_n^{\alpha\beta}(s,a,b)_q$ and $n = j_1 + j_2 - j_{12}$, while s, a, b, α and β are as before. Moreover, the duality relation for the Racah polynomials is equivalent to the symmetry property of the q–$6j$-symbols

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q = \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12}\end{Bmatrix}_q\,. \tag{86}$$

7. Conclusions

In this paper, we show that for any sequence of orthogonal polynomials $(y_n)_n$ with a weight function supported on a finite set of points two sets of dual (to this family) polynomials $(z_k)_k$ can be obtained. These pairs were obtained explicitly for the Hahn and Racah polynomials as well as for their corresponding q-analogs, i.e., we have found explicitly the formulas connecting the dual set $z_k(t)$ with the starting family of polynomials $y_n(s)$

$$z_k(t) = D_{kn} y_n(s)\,. \tag{87}$$

Moreover, if we substitute the inverse of the last expression

$$y_n(s) = D^{-1}_{kn} z_k(t)$$

into the difference equation (SODE) (1) that the polynomials y_n satisfy, we recover the three-term recurrence relation (TTRR) (8). And vice versa, if we substitute it on the three-term recurrence relation (8) for the polynomials y_n, we obtain the difference equation (1) for the dual ones z_k. The same happens when we start with the formula (87). The SODE for the dual polynomials z_k becomes the TTRR for the y_n, whereas the TTRR of the the dual polynomials z_k transforms into the SODE for the starting family y_n.

In particular, using the obtained formulas connecting the different families discussed above, we can extend the group-theoretical-representation interpretation of the Clebsch–Gordan coefficients and the $6j$-symbols as well as their q-analogs.

To conclude this paper, let us mention that all formulas connecting the different families of orthogonal polynomials here, as well as the new expressions for the Clebsch–Gordan coefficients and the $6j$-symbols, can be obtained by using the Whipple's transformation or Sear's transformation for hypergeometric and basic hypergeometric series. Our main aim here, however, is to show that it can be done using the already classical theory of orthogonal polynomials of discrete variables developed in [7,8] in a completely equivalent way.

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