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On characterizations of classical polynomials

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Abstract

It is well known that the classical families of Jacobi, Laguerre, Hermite, and Bessel polynomials are characterized as eigenvectors of a second order linear differential operator with polynomial coefficients, Rodrigues formula, etc. In this paper we present a unified study of the classical discrete polynomials and q-polynomials of the q-Hahn tableau by using the difference calculus on linear-type lattices. We obtain in a straightforward way several characterization theorems for the classical discrete and q-polynomials of the "q-Hahn tableau". Finally, a detailed discussion of a characterization by Marcellán et al. is presented.

MSC: 33C45; 33D45

Keywords: Classical polynomials; q-Hahn tableau; Discrete polynomials; Characterization theorems

1. Introduction

The well known families of Hermite, Laguerre, Jacobi, and Bessel polynomials (usually called the classical orthogonal polynomials) are the most important instances of orthogonal polynomials. One of the reasons is because they satisfy not only a three-term recurrence relation (TTRR)

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad \gamma_n \neq 0, \quad P_{-1}(x) = 0, \quad P_0(x) = 1, \tag{1.1}$$

but also other useful properties: they are the eigenfunction of a second order linear differential operator with polynomial coefficients, their derivatives also constitute an orthogonal family, their generating functions can be given explicitly, among others (see for instances [1,8,24,25] or the more recent work [3]). Among all these properties there are very important ones that characterize these families of polynomials.

In fact not every property characterizes a sequence of classical orthogonal polynomials. The simplest example is the TTRR (1.1). It is well known (see e.g., [8]) that the TTRR characterizes the sequences of orthogonal polynomials if $\gamma_n \neq 0$ for all $n \in \mathbb{N}$. This is the so-called Favard Theorem (for a review see [18]). Nevertheless there exist several families that satisfy the TTRR but not a second order linear differential equation with polynomial coefficients, or a Rodrigues-type formula. In fact only the aforesaid families of orthogonal polynomials satisfy these properties.

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Table 1
The classical OPS

$P_n(x)$	Hermite $H_n(x)$	Laguerre $L_n^\alpha(x)$	Jacobi $P_n^{\alpha,\beta}(x)$	Bessel $B_n^\alpha(x)$
(a,b)	$\mathbb{R}$	$[0,\infty)$	$[-1,1]$	$\mathbb{T}:=\{\lvert z\rvert=1, z\in\mathbb{C}\}$
$\sigma(x)$	1	x	$1-x^2$	x^2
$\tau(x)$	$-2x$	$-x+\alpha+1$	$-(\alpha+\beta+2)x+\beta-\alpha$	$(\alpha+2)x+2$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$	$-n(n+\alpha+1)$
$\rho(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1+x)^\beta$	$\frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\Gamma(a+2)}{\Gamma(a+k+1)}\left(-\frac{2}{x}\right)^k$
		$\alpha>-1$	$\alpha,\beta>-1$	$\alpha>-2$

Definition 1.1. We say that the orthogonal polynomial sequence (OPS) (Table 1) $(P_n)_n$ is a classical OPS with respect to the weight function ρ if

$$\int_a^b P_n(x)P_m(x)\rho(x)\,dx=\delta_{mn}d_n^2, \tag{1.2}$$

where δ_{mn} is the Kronecker symbol $\delta_{mn}=1$ if $n=m$ and 0 otherwise, d_n is the norm of the polynomial P_n, ρ is the solution of the Pearson equation

$$[\sigma(x)\rho(x)]'=\tau(x)\rho(x), \tag{1.3}$$

where σ and τ are fixed polynomials of degrees at most 2 and exactly 1, respectively, such that the following boundary conditions hold[1]

$$\sigma(a)\rho(a)=\sigma(b)\rho(b)=0.$$

It can be shown (see e.g., [3,8,25]) that the only families satisfying the above definition are the Hermite, Laguerre, Jacobi, and Bessel polynomials. Nevertheless there are other properties characterizing such families and that can be used to define the classical OPS. The oldest one is the so called Hahn characterization—unless this was firstly observed and proved for the Jacobi, Laguerre, and Hermite polynomials by Sonin in 1887. In [12], Hahn proved the following,

Theorem 1.2 (*Sonin–Hahn [12,19]*). *A given sequence of orthogonal polynomials $(P_n)_n$, is a classical sequence if and only if the sequence of its derivatives $(P_n')_n$ is an orthogonal polynomial sequence.*

Furthermore, we have the following (see e.g., [1,8,19,20]).

Theorem 1.3. *Let $(P_n)_n$ be an OPS. The following statements are equivalent*:

(1) *$(P_n)_n$ is a classical orthogonal polynomial sequence* (*COPS*) (*Hildebrandt* [14]).
(2) *The sequence of its derivatives $(P_n')_{n\geqslant 1}$ is an OPS*[2] *with respect to the weight function $\rho_1(x)=\sigma(x)\rho(x)$, where ρ satisfies* (1.3).
(3) *$(P_n)_n$ satisfies the second order linear differential equation with polynomial coefficients* (*Bochner* [7])

$$\sigma(x)P_n''(x)+\tau(x)P_n'(x)+\lambda_nP_n(x)=0, \tag{1.4}$$

where $\deg(\sigma)\leqslant 2$, $\deg(\tau)=1$, *and are independent of n, and λ_n is a constant independent of x.*

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[1] These conditions follow from the fact that for the classical polynomials the moments $\mu_n=\int_a^b x^n\rho(x)\,dx$, $n\geqslant 0$, of the measure associated with $\rho(x)$ are be finite. Furthermore, here we will restrict ourselves to the Jacobi, Laguerre, Hermite cases. The Bessel case is a little more complicated and we should use a different boundary condition. For more details see e.g., [3,11].

[2] Notice that this is not the Hahn theorem (1.2). In the Hahn theorem the orthogonality of both sequences is imposed whereas here a more restrictive condition is supposed.

(4) $(P_n)_n$ *can be expressed by the Rodrigues formula* (*Tricomi* [28] *and Cryer* [9])

$$P_n(x)=\frac{B_n}{\rho(x)}\frac{d^n}{dx^n}[\sigma^n(x)\rho(x)].$$

(5) *There exist three sequences of complex numbers* $(a_n)_n$, $(b_n)_n$, $(c_n)_n$, *and a polynomial* σ, $\deg(\sigma)\leqslant 2$, *such that* [2]

$$\sigma(x)P_n'(x)=a_nP_{n+1}(x)+b_nP_n(x)+c_nP_{n-1}(x),\quad n\geqslant 1. \tag{1.5}$$

(6) *There exist two sequences of complex numbers* $(f_n)_n$ *and* $(g_n)_n$ *such that the following relation for the monic polynomials holds* (*Marcellán et al.* [19])

$$P_n(x)=\frac{P_{n+1}'(x)}{n+1}+f_nP_n'(x)+g_nP_{n-1}'(x),\quad g_n\neq\gamma_n,\quad n\geqslant 1, \tag{1.6}$$

where γ_n *is the corresponding coefficient of the TTRR* (1.1).

For the sake of completeness we include a proof of this theorem in the Appendix A.

Remark 1.4. Notice that the only orthogonal polynomial solutions of the differential equation (1.4) are the Jacobi, Laguerre, Hermite, and Bessel polynomials (see e.g., [3,7,8,25]). Therefore, the statements of Theorems 1.2 and 1.3 are equivalent, i.e., they both characterize the same families of orthogonal polynomials.

A natural extension of the classical polynomials are the so-called discrete polynomials (those of Charlier, Meixner, Kravchuk, and Hahn, see e.g., [8,24,25]) and the q-polynomials (see e.g., [6,24,25]). In this respect, Hahn in 1949 [13] posed the problem of finding all the orthogonal polynomial sequences satisfying the Theorem 1.2 and the conditions 3 and 4 (among others) in Theorem 1.3 but instead of the derivatives, the linear operator $L_{q,w}$

$$L_{q,w}f(x)=\frac{f(qx+w)-f(x)}{(q-1)x+w},\quad q,w\in\mathbb{R}^+, \tag{1.7}$$

is considered. Hahn solved the problem for the case when $q\in(0,1)$ and $w=0$, that leads to the "q-Hahn tableau" (see e.g., [16,5]). In this case $L_{q,0}=\mathscr{D}_q$, where

$$\mathscr{D}_\varsigma P(x)=\frac{P(\varsigma x)-P(x)}{x(\varsigma-1)},\quad \varsigma\neq 0,\pm 1, \tag{1.8}$$

is the ς-Jackson derivative. In particular, Hahn found the more general sequence of orthogonal polynomials $(P_n)_n$ such that the sequence of its q-derivatives $(\mathscr{D}_qP_n)_n$ was also an OPS, the so-called big q-Jacobi polynomials (see e.g., [15,16], and Section 4.1), and proved that they satisfy a second order difference equation of the form (here we use the equivalent equation obtained in [23])

$$\sigma(x)\mathscr{D}_q\mathscr{D}_{1/q}P_n(x)+\tau(x)\mathscr{D}_qP_n(x)+\lambda_nP_n(x)=0, \tag{1.9}$$

where σ and τ are polynomials independent of n, $\deg(\sigma)\leqslant 2$, $\deg(\tau)=1$, and λ_n is a constant independent of x. Let us point out that the other polynomial solutions of (1.9) can be obtained from the big q-Jacobi polynomials via certain limit processes (see e.g., [5,6,15,26]). The properties of the corresponding families of these q-polynomials can be found in [5,15,23].

After the work by Hahn, the study of such polynomials has known an increasing interest (for recent reviews see [3,6,15]). Indeed the first systematic approach for q-polynomials comes from the fact that they are basic (terminating) hypergeometric series

$${}_r\varphi_p\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_p\end{matrix};q,z\right)=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\frac{z^k}{(q;q)_k}[(-1)^kq^{(k(k-1))/2}]^{p-r+1}, \tag{1.10}$$

being $(a;q)_k=\prod_{m=0}^{k-1}(1-aq^m)$, $(a;q)_0:=1$, the q-shifted factorial. For a complete set of references on this see [6,15]. Another point of view was developed by the Russian (former Soviet) school of mathematicians starting from a work by

Nikiforov and Uvarov in 1983. It was based on the idea that the q-polynomials are the solution of a second-order linear difference equation with certain properties: the so-called difference equation of hypergeometric type on non-uniform lattices [24,25]

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n[x(s)]}{\nabla x(s)}+\tau(s)\frac{\Delta P_n[x(s)]}{\Delta x(s)}+\lambda_n P_n[x(s)]=0, \tag{1.11}$$

$$\sigma(s)=\tilde{\sigma}(x(s))-\tfrac{1}{2}\tau(s)\Delta x(s-\tfrac{1}{2}),\quad x(s)=c_1(q)q^s+c_2(q)q^{-s}+c_3(q),$$

where $\tilde{\sigma}$ and τ are polynomials in $x(s)$ of degrees at most 2 and 1, respectively, and λ_n is a constant independent of s. Here $\Delta f(s)=f(s+1)-f(s)$ and $\nabla f(s)=f(s)-f(s-1)$ denote the forward and backward difference operators, respectively. One of the properties of the above equation is that its polynomial solutions can be expressed as basic hypergeometric series. In particular, when the lattice function is $x(s)=q^s$ it becomes into the Hahn q-difference equation (1.9). This approach based on the difference equation is usually called the Nikiforov–Uvarov scheme of q-polynomials [26] (for more details see e.g., [3,6,24,27]).

The case $w=q=1$ in (1.7), leads to the classical discrete polynomials of Charlier, Meixner, Kravchuk, and Hahn (see [8,17,24,25]). In particular, these families of discrete polynomials satisfy the difference equation (1.11) in the linear lattice $x(s)=s$, i.e.,

$$\sigma(s)\Delta\nabla P_n(s)+\tau(s)\Delta P_n(s)+\lambda_n P_n(s)=0,$$

where $\deg(\sigma)\leqslant 2$, $\deg(\tau)=1$, and are independent of n, and λ_n is a constant independent of s. They are orthogonal on the integers in $[a,b-1]$ with respect to the weight function $\rho(s)$, i.e.,

$$\sum_{x=a}^{b-1}P_n(s)P_m(s)\rho(s)=\delta_{mn}d_n^2,$$

provided that the boundary condition $\sigma(s)\rho(s)s^k|_{s=a,b}=0$, for all $k\geqslant 0$, holds, where d_n is the norm of the polynomial $P_n(s)$, and ρ satisfies the Pearson type equation $\Delta[\sigma(s)\rho(s)]=\tau(s)\rho(s)$.

A complete study of the characterization theorems for these two cases has been performed using a functional approach in the papers [10] (discrete case) and [23] ("q" case). The main aim of the present paper is twofold: on the one hand to present a simple and unified approach to the aforesaid two cases using the theory of difference equations on lattices presented in [24,25], and on the other hand to complete the study started in [10,20,23].

The structure of the paper is as follows: In Section 2 we introduce the "linear" lattices $x(s)$ and characterize them. In Section 3 the characterization theorem is presented and proved for any linear-type lattice and, as a direct consequence, the corresponding theorems for the uniform lattice $x(s)=s$ and the q-linear lattice $x(s)=c_1q^s+c_2$ are obtained. Finally, in Section 4, we discuss each case in details as well as the classical case (that can be obtained taking an appropriate limit $q\to 1^-$). In particular, some problems related to the characterization by Marcellán et al. [19] are discussed.

2. The linear-type lattices $x(s)$

Definition 2.1. A complex function $x(z)$ of the complex variable z is said to be a linear-type lattice if

$$x(z+\zeta)=F(\zeta)x(z)+G(\zeta)\quad\forall z,\zeta\in\mathbb{C},\quad F(\zeta)\neq 0, \tag{2.1}$$

being F and G two complex functions independent of z.

Obviously for the linear lattice $x(s)=s$ we have $F(\zeta)=1$ and $G(\zeta)=\zeta$. Another important instance of the linear-type lattice is the q-linear lattice, ($q\neq\{0,\pm 1\}$), i.e., the functions of the form $x(s)=Aq^s+B$, A, B some constants. In this case $x(s+\zeta)=F(\zeta)x(s)+G(\zeta)$, where $F(\zeta)=q^\zeta$ and $G(\zeta)=B(1-q^\zeta)$.

Proposition 2.2. *Let $q\in\mathbb{C}$, $q\neq\{0,\pm 1\}$. The function $x(z)$ is a q-linear lattice of z if and only if it satisfies $x(z+1)=qx(z)+C$, where C is a complex number.*

Proof. A straightforward computation shows that if $x(z)$ is a q-linear function of z, i.e., $x(z)=cq^z+d$ then it satisfies the recurrence formula $x(z+1)=qx(z)+C$, where $C=d(1-q)$ is a constant. But the general solution of the difference equation $x(z+1)=qx(z)+C$ is $x(z)=Aq^z+D$, where A and D are, in general, non-zero constants. □

Notice that for the linear-type lattices, if $Q_m(x(s))$ is a polynomial of degree m in $x(s)$, $Q_m(x(s+\alpha))$ is also a polynomial of degree m in $x(s)$, i.e., $Q_m(x(s+\alpha))=\widetilde{Q}_m(x(s))$. Moreover, for the linear-type lattices we have the following.

Lemma 2.3. *Let $x(s)$ be a linear-type lattice and $Q_m(x(s))$ a polynomial of degree m in $x(s)$. Then*

$$\frac{\Delta Q_m(x(s+\alpha))}{\Delta x(s+\beta)}=R_{m-1}(x(s)) \quad \forall \alpha,\beta\in\mathbb{C},$$

where $R_{m-1}(x(s))$ is again a polynomial in $x(s)$ but of degree $m-1$.

Proof. It is sufficient to prove the lemma for the powers $x^n(s)$. Since $x(s)$ is a linear-type lattice

$$\frac{\Delta x^n(s+\alpha)}{\Delta x(s+\beta)}=\frac{\Delta(F(\alpha)x(s)+G(\alpha))^n}{F(\beta)\Delta x(s)}=\sum_{k=0}^{n}\binom{n}{k}\frac{F(\alpha)^kG(\alpha)^{n-k}}{F(\beta)}\frac{\Delta x^k(s)}{\Delta x(s)}.$$

But $\Delta x^k(s)/\Delta x(s)$ is a polynomial of degree $k-1$ in $x(s)$ and therefore $\Delta x^n(s+\alpha)/\Delta x(s+\beta)$ also is. □

Remark 2.4. From Proposition 2.2 and Definition 2.1 it follows that the only linear-type lattices are those corresponding to $F(1)=1$ (the linear lattice $x(s)=C_1s+C_2$) and the ones when $F(1)=q$ for some $q\neq 0,\pm 1$ (the q-linear lattices $x(s)=c_1q^s+c_2$).

3. The characterization theorem for classical polynomials

In the sequel we will assume that $(P_n[x(s)])_n$ is an orthogonal polynomial sequence on a linear-type lattice $x(s)$ with respect to the weight function $\rho(s)$, i.e.,

$$\sum_{s=a}^{b-1}P_n(s)P_m(s)\rho(s)\nabla x_1(s)=\delta_{mn}d_n^2, \quad x_k(s)=x(s+k/2), \tag{3.1}$$

where δ_{mn} is the Kronecker symbol and d_n is the norm of the polynomials.

For the sake of simplicity we will denote $P_n(s):=P_n[x(s)]$. Since the polynomials $P_n(s)$, $n=0,1,2,\dots,$ are orthogonal they satisfy the TTRR

$$x(s)P_n(s)=\alpha_nP_{n+1}(s)+\beta_nP_n(s)+\gamma_nP_{n-1}(s), \quad P_{-1}(s)=0, \quad P_0(s)=1. \tag{3.2}$$

Let us point out that if $\gamma_n\neq 0$, for all $n\in\mathbb{N}$, then the above TTRR defines an OPS. Nevertheless there are several examples for which $\gamma_n=0$ for some $n_0\in\mathbb{N}$ (e.g., the Hahn and q-Hahn polynomials). In this case we have a finite family of orthogonal polynomials (see e.g., [8,25]). In the first case, i.e., when $\gamma_n\neq 0$, for all $n\in\mathbb{N}$ we say that it is a quasi-definite case [8] (also called the regular case) whereas in the second one, we get a weak-quasi-definite case or weak-regular case. Here we will assume that $(P_n)_n$ is a normal sequence, i.e., $\deg(P_n)=n$, and that $\gamma_n\neq 0$ for all $n\in\mathscr{N}$ where by $\mathscr{N}$ we denote either the set $\mathscr{N}=\{1,2,\dots,n_0\}$ for some $n_0\in\mathbb{N}$ or $\mathscr{N}=\mathbb{N}$.

Here we will use the notation of the theory of difference calculus on non-uniform lattices (for more details see [25, Section 13] or [24, Chapter 3]).

Let $s=a,a+1,a+2,\dots$. We will define the forward and backward differences in $x(s)$ by

$$\frac{\Delta y[x(s)]}{\Delta x(s)}, \quad \frac{\nabla y[x(s)]}{\nabla x(s)},$$

respectively, where $\nabla f(s)=f(s)-f(s-1)$, $\Delta f(s)=f(s+1)-f(s)$.

Table 2
Classical discrete OPS

P_n	Hahn $h_n^{\alpha,\beta}(s;N)$	Meixner $M_n^{\gamma,\mu}(s)$	Kravchuk $K_n^p(s)$	Charlier $C_n^\mu(s)$
$[a,b]$	$[0,N]$	$[0,\infty)$	$[0,N+1]$	$[0,\infty)$
σ	$s(N+\alpha-s)$	s	s	s
τ	$(\beta+1)(N-1)-(\alpha+\beta+2)s$	$(\mu-1)s+\mu\gamma$	$\frac{Np-s}{1-p}$	$\mu-s$
λ_n	$n(n+\alpha+\beta+1)$	$(1-\mu)n$	$\frac{n}{1-p}$	n
ρ	$\frac{\Gamma(N+\alpha-s)\Gamma(\beta+s+1)}{\Gamma(N-s)\Gamma(s+1)}$ $\alpha,\beta\geqslant-1, n\leqslant N-1$	$\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(\gamma)\Gamma(s+1)}$ $\gamma>0, 0<\mu<1$	$\frac{N!p^s(1-p)^{N-s}}{\Gamma(N+1-s)\Gamma(s+1)}$ $0<p<1, n\leqslant N-1$	$\frac{e^{-\mu}\mu^s}{\Gamma(s+1)}$ $\mu>0$

For the operator Δ we have

$$\Delta\{f(s)g(s)\}=g(s)\{\Delta f(s)\}+f(s+1)\{\Delta g(s)\}. \tag{3.3}$$

Thus, the following formula of summation by parts holds

$$\sum_{s=a}^{b} f(s)\Delta g(s)=f(s)g(s)\Big|_a^{b+1}-\sum_{s=a}^{b}(\Delta f(s))g(s+1). \tag{3.4}$$

Also we define the kth forward difference of a function $f(s)$ by

$$\Delta^{(k)}f(s):=\frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)}f(s), \quad x_m(s)=x\left(s+\frac{m}{2}\right).$$

Remark 3.1. Notice that the differences $\Delta^{(k)}P_n(s)$ can be written in the linear-type lattice, up to a constant factor, as $(\Delta/\Delta x(s))^k P_n(s)$. Moreover, the operator $\Delta/\Delta x(s)$ for the q-linear lattice $x(s)=c_1q^s$ becomes the Jackson q-derivative $\mathscr{D}_q$ (1.8).

Definition 3.2. We say that an OPS $(P_n)_n$ is a classical OPS on the linear-type lattice if they satisfy (3.1) where ρ is the solution of the Pearson-type equation

$$\Delta[\sigma(s)\rho(s)]=\tau(s)\rho(s)\nabla x_1(s), \tag{3.5}$$

and σ and τ are fixed polynomials on $x(s)$ of degrees at most 2 and exactly 1, respectively, and such that the following boundary conditions hold

$$x^k(a)\sigma(a)\rho(a)=x^k(b)\sigma(b)\rho(b)=0 \quad \forall k=0,1,2,\ldots. \tag{3.6}$$

Next we state the Hahn–Lesky theorem.

Theorem 3.3. *A given sequence of orthogonal polynomials $(P_n)_n$, is a classical sequence if and only if*

- *The sequence of its differences $(\Delta P_n)_n$ is an OPS* [10,17].
- *The sequence of its q-differences $(\mathscr{D}_q P_n)_n$ is an OPS* [13,23].

Let us remember here that the first case lead to the classical discrete polynomials (see Table 2), whereas the second one, leads to the "q-Hahn tableau".

Notice that since we are dealing with linear lattices the statement of the theorem can be replaced by the following equivalent one:

Theorem 3.3. *A sequence of orthogonal polynomials $(P_n)_n$ is classical if and only if the sequence of their finite differences $([\Delta/\Delta x(s)]P_n)_n$ is an OPS, $x(s)$ being a linear-type lattice.*

The standard proof of this theorem can be found in [17] for the linear lattice $x(s)=s$, and in [10] using the functional technique developed by Maroni. For the q-linear lattice $x(s)=q^s$ it has been done by Hahn in [13] and using a functional approach in [23].

We are now in a position to state the main result of the paper.

Theorem 3.4. *Let $(P_n)_n$ be an OPS on a linear-type lattice $x(s)$ satisfying* (3.1) *and let $\sigma(s)$ and $\rho(s)$ be two functions such that* (3.6) *holds. Then, the following statements are equivalent*

(1) *$(P_n)_n$ is a classical OPS.*

(2) *The sequence of its differences $(\Delta P_n/\Delta x(s))_n$ also is an OPS with respect to the weight function $\rho_1(s) = \sigma(s+1)\rho(s+1)$, where ρ satisfy* (3.5).

(3) *$(P_n)_n$ satisfies the second order linear difference equation with polynomial coefficients*

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta P_n(s)}{\Delta x(s)} + \lambda_n P_n(s) = 0, \tag{3.7}$$

where $\deg(\sigma)\leqslant 2$, $\deg(\tau)=1$, *are independent of n and λ_n is a constant independent of x.*

(4) *$(P_n)_n$ can be expressed by the Rodrigues-type formula*

$$P_n(s) = \frac{B_n}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}[\rho_n(s)], \tag{3.8}$$

where $\rho_n(s) = \rho(s+n)\prod_{m=1}^{n}\sigma(s+m)$ and B_n is a constant non-equal to 0.

(5) *There exist three sequences of complex numbers $(a_n)_n$, $(b_n)_n$, $(c_n)_n$, and a polynomial ϕ,* $\deg(\phi)\leqslant 2$, *such that*

$$\phi(x)\frac{\Delta P_n(s)}{\Delta x(s)} = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n\geqslant 1. \tag{3.9}$$

(6) *There exist three sequences of complex numbers $(e_n)_n$, $(f_n)_n$, $(g_n)_n$ such that the following relation holds for all $n\geqslant 1$*

$$P_n(x) = e_n\frac{\Delta P_{n+1}(s)}{\Delta x(s)} + f_n\frac{\Delta P_n(s)}{\Delta x(s)} + g_n\frac{\Delta P_{n-1}(s)}{\Delta x(s)}, \tag{3.10}$$

where $e_n \neq 0$, $g_n \neq \gamma_n$, for all $n \in \mathcal{N}$, and γ_n is the corresponding coefficient of the TTRR (1.1).

As a simple consequence of the above theorem we have the following.

Corollary 3.5 (*García et al. [10], Medem et al. [23]*). *The discrete polynomials on the linear lattice $x(s)=s$ are classical. The q-polynomials in the q-linear lattice* (*or exponential lattice*) *$x(s)=c_1q^s+c_2$ are classical.*

Proof. It follows from the fact that $x(s)=s$ and $x(s)=c_1q^s+c_2$ are linear-type lattices. □

Remark 3.6. Notice that the more general orthogonal polynomial solutions of the difference equation (3.7) are the big q-Jacobi polynomials for the q-linear lattice and the Hahn polynomials for the linear one (see e.g., [3,6,13,24]), then the statement (3) of Theorem 3.4 and the Hahn–Lesky Theorem 3.3 are equivalent, i.e., they both characterize the same families of orthogonal polynomials.

Let us prove the Theorem 3.4. We start proving that (1) → (2):

Proposition 3.7 ((1) → (2)). *Let $x(s)$ be a linear-type lattice and let $(P_n)_n$ be a classical OPS orthogonal, i.e., an OPS with respect to a weight function ρ, solution of the Pearson-type equation* (3.5) *with the boundary conditions*[3] (3.6). *Then the sequence $(\Delta P_n(s)/\Delta x(s))_n$ is also a classical OPS with respect to the weight function $\rho_1(s)=\sigma(s+1)\rho(s+1)$.*

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[3] This condition leads to the so-called discrete orthogonal polynomials, i.e., polynomials with a discrete orthogonality of the form (3.1). For the q-linear lattices (3.1) becomes into the q-Jackson integral (see e.g., [5,15,16]). For the continuous orthogonality see [24, Section 3.10].

Proof. Let $Q_k(s)$ be an arbitrary kth degree polynomial on $x(s)$, $k<n$. The orthogonality conditions for $(P_n)_n$ yield, for all $k<n$,

$$\begin{aligned} 0 &= \sum_{s=a}^{b-1} P_n(s)Q_{k-1}(s)\tau(s)\rho(s)\nabla x_1(s) \quad \text{(from (3.5))} \\ &= \sum_{s=a}^{b-1} P_n(s)Q_{k-1}(s)\Delta(\sigma(s)\rho(s)) \quad \text{(from (3.4) and (3.6))} \\ &= -\sum_{s=a}^{b-1} \Delta(P_n(s)Q_{k-1}(s))\sigma(s+1)\rho(s+1). \end{aligned}$$

Applying the Leibniz rule (3.3)

$$\begin{aligned} 0 &= -\sum_{s=a}^{b-1}(\Delta P_n(s))Q_{k-1}(s)\sigma(s+1)\rho(s+1) \\ &\quad + \sum_{s=a}^{b-1} P_n(s+1)(\Delta Q_{k-1}(s))\sigma(s+1)\rho(s+1) \quad (s\to s-1,\ \text{and (3.6)}) \\ &= -\sum_{s=a}^{b-2}\left(\frac{\Delta P_n(s)}{\Delta x(s)}\right) Q_{k-1}(s)\sigma(s+1)\rho(s+1)\nabla x_1(s+\tfrac{1}{2}) \\ &\quad + \sum_{s=a+1}^{b} P_n(s)\left(\frac{\Delta Q_{k-1}(s-1)}{\Delta x(s-1/2)}\right)\sigma(s)\rho(s)\nabla x_1(s). \end{aligned}$$

Next, we use Lemma 2.3 as well as the conditions (3.6), then

$$\begin{aligned} 0 &= -\sum_{s=a}^{b-2}\left(\frac{\Delta P_n(s)}{\Delta x(s)}\right) Q_{k-1}(s)\sigma(s+1)\rho(s+1)\nabla x_1(s+\tfrac{1}{2}) \\ &\quad + \sum_{s=a}^{b-1} P_n(s)\underbrace{(R_{k-2}(s))\sigma(s)}_{\text{degree}\leqslant n}\rho(s)\nabla x_1(s) \quad \text{(from (3.6) and (3.1))} \\ &= -\sum_{s=a}^{b-2}\left(\frac{\Delta P_n(s)}{\Delta x(s)}\right) Q_{k-1}(s)\sigma(s+1)\rho(s+1)\nabla x_1(s). \end{aligned}$$

Thus, $\Delta P_n(s)/\Delta x(s)$ is orthogonal with respect to $\rho_1(s)\nabla x_1(s+\frac{1}{2})=\sigma(s+1)\rho(s+1)\Delta x(s)$. We only need now to prove that $\Delta^{(1)}P_n(s)$ is a classical OPS. For doing this notice that the weight function $\rho_1(s)$ satisfy the Pearson type equation (see e.g., [24, Section 3.2.2])

$$\frac{\Delta}{\Delta x_1(s-1/2)}[\sigma(s)\rho_1(s)] = \tau_1(s)\rho_1(s),$$

where τ_1 is a first degree polynomial on $x(s)$ given by

$$\tau_1(s) = \frac{\sigma(s+1)-\sigma(s)+\tau(s+1)\Delta x_1(s)}{\Delta x(s)}.$$

Thus, ρ_1 satisfies a difference equation of the form (3.5). This completes the proof. □

In the same way, using induction we have:

Corollary 3.8. *Let $x(s)$ be a linear-type lattice and let $(P_n)_n$ be a classical OPS. Then, the sequence of their kth finite differences $\Delta^{(k)}P_n(s)$, where*

$$\Delta^{(k)} := \frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)},$$

also is a classical OPS orthogonal with respect to the weight function $\rho_k(s)=\rho(s+k)\prod_{m=1}^{k}\sigma(s+m)$.

Proposition 3.9 ((2) → (3)). *Let $x(s)$ be a linear-type lattice and $(P_n)_n$ a sequence of polynomials. If the sequences $(\Delta^{(1)}P_n)_n$ is orthogonal with respect to the function $\rho_1(s)\Delta x(s)$, where $\rho_1(s)=\sigma(s+1)\rho(s+1)$, and $\rho(s)$ satisfy* (3.5), *then $(P_n)_n$ satisfies the second order linear difference equation of hypergeometric type* (3.7).

Proof. Let $k<n$. Then, using the orthogonality of $\Delta^{(1)}P_n$,

$$\begin{aligned}
0 &= \sum_{s=a}^{b-2}\frac{\Delta P_n(s)}{\Delta x(s)}\frac{\Delta Q_k(s)}{\Delta x(s)}\sigma(s+1)\rho(s+1)\nabla x_1(s+\tfrac{1}{2}) \quad \text{(from (3.6))}\\
&= \sum_{s=a}^{b-1}\frac{\Delta P_n(s)}{\Delta x(s)}\Delta Q_k(s)\sigma(s+1)\rho(s+1) \quad \text{(from (3.4) and (3.6))}\\
&= -\sum_{s=a}^{b-1}Q_k(s)\Delta\left(\frac{\Delta P_n(s-1)}{\Delta x(s-1)}\sigma(s)\rho(s)\right) \quad (\Delta f(s)=\nabla f(s+1))\\
&= -\sum_{s=a}^{b-1}Q_k(s)\Delta\left(\frac{\nabla P_n(s)}{\nabla x(s)}\sigma(s)\rho(s)\right) \quad \text{(from (3.3))}\\
&= -\sum_{s=a}^{b-1}Q_k(s)\left(\sigma(s)\rho(s)\Delta\frac{\nabla P_n(s)}{\nabla x(s)}+\frac{\nabla P_n(s+1)}{\nabla x(s+1)}\Delta[\sigma(s)\rho(s)]\right) \quad \text{(from (3.5))}\\
&= -\sum_{s=a}^{b-1}Q_k(s)\left(\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n(s)}{\nabla x(s)}+\tau(s)\frac{\Delta P_n(s)}{\Delta x(s)}\right)\rho(s)\nabla x_1(s).
\end{aligned}$$

But, since the lattice $x(s)$ is of the linear type,

$$Q(s) := \sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n(s)}{\nabla x(s)}+\tau(s)\frac{\Delta P_n(s)}{\Delta x(s)}$$

is a polynomial of degree n in $x(s)$. Therefore, it should be, up to a constant factor (in general depending on n) the polynomial $P_n(s)$. Thus $Q(s)=-\lambda_n P_n(s)$ where λ_n is independent of s. □

Remark 3.10. The proof of the last proposition in the linear lattice $x(s)=s$ can be found in the first Russian edition of the book [25].

(3) → (4): The last proposition is very important because it gives a very simple method for finding the classical polynomials on the linear-type lattice: solving the difference equation (3.7). In fact, it was the key in the proofs of Theorem 3.3 (see Remark 3.6). The solutions of (3.7) have been extensively studied (see e.g., [6,24,25]). In particular, they can be written by the Rodrigues-type formula (3.8) [24,25], so (3) → (4). Let us mention that from the

Rodrigues-type formula (3.8) one can obtain an explicit expression for the classical polynomials in terms of the hypergeometric or basic hypergeometric series as it is shown in several previous works (see e.g., [3,6,24]).

(4) → (1): Another consequence of the Rodrigues formula is the following: By setting $n=1$ in (3.8) we obtain

$$P_1(s)=\frac{B_1}{\rho(s)}\frac{\Delta}{\nabla x_1(s)}[\sigma(s)\rho(s)] \quad\Rightarrow\quad \Delta[\sigma(s)\rho(s)]=\rho(s)\tau(s)\nabla x_1(s),$$

i.e., the Pearson-type equation (3.5) thus (4) → (1).

Remark 3.11. Notice that from the above results the equivalence of (1)–(4) in Theorem 3.4 follows. Moreover, since Remark 3.6, the statement (1)–(4) are equivalent to the Hahn–Lesky Theorem 3.3.

Proposition 3.12 ((5) → (1)). *Let $x(s)$ be a linear-type lattice and $\phi(s)$ a polynomial such that* $\deg(\phi)\leqslant 2$. *If $(P_n)_n$ is an OPS and there exist three sequences of complex numbers $(a_n)_n$, $(b_n)_n$, and $(c_n)_n$, such that* (3.9) *holds,*

$$\phi(x)\frac{\Delta P_n(s)}{\Delta x(s)}=a_nP_{n+1}(s)+b_nP_n(s)+c_nP_{n-1}(s)\quad \forall n\in\mathscr{N}$$

then $(P_n)_n$ is a classical OPS.

Proof. We start computing the following sum for all $k<n-1$

$$\begin{aligned}
&\sum_{s=a}^{b-1}Q_k(s)\frac{\Delta P_n(s)}{\Delta x(s)}\phi(s)\rho(s)\Delta x(s)\\
&\quad=\sum_{s=a}^{b-1}Q_k(s)[a_nP_{n+1}(s)+b_nP_n(s)+c_nP_{n-1}(s)]\rho(s)\Delta x(s)\\
&\quad=F\left(-\frac{1}{2}\right)\sum_{s=a}^{b-1}Q_k(s)[a_nP_{n+1}(s)+b_nP_n(s)+c_nP_{n-1}(s)]\rho(s)\nabla x_1(s)=0.
\end{aligned}$$

Therefore, the sequence $(\Delta P_n(s)/\Delta x(s))_n$ is an OPS, and then by Theorem 3.3 and Remark 3.11 P_n is a classical OPS. □

Remark 3.13. From the above proposition it follows that $\phi(s)\rho(s)=\rho_1(s)=\sigma(s+1)\rho(s+1)$. Therefore, comparison with the Pearson-type equation leads to the expression $\phi(s)=\sigma(s)+\tau(s)\nabla x_1(s)$. Notice also that since $(P_n)_n$ is an OPS then, the relation (3.9), the so-called structure relation of Al-Salam and Chihara type, is equivalent to the following relations (I is the identity operator)

$$L_nP_n(x):=\left(\phi(x)\frac{\Delta}{\Delta x(s)}+\psi_1(x,n)I\right)P_n(x)=\widetilde{c}_nP_{n-1}(x),\quad \deg(\psi_1)=1,$$

$$R_nP_n(x):=\left(\phi(x)\frac{\Delta}{\Delta x(s)}+\psi_2(x,n)I\right)P_n(x)=\widetilde{a}_nP_{n+1}(x),\quad \deg(\psi_2)=1.$$

The operators L_n and R_n are usually called the lowering and raising operators for the polynomial family $(P_n)_n$.

Proposition 3.14 ((6) → (1)). *Let $x(s)$ be a linear-type lattice. If $(P_n)_n$ is a monic OPS and there exist three sequences of complex numbers $(e_n)_n$, $(f_n)_n$, and $(g_n)_n$, $e_n\neq 0$, $g_n\neq\gamma_n$, $\forall n\in\mathscr{N}$, such that* (3.10) *holds, i.e.,*

$$P_n(x)=e_n\frac{\Delta P_{n+1}(s)}{\Delta x(s)}+f_n\frac{\Delta P_n(s)}{\Delta x(s)}+g_n\frac{\Delta P_{n-1}(s)}{\Delta x(s)},$$

then $(P_n)_n$ is a classical OPS.

Proof. For a sake of simplicity we will suppose that $(P_n)_n$ is a monic sequence. Since $(P_n)_n$ is an OPS they satisfy a TTRR (3.2). Taking the difference to both sides of (3.2), using (3.3) as well as the linearity property (2.1) we get

$$P_n(s) + [F(1)x(s) + G(1)]\frac{\Delta P_n(s)}{\Delta x(s)} = \frac{\Delta P_{n+1}(s)}{\Delta x(s)} + \beta_n \frac{\Delta P_n(s)}{\Delta x(s)} + \gamma_n \frac{\Delta P_{n-1}(s)}{\Delta x(s)}.$$

Then, substituting the value of $P_n(s)$ from (3.10) we find

$$F(1)x(s)\frac{\Delta P_n(s)}{\Delta x(s)} = (1-e_n)\frac{\Delta P_{n+1}(s)}{\Delta x(s)} + (\beta_n - G(1) - f_n)\frac{\Delta P_n(s)}{\Delta x(s)} + (\gamma_n - g_n)\frac{\Delta P_{n-1}(s)}{\Delta x(s)}.$$

If $g_n \neq \gamma_n$, $\forall n \in \mathcal{N}$, then from the Favard theorem (see e.g., [8]) the sequence $(\Delta P_n(s)/\Delta x(s))_n$ is an OPS, and therefore by Theorem 3.3 and Remark 3.11 P_n is a classical OPS. □

To conclude the proof we should show that if $(P_n)_n$ is a classical OPS, then (3.9) and (3.10) takes place. The first one follows directly from the Rodrigues-type formula as it is shown in [3,4] so (4) → (5), and the second one follows from the first one, i.e., (5) → (6) (see [3,4]). For the sake of completeness we will present it here and alternative proof for the second case taken from [3] (the first relation can be proven using the same ideas and we leave it as an exercise to the reader). In fact we will show that (2) → (6).

Let be $Q_n(s) = \Delta P_{n+1}(s)/\Delta x(s)$. Using the linearity of $x(s)$ we have $P_n(s) = \sum_{k=0}^{n} c_{n,k} Q_k(s)$. Since $(Q_n)_n$ is a classical OPS

$$c_{n,k} = \frac{\left(\sum_{s=a}^{b-2} P_n(s) Q_k(s)\rho_1(s)\Delta x(s)\right)}{d_{1\,k}^2}, \quad \rho_1(s) = \rho(s+1)\sigma(s+1),$$

where $d_{1\,k}^2$ is the square of the norm of Q_k. Using the condition (3.6) the numerator becomes

$$\begin{aligned}
\sum_{s=a-1}^{b-2} P_n(s)Q_k(s)\rho_1(s)\Delta x(s) &= \sum_{s=a-1}^{b-2} P_n(s)\Delta[P_{k+1}(s)]\rho_1(s) \\
&= P_n(s)P_{k+1}(s)\rho_1(s)\Big|_{a-1}^{b-1} - \sum_{s=a-1}^{b-2} P_{k+1}(s+1)\Delta[P_n(s)\rho_1(s)] \\
&= -\sum_{s=a-1}^{b-2} P_{k+1}(s+1)P_n(s+1)\Delta[\rho_1(s)] - \sum_{s=a-1}^{b-2} P_{k+1}(s+1)\Delta[P_n(s)]\rho_1(s) \\
&= -\sum_{s=a}^{b-1} P_{k+1}(s)P_n(s)\tau(s)\rho(s)\nabla x_1(s) - \sum_{s=a}^{b-2} P_{k+1}(s+1)\frac{\Delta P_n(s)}{\Delta x(s)}\rho_1(s)\Delta x(s),
\end{aligned}$$

where we use the condition (3.6), the formula (3.3) as well as the Pearson-type equation (3.5). Now, from the orthogonality of the polynomials P_n we conclude that the first sum vanishes for all $k < n-2$. But the second one also vanishes for all $k < n-2$ since $\Delta P_n(s)/\Delta x(s)$ is an orthogonal sequence with respect to $\rho_1(s)\Delta x(s)$ and $P_{k+1}(s+1)$ is a polynomial of degree $k+1$ in $x(s)$. This completes the proof of Theorem 3.4. □

Remark 3.15. Notice that if we consider monic polynomials, then for the linear lattice $x(s)$, $e_n = 1/(n+1) \neq 0$ and $F(1) = 1$ and for the q-linear one $e_n = (1-q)/(1-q^{n+1}) \neq 0$ and $F(1) = q$.

It is important to notice that in the proof of Proposition 3.12 there is not any restriction on c_n but for the classical “continuous”, discrete and q cases the condition $c_n \neq 0$ was imposed (see e.g., [10,19,23]). A similar situation happens in the proof of the Proposition 3.14, in the same aforesaid papers the condition $g_n \neq 0$ is imposed. Nevertheless, we see from the proof presented here that a more restricted condition should be imposed: $g_n \neq \gamma_n$. Notice that since $\gamma_n \neq 0$

(by Favard theorem) the last condition implies the first one $\gamma_n \neq 0$. In the next section we will discuss what happens if these conditions are not fulfilled.

4. The classical polynomials: further discussion

4.1. The q-linear lattices: The "q-Hahn tableau"

Here we will discuss the q-case. The classical case follows from the limit $q \to 1^-$. For the sake of simplicity and without loss of generality we will consider the most simple q-lattice $x(s) = q^s$. Hereafter, we will use the notation

$$\mathscr{D}_q P(x) = \frac{\Delta P(s)}{\Delta x(s)}, \quad \mathscr{D}_{1/q} P(x) = \frac{\nabla P(s)}{\nabla x(s)}, \quad x(s) := x = q^s,$$

where $\mathscr{D}_\varsigma$ denotes, as before, the classical q-Jackson derivative (1.8). With this notation we have that (3.7), (3.9), and (3.10) become

$$\phi(x)\mathscr{D}_q P_n(x) - \sigma(x)\mathscr{D}_{1/q} P_n(x) - x(1-q)q^{-1/2}\lambda_n P_n(x) = 0, \quad x := q^s, \tag{4.1}$$

$$\phi(x)\mathscr{D}_q P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad x := q^s, \tag{4.2}$$

$$P_n(x) = \frac{1}{[n+1]_q}\mathscr{D}_q P_{n+1}(x) + f_n \mathscr{D}_q P_n(x) + g_n \mathscr{D}_q P_{n-1}(x), \quad x := q^s, \tag{4.3}$$

respectively, being $\phi(x) = \sigma(x) - q^{-1/2}\tau(x)x(1-q)$.

The general polynomial solution of (4.1) is [6,24].

$$P_n(s) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{s_1+s_2-\bar{s}_1-\bar{s}_2+n-1}, x\, q^{-\bar{s}_2} \\ q^{s_1-\bar{s}_2}, q^{s_2-\bar{s}_2} \end{matrix} \,\middle|\, q, q \right), \tag{4.4}$$

where the basic hypergeometric series ${}_3\varphi_2$ is defined by (1.10). It corresponds to the functions

$$\sigma(x) = C(x - q^{s_1})(x - q^{s_2}), \quad \phi(x) = C'(x - q^{\bar{s}_1})(x - q^{\bar{s}_2}), \quad C q^{s_1} q^{s_2} = C' q^{\bar{s}_1} q^{\bar{s}_2},$$

and the eigenvalues are given by

$$\lambda_n = -\frac{C\, q^{-n+(3/2)}}{c_1^2(1-q)^2}(1-q^n)(1-q^{s_1+s_2-\bar{s}_1-\bar{s}_2+n-1}).$$

In particular, choosing $\phi = aq(x-1)(bx-c)$ and $\sigma = q^{-1}(x-aq)(x-cq)$, we find

$$\lambda_n = -q^{-n+1/2}\frac{1-q^n}{1-q}\frac{1-abq^{n+1}}{1-q}$$

and we obtain the big q-Jacobi polynomials introduced by Hahn in [13], i.e.,

$$p_n(x; a, b, c; q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \,\middle|\, q; q \right).$$

In the particular case $c = q^{-N-1}$, the aforesaid q-Hahn polynomials $Q_n(x; a, b, N|q)$ are deduced.

Remark 4.1. The general solution of Eq. (4.1) defines the so-called "q-Hahn tableau" [16]. A detailed study of this class has been done in [5]. In particular, in [5] comparison with the q-analog of the Askey tableau [15] and the Nikiforov and Uvarov tableau [26] has been performed and all possible limit cases obtained from (4.4) have been analyzed, identifying them with several known classical families of q-polynomials as well as two new ones.

On the sequel we will use the notation introduced in [23]

$$\phi(x)=\widehat{a}x^2+\overline{a}x+\widetilde{a},\quad \psi(x):=q^{-1/2}\tau(s)=\widehat{b}x+\bar{b},\quad \widehat{b}\neq 0. \tag{4.5}$$

In the paper [23] the values of the coefficients of the TTRR (3.2), and the structure relations (4.2) and (4.3) have been obtained in terms of the coefficients of ϕ and ψ defined in (4.5). In particular,

$$\gamma_n=-\frac{q^{n-1}[n]_q([n-2]_q\widehat{a}+\widehat{b})}{([2n-1]_q\widehat{a}+\widehat{b})([2n-2]_q\widehat{a}+\widehat{b})^2([2n-3]_q\widehat{a}+\widehat{b})}$$
$$\times[q^{n-1}([n-1]_q\overline{a}+\overline{b})(q^{n-1}\widehat{a}\overline{b}-\overline{a}([n-1]_q\widehat{a}+\widehat{b}))+\widetilde{a}([2n-2]_q\widehat{a}+\widehat{b})^2],\quad n\geqslant 1, \tag{4.6}$$

$$c_n=-\frac{[n]_{q^{-1}}([n-1]_q\widehat{a}+\widehat{b})}{[n]_q}\gamma_n,\quad n\geqslant 1, \tag{4.7}$$

and

$$g_n=-\frac{q^{n-2}[n-1]_q\widehat{a}}{[n-2]_q\widehat{a}+\widehat{b}}\gamma_n,\quad n\geqslant 2, \tag{4.8}$$

where we use the standard notation for the q-numbers

$$[x]_\varsigma=\frac{\varsigma^x-1}{\varsigma-1}.$$

From the above relations it follows that if we want to have an infinite orthogonal polynomial sequence $(P_n)_{n\geqslant 0}$ (the so called *quasi-definite* or regular case) γ_n should be different from zero for all $n\geqslant 0$. But, as we already pointed out, there exist some examples when $\gamma_n=0$ for some n_0 (e.g., the q-Hahn and q-Kravchuk polynomials for $n=N+1$). In these cases we have a *finite* family of polynomials (strictly speaking this case does not constitute a regular case) that corresponds to a *weak-regular* case. Notice that from formula (4.6) it follows that the corresponding family exists, at least in the weak-regular sense, if the square bracket in (4.6) is different from zero and a sufficient condition is

$$[n]_q\widehat{a}+\widehat{b}\neq 0\quad \text{for } n\in\mathcal{N}. \tag{4.9}$$

The last condition is usually called the *admissibility condition* (for a detailed study of this condition see [21,22] and references therein). That this condition was necessary was established in [23].

Now, from the expression (4.7) and taking into account that $\gamma_n\neq 0$ for all $n\in\mathcal{N}$, the condition $c_n\neq 0$, for all $n\in\mathcal{N}$, follows. This condition is equivalent to the admissibility condition.

Let us now analyze the expression (4.8). In this case we see that for the quasi-definite case $g_n\neq 0$. But in our proof we see that $g_n\neq\gamma_n$ for all $n\in\mathcal{N}$. Thus, the following question arises: what happens if $g_n=\gamma_n$ for $n=1,2,\dots,n_0$?

To answer this question we use (4.8). Then

$$g_n=\gamma_n\iff [2n-3]_q\widehat{a}+\widehat{b}=0\quad \forall n=2,3,\dots,$$

which is in contradiction with the admissibility condition (4.9).

Remark 4.2. In [23] the condition $g_n\neq 0$ for all $n\in\mathcal{N}$ was imposed but not the more restrictive one $g_n\neq\gamma_n$, from where the first one immediately follows. Of course in [23] the admissibility condition $[n]_q\widehat{a}+\widehat{b}\neq 0$ is assumed and it implies that $g_n\neq\gamma_n$ for all $n\in\mathcal{N}$.

From the above discussion it follows that the q-classical polynomials are completely characterized by the relation (4.3) with the restriction $g_n\neq\gamma_n$ for all $n\in\mathcal{N}$. Moreover, if $g_n=\gamma_n$ for all $n=1,2,\dots,n_0$, then the corresponding orthogonal polynomial sequence, if such a sequence exists, is not a classical one.

4.2. The linear lattice $x(s)=s$

For the linear lattice $x:=x(s)=s$ the second order linear difference equation is

$$[\sigma(x)+\tau(x)]\Delta P_n(x)-\sigma(x)\nabla P_n(x)+\lambda_n P_n(x)=0, \tag{4.10}$$

where

$$\sigma(x)=A(x-x_1)(x-x_2), \quad \phi(x):=\sigma(x)+\tau(x)=A(x-\bar{x}_1)(x-\bar{x}_2),$$

and its general solution is of the form

$$P_n(x)={}_3\mathrm{F}_2\left(\begin{matrix}-n, x_1+x_2-\bar{x}_1-\bar{x}_2+n-1, x_1-x\\ x_1-\bar{x}_1, x_1-\bar{x}_2\end{matrix}\,\middle|\,1\right), \tag{4.11}$$

and

$$\lambda_n=-A\,n\,(x_1+x_2-\bar{x}_1-\bar{x}_2+n-1).$$

Here ${}_3\mathrm{F}_2$ is the generalized hypergeometric series

$${}_p\mathrm{F}_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\,\middle|\,x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!},$$

where $(a)_k=\prod_{m=0}^{k-1}(a+m)$, $(a)_0:=1$, is the Pochhammer symbol.

A particular choice $x_1=0$, $x_2=N+\alpha$, $\bar{x}_1=-\beta-1$, and $\bar{x}_2=N-1$ leads to the Hahn polynomials. Taking several limits from (4.11) we can obtain the other classical families: Kravchuk, Meixner, and Charlier (see e.g., [3,15,24,26]). In this case the structure relations are

$$\phi(x)\Delta P_n(x)=a_n P_{n+1}(x)+b_n P_n(x)+c_n P_{n-1}(x),$$

$$P_n(x)=\frac{1}{n+1}\Delta P_{n+1}(x)+f_n\Delta P_n(x)+g_n\Delta P_{n-1}(x). \tag{4.12}$$

Next we compute γ_n. For this purpose, since we are dealing with monic polynomials we can set $\gamma_n=l_n-l_{n+1}-k_n\beta_n$ (it can be obtained by identifying the coefficients of x^{n-1} in the TTRR (1.1)), where k_n and l_n are the coefficients of the monomials x^{n-1} and x^{n-2} in $P_n(x)=x^n+k_nx^{n-1}+l_nx^{n-2}+\cdots$, $n\geqslant 3$. To compute the values of k_n and l_n we substitute P_n in the second order linear difference equation (4.10) and identify the coefficients of the monomials x^{n-1} and x^{n-2} (for more details see [3]). This yields

$$\begin{aligned}\gamma_n=&-\frac{(p+a(n-2))n}{(p+2a(n-1))^2(p+a(2n-3))(p+a(2n-1))}\\ &\times[c(p+2a(n-1))^2-bp(q+p(n-1)+a(n-1)^2).\\ &+a(q+p(n-1)a(n-1)^2)^2-b^2(p+a(n-1))(n-1)],\quad n\geqslant 1,\end{aligned}$$

where the expressions $\sigma(x)=ax^2+bx+c$ and $\tau(x)=px+q$ have been used.

From the above expression we see that the corresponding orthogonal polynomial sequence exists (at least in the weak-regular sense) provided that the expression in the square bracket is different from zero and a sufficient condition for this is $p+na\neq 0$, for all $n\in\mathcal{N}$.

But now, using the expression (see e.g., [3, p. 108]) $c_n=\lambda_n\gamma_n/n$, we see that for all $n\geqslant 1$, $c_n\neq 0$. The condition $p+na\neq 0$ for all $n\in\mathcal{N}$ is the admissibility condition in this case.

Let us now analyze the structure relation (4.12). In this case [3, p. 109] $g_n=-((n-1)a\gamma_n)/(p+(n-2)a)$, therefore in the quasi-definite case $g_n\neq 0$. If $\gamma_n=g_n$ for all n, then we obtain that $p+(2n-3)a=0$, for all n which is in contradiction with the admissibility condition.

Remark 4.3. In [10] the condition $g_n \neq 0$ for all $n \in \mathscr{N}$ was imposed but not the more restrictive one $g_n \neq \gamma_n$, from where the first one immediately follows. For the discrete case in [10] the admissibility condition $p+na \neq 0$ is assumed and therefore $g_n \neq \gamma_n$ for all $n \in \mathscr{N}$.

From the above discussion also it follows that the classical discrete polynomials are completely characterized by the relation (4.12) with the restriction $g_n \neq \gamma_n$ for all $n \in \mathscr{N}$. Moreover, if $g_n = \gamma_n$ for all $n \in \mathscr{N}$, then the corresponding orthogonal polynomial sequence, if such a sequence exists, is not a classical one.

4.3. *The classical case*

The classical case can be obtained from the q-case taking the limit $q \to 1^-$. Nevertheless the Theorem 1.3 can be proved by using the same scheme as in Section 3. The only difference is that here one uses the standard integral calculus and integration by parts instead of the calculus with the difference operator. Of particular interest is the proof of property 6 so we will provide it here: taking derivatives of the TTRR (1.1) and using (1.6), we have the expression

$$xP'_n(x) = \frac{n}{n+1}P'_{n+1}(x) + (\beta_n - f_n)P'_n(x) + (\gamma_n - g_n)P'_{n-1}(x), \tag{4.13}$$

from where, if $g_n \neq \gamma_n$, $\forall n \in \mathbb{N}$, and using the Favard theorem the sequence $(P'_n)_n$ is an OPS, and therefore by the Sonin–Hahn Theorem 1.2 P_n is a classical OPS. Notice again that the condition $g_n \neq \gamma_n$ should be imposed. Using the formulas in [20] it is easy to see that this condition is equivalent to the condition $n\sigma''/2 + \tau' = 0$ which is nothing else that the admissibility condition for the classical polynomials [20]. Let us point out that the more restrictive condition $\gamma_n \neq g_n$ for all $n \in \mathbb{N}$ was not considered in [19] (they considered only the regular case, i.e., $\gamma_n \neq 0$). As in the cases already discussed we conclude that the classical *continuous* polynomials are completely characterized by the relation (1.6) with the restriction $g_n \neq \gamma_n$ for all $n \in \mathbb{N}$. Moreover, if $g_n = \gamma_n$ for $n = 1, 2, \ldots, n_0$, then the corresponding orthogonal polynomial sequence, if such a sequence exists, is not a classical one.

4.4. *The characterization by Marcellán et al.*

At this point the following question arises: what happens if we do not impose the condition $g_n \neq \gamma_n, \forall n=1, 2, \ldots, n_0$? There is any family of orthogonal polynomials, necessarily non-classical, that satisfies the TTRR (1.1) where $\gamma_n \neq 0$ for $n \in \mathscr{N}$, and the relation (1.6) with $g_n = \gamma_n$ for all $n \in \mathscr{N}$? i.e.,

$$P_n(x) = \frac{P'_{n+1}(x)}{n+1} + f_n P'_n(x) + \gamma_n P'_{n-1}(x). \tag{4.14}$$

To answer this question we can use (4.13) but rewritten in the form[4]

$$P'_{n+1}(x) = \frac{n+1}{n}(x - \beta_n + f_n)P'_n(x),$$

that leads to

$$P'_n(x) = n\prod_{j=1}^{n-1}(x - \beta_j + f_j), \quad n \geqslant 2.$$

Therefore, substituting the last expression in (4.14) we get, denoting $\xi_j = \beta_j - f_j$,

$$P_n(x) = [(x-\xi_n)(x-\xi_{n-1}) + nf_n(x-\xi_{n-1}) + (n-1)\gamma_n]\prod_{j=1}^{n-2}(x-\xi_j).$$

But this implies that for $n \geqslant 3$, two consecutive polynomials have common zeros that is a contradiction. Therefore, there is not any family of orthogonal polynomials that satisfy (4.14).

[4] As in Section 4.3 we will take the derivative of the TTRR (1.1) but now use (4.14).

For the linear lattices $x(s)=s$ and $x(s)=q^s$ the situation is the same. We present here the computations only for the q-case, the other case is analogous—in fact the final expression for the polynomials P_n coincide with the one in the classical "continuous" case.

For the q-case we proceed as before, i.e., we take the q-derivatives of the TTRR (3.2) and use the relation (4.3) where $e_n = 1/[n]_q$, $g_n=\gamma_n$, $F(1)=1$, $G(1)=0$, we obtain

$$\mathscr{D}_q P_{n+1}(x) = \frac{[n+1]_q}{[n]_q}(x-\xi_n/q)\mathscr{D}_q P_n(x), \quad \xi_j = \beta_j - f_j.$$

Substituting it in (4.3) when $g_n=\gamma_n$ we obtain the following expression for the polynomials P_n

$$P_n(x) = [(x-\xi_n/q)(x-\xi_{n-1}/q) + [n]_q f_n (x-\xi_{n-1}/q) + [n-1]_q\gamma_n] \prod_{j=1}^{n-2}(x-\xi_j/q).$$

As before, from this expression follows that for $n \geqslant 3$, two consecutive polynomials have common zeros, that is in contradiction with the fact that they constitutes an orthogonal sequence.

From the above discussion follows that the structure relation (3.10) when $g_n \neq \gamma_n$ for all $n \in \mathcal{N}$ completely characterizes the classical orthogonal polynomials.

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Appendix A. The classical polynomials

In this appendix we will present the proof of the Theorem 1.3. We will follow the same scheme in Section 3. Our starting point will be the Definition 1.1.

(1) → (2): Using the orthogonality of the classical OPS $(P_n)_n$ with respect to ρ (1.2) we have that for any polynomial of degree less than or equal to $k-1$, Q_{k-1}, with $k<n$,

$$\begin{aligned} 0 &= \int_a^b P_n(x)\underbrace{Q_{k-1}(x)\tau(x)}_{\text{degree}\leqslant k<n}\rho(x)\,\mathrm{d}x = \int_a^b P_n(x)Q_{k-1}(x)[\sigma(x)\rho(x)]'\,\mathrm{d}x \\ &= \underbrace{P_n(x)Q_{k-1}(x)\sigma(x)\rho(x)|_a^b}_{=0} - \int_a^b [P_n(x)Q_{k-1}(x)]'\sigma(x)\rho(x)\,\mathrm{d}x \\ &= -\underbrace{\int_a^b P_n(x)\overbrace{Q'_{k-1}(x)\sigma(x)}^{\text{degree}<n}\rho(x)\,\mathrm{d}x}_{=0} - \int_a^b P'_n(x)Q_{k-1}(x)[\sigma(x)\rho(x)]\,\mathrm{d}x. \end{aligned}$$

Thus P'_n is orthogonal to any polynomial of degree $k-1<n-1$, i.e., $(P'_n)_n$ is also an orthogonal family. Furthermore, since the weight function for the sequence $(P'_n)_n$ is $\rho_1(x)=\sigma(x)\rho(x)$, we have that they satisfy the equation $[\sigma(x)\rho_1(x)]' = [\tau(x)+\sigma'(x)]\rho_1(x)$, i.e., a Pearson equation (1.3).

(2) $\to$ (3): We use now that $(P_n')_n$ is an orthogonal family with respect to the weight function $\rho_1(x)=\sigma(x)\rho(x)$ where ρ satisfies the Pearson equation (1.3). Thus,

$$\begin{aligned} 0 &= \int_a^b P_n'(x)Q_k'(x)\rho_1(x)\,\mathrm{d}x \\ &= \underbrace{P_n'(x)Q_k(x)\sigma(x)\rho(x)|_a^b}_{=0} - \int_a^b [\sigma(x)\rho(x)P_n'(x)]'Q_k(x)\,\mathrm{d}x \\ &= -\int_a^b Q_k(x)\left\{\underbrace{[\sigma(x)\rho(x)]'}_{=\tau(x)\rho(x)}P_n'(x)+\sigma(x)\rho(x)P_n''(x)\right\} \\ &= \int_a^b Q_k(x)[\sigma(x)P_n''(x)+\tau(x)P_n'(x)]\rho(x)\,\mathrm{d}x. \end{aligned}$$

But since the last integral vanishes for every polynomial Q_k of degree $k<n$ then $\sigma(x)P_n''(x)+\tau(x)P_n''(x)$ should be proportional to P_n, i.e., $\sigma(x)P_n''(x)+\tau(x)P_n''(x)=-\lambda_n P_n$, where λ_n is a constant, in general depending on n.

(3) $\to$ (4): The solution of the above differential equation can be written in the following compact form (see e.g., [25, Section 2] or [24, Section 1.2]) usually called the Rodrigues formula

$$P_n(x)=\frac{B_n}{\rho(x)}\frac{d^n}{dx^n}[\sigma^n(x)\rho(x)],$$

where B_n is a constant.

(4) $\to$ (1): It follows from the Rodrigues formula just putting $n=1$.

(4) $\to$ (5): From the Rodrigues formula the following expression (see e.g., [25, Eq. (7) p. 25]) immediately follows

$$\sigma(x)P_n'(x)=\frac{\lambda_n}{n\tau_n'}\left[\tau_n(x)P_n(x)-\frac{B_n}{B_{n+1}}P_{n+1}(x)\right],\qquad \tau_n(x)=\tau(x)+n\sigma'(x),$$

from where, using the three-term recurrence relation for the family $(P_n)_n$ the structure relation (1.5) follows.

(5) $\to$ (1): Suppose that (1.5) holds where $\deg\sigma\leqslant 2$ and $(P_n)_n$ is an orthogonal family. Notice that the integral

$$\int_a^b Q_k(x)P_n'(x)\sigma(x)\rho(x)\,\mathrm{d}x=\int_a^b Q_k(x)\rho(x)[a_nP_{n+1}(x)+b_nP_n(x)+c_nP_{n-1}(x)]\,\mathrm{d}x$$

vanishes for all $k<n-1$. Then $(P_n')_n$ is an orthogonal family with respect to the weight function $\rho_1(x)=\sigma(x)\rho(x)$ and therefore by the Sonin–Hahn Theorem 1.2 $(P_n)_n$ is a classical OPS.

(2) $\to$ (6): For proving this we suppose that $(P_n)_n$ and $(P_n')_n$ are orthogonal with respect to $\rho(x)$ and $\rho_1(x)=\sigma(x)\rho(x)$, respectively. If $(P_n)_n$ is a monic sequence then

$$P_n(x)=\frac{1}{n+1}P_{n+1}'+f_nP_n'(x)+g_nP_{n-1}'(x)+\sum_{k=1}^{n-2}c_k(n)P_k'(x).$$

But

$$c_k(n)=\frac{\int_a^b P_n(x)P_k'(x)\sigma(x)\rho(x)\,\mathrm{d}x}{\int_a^b[P_k'(x)]^2\sigma(x)\rho(x)\,\mathrm{d}x}=0,$$

since $\deg P_k'\sigma\leqslant k+1<n-2$ and $(P_n)_n$ is and orthogonal family with respect to $\rho(x)$.

Finally the proof (6) $\to$ (1) is presented in Section 4.3.

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