Computational Methods in Conservative Dynamical Systems and Advanced Examples

FisMat 2015

Obverse and reverse of the same coin [head and tails]

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Outline of the Course

Goal

Introduce computational tools to analyze systems in which at least one conserved quantity is present.

- Lecture 3: What are conservative systems? How do we analyze the dynamical behavior.
- Lecture 4: Advanced Applications in Celestial Mechanics.

Content

Lecture 3: Conservative Systems

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  - what is a conservative system?
  - why do we care about them?

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  - Eigenvalues and Floquet Multipliers

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- Conserved quantities, symmetries and reversibility.
  - How do we find conserved quantities?

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  - How do we find conserved quantities?
- Not so simple example.

A simple example

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$E(\theta, \dot{\theta}) = K + V$ is the energy with

$E(0,0) = 0$ and $E(\pi, 0) = 2$.

Consequences

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Where is the catch?

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Denoting by $q = \theta$, $p = \dot{\theta}$, $E(q, p) = H(q, p) = \frac{1}{2}p^2 + (1 - \cos q)$

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$[q,p] = u \in U \subset \mathbb{R}^{2n}$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ the symplectic matrix, we derive the Hamiltonian Equations of motion.

$$\dot{u} = J \nabla H(u)$$

Properties of J: $J^t = -J$, $J^{-1} = J^t$, $J^t J = I_{2n}$ and $\det(J) = \pm 1$.

How are the eigenvalues and Floquet multipliers?

- The structure of the equations of motion have a strong influence on the stability indicators of equilibria and periodic orbits.
- The characteristic polinomial $p(\lambda) = p(-\lambda)$ and, consequently,

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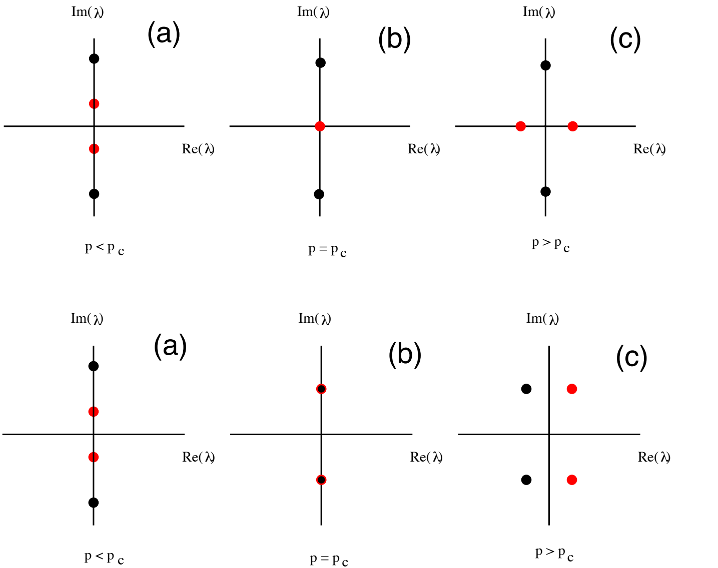
- There are **no attractors** in Hamiltonian systems.
- An analogous result holds for Floquet Multipliers:

$$\text{if } \mu \text{ is a multiplier} \Rightarrow \frac{1}{\mu} \text{ is also a multiplier.}$$

+1 multiplier appears always in pairs.

Eigenvalues for conservative systems



Geometrical meaning of conserved quantities

F is a conserved quantity of $\dot{x} = f(x)$ if

$$\dot{F}(x) = \nabla F(x)^t f(x) = 0$$

in words, the vector $\nabla F(x)$ is **orthogonal** to the flow of the vector field.

How do we obtain conserved quantities?

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- For each one parameter symmetry of a Lagrangian there exists an associated conserved quantity.
- Claim: If $q \to q(s)$ leaves the Lagrangian invariant: i.e. $\frac{d}{ds}\mathcal{L}(q(s),\dot{q}(s)) = 0$, then, $C = p\frac{dq(s)}{ds}$ is a conserved quantity.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \leftrightarrow \dot{p} = \frac{\partial \mathcal{L}}{\partial q}$$

$$\frac{dC}{dt} = \dot{p}\frac{dq(s)}{ds} + p\frac{d\dot{q}(s)}{ds} =$$

$$\frac{\partial \mathcal{L}}{\partial q}\frac{dq(s)}{ds} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}(s)}{ds} = \frac{d}{ds}\mathcal{L}(q(s),\dot{q}(s)) = 0$$

Conserved quantities and symmetries

- If $\mathcal{L}$ is time invariant, then $C =$ energy
- If $\mathcal{L}$ is invariant under translations, then $C =$ Linear momentum
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Message

There is a direct relation between conserved quantities and continuous symmetries

Remark: Also holds for Quantum Mechanics

- Hamiltonian equations of **Classical Mechanics**:

$$\dot{u} = J \nabla H(u)$$

- Schrödinger equation with $\Psi = \Psi_{RE} + i\Psi_{IM}$

$$i \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

Splitting in real and imaginary parts

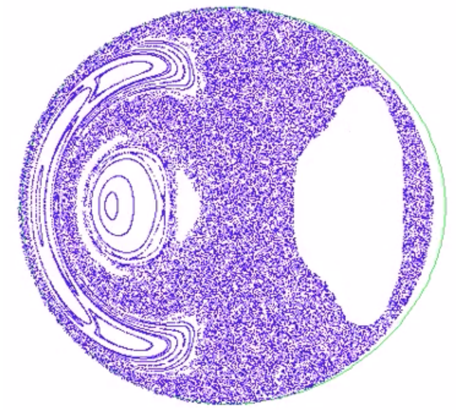
$$\begin{cases} \frac{\partial \Psi_{RE}}{\partial t} & = & \hat{H}\, \Psi_{IM} \\ \frac{\partial \Psi_{IM}}{\partial t} & = & -\hat{H}\, \Psi_{RE} \end{cases}$$

with $u = [\Psi_{RE}\ \Psi_{IM}]^t$ we get

$$\dot{u} = J \hat{H} u$$

What about Computational Tools ?

- Excellent IVP integrators (Dstool, Taylor, Tides, Matlab)
- Compute Lyapunov exponents or other chaos indicators.
- Compute Poincaré sections: combination of IVP integration + reduction + event location (fingerprints of chaotic motion)



Poincaré: periodic orbits in Hamiltonian systems

Les méthodes nouvelles de la mécanique céleste, 1899

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Indeed, the probability is zero for the initial condition to correspond precisely to those of a periodic solution. . . . (Poincare's conjecture) . . . what renders these periodic solutions so precious is that they are, so to speak, the only opening through which we may try to penetrate into the fortress which has the reputation of being impregnable"

Content

Lecture 3: Continuation Theorems

- Reduction vs unfolding.
- Continuation Theorems.
- Examples: the spring pendulum and localized NLS.

Reduction vs unfolding

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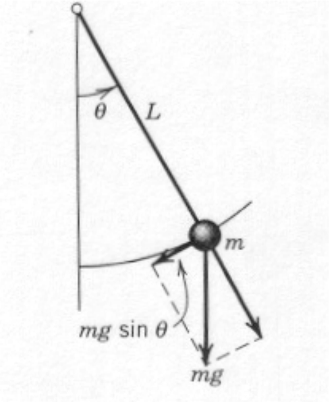
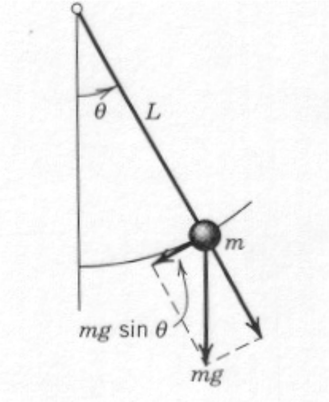
4. Continuation in conservative systems or continuation without parameters; an alternative to reduction methods.
5. Applications to the elastic pendulum and quantum wells (NLS).

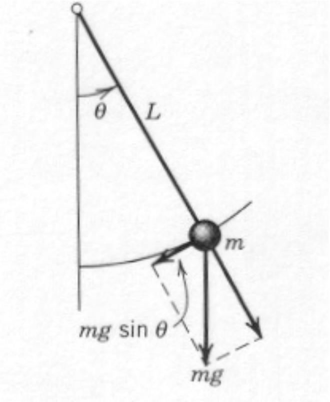


Galileo's pendulum



Galileo's pendulum

- 3 parameters: L, m, g



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- Newton's second law:

$$mL\ddot{\theta} + mg\sin\theta = 0$$

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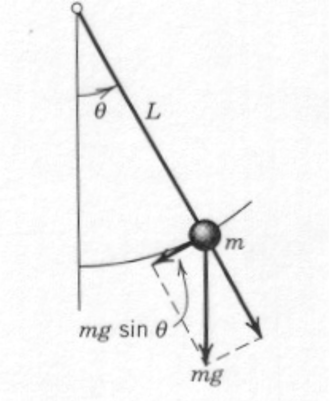

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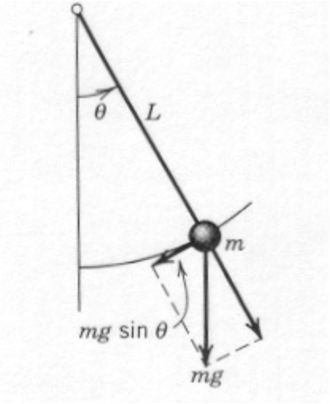

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- Rescaling time with $\tau = \sqrt{\frac{L}{g}}$.
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Galileo's Pendulum Equation

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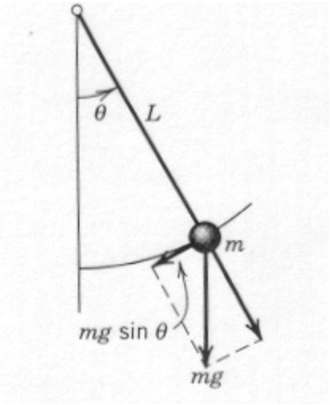

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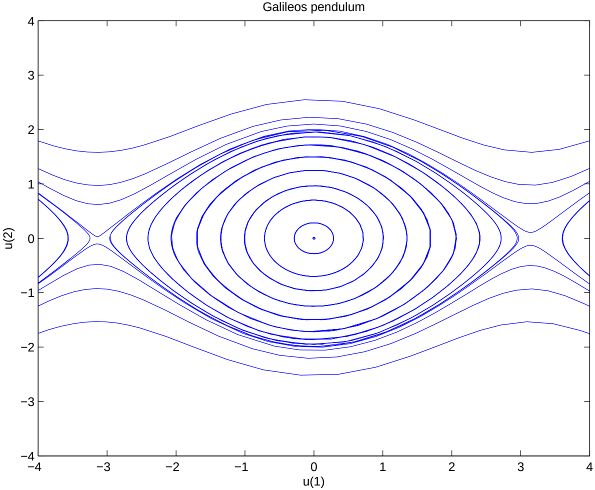
- One dof ODE without parameters with two equilibria: $\theta = 0$ (S) and $\theta = \pi$ (U) and a one parameter family of periodic orbits.

Phase portrait of Galileo’s pendulum



The reduction method

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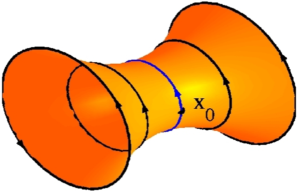
- We have introduced now E as an internal parameter that can be used for continuation (and lowered the dimension).

The general picture for Hamiltonian systems

U open set in $\mathbb{R}^{2n}$, $H \in \mathcal{C}^1(U)$ con $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

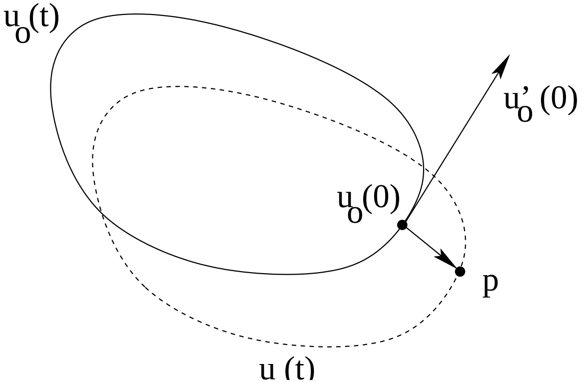
$$u' = J \nabla H(u)$$

- ODE without explicit parameters.
- H is a conserved quantity.
- Periodic orbits are not isolated (cylinder theorem).

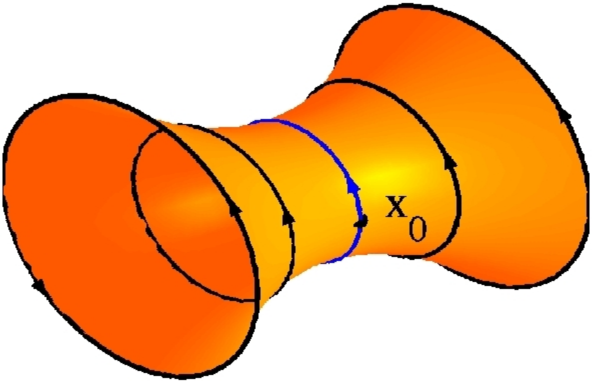


Continuation of periodic orbits

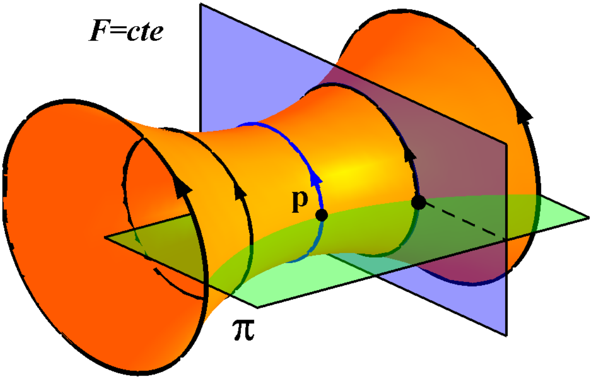
Let $u_0(t)$ be a T_0-periodic solution.
Is it possible to find another periodic solution *close* to $u_0(t)$ by changing the natural parameter H?



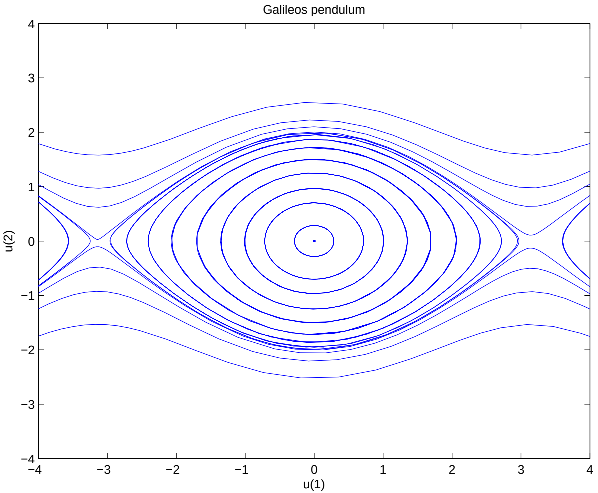
Geometrical picture: Cylinder Theorem



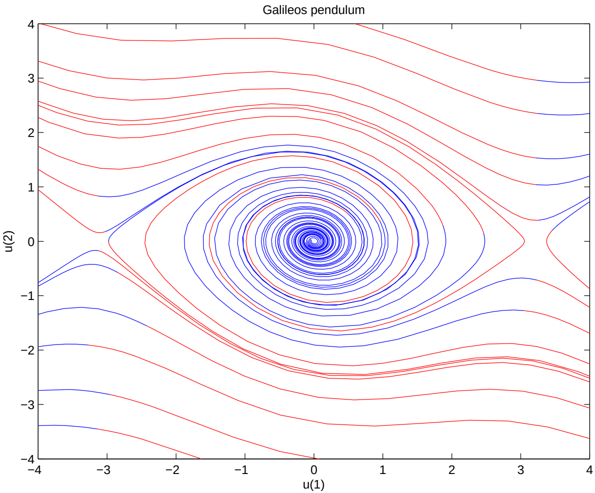
Geometrical picture: Reduction



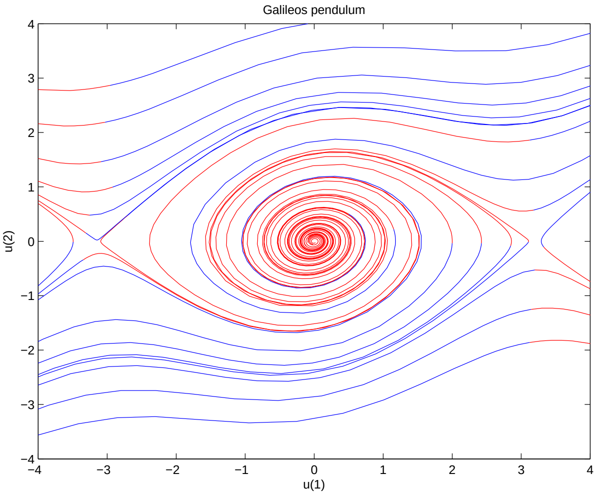
Alternative method: Increase the dimension!



Alternative method: positive dissipation



Alternative: negative dissipation



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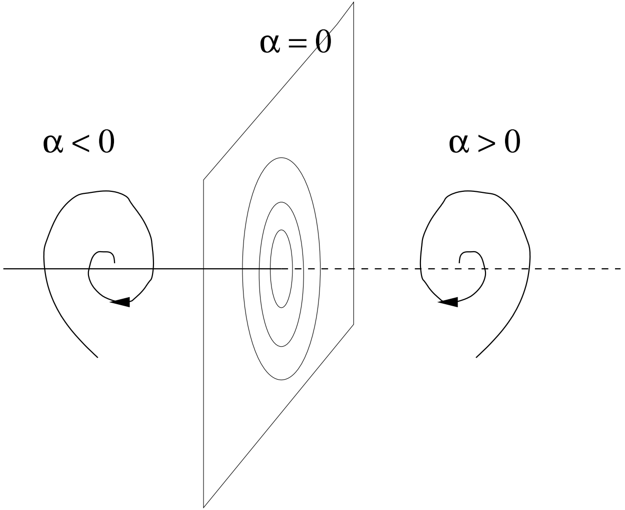

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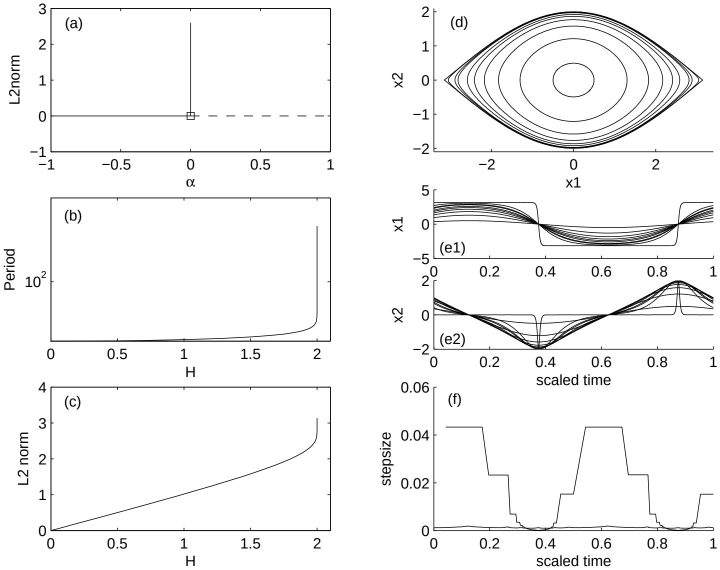
We have changed the $\boxed{E}$ parameter by an $\boxed{\alpha}$ parameter.

AUTO results



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3. Bifurcations can be detected and followed.
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5. The computation preserves the simplectic character of the problem (Hamiltonian case).

Remarks

1. It is straightforward to implement
(if we know the unfolding term) [Physica D **181** (2001)].
2. It can be extended to k independent conserved quantities.
3. Bifurcations can be detected and followed.
4. We can detect homo- and heteroclinic connections.
5. The computation preserves the simplectic character of the problem (Hamiltonian case).
6. For reversible system there are further simplifications.

Theory: BVP Formulation

$$u' = T(J\nabla H(u(t)) + \alpha \nabla H(u(t))), \qquad u(1) = u(0). \quad (1)$$

with u, α and T as unknowns. Finding a T-periodic orbit of $u' = J\nabla H(u)$ is equivalent to finding a solution of (1) if $\alpha = 0$. We have to include a phase condition to fix the time origin.

$$(u(0) - u_0(0))^* u_0'(0) = 0. \quad (2)$$

Theorem

Let $u_0(t)$ be a periodic solution with period $0 < T_0 < +\infty$ whose monodromy matrix has 1 as an eigenvalue with **geometric multiplicity one** or **algebraic multiplicity two**. Then, there exists a unique branch of solutions of (1) and (2) in a neighborhood of $(u, T, \alpha) = (u_0, T_0, 0)$. Moreover, along the branch $\alpha = 0$.

- The proof is a direct application of IFT and the fact that $H(u(t))$ is constant along the periodic orbit.

Generalization

- Let $\mathcal{W}_{\mathbf{p}} = \{\nabla F(\mathbf{p}) \, : \, F$ first ontegral of $\dot{x} = f(x)\}$, $\dim(\mathcal{W}_{\mathbf{p}}) = k$, $\varphi_t(\mathbf{x}, \boldsymbol{\alpha})$ the flow and $\mathrm{orb}_{\varphi}(\mathbf{p})$ the orbit.
- $\dot{x} = f(x) \quad \to \quad \dot{x} = f(x) + \alpha_1 \nabla F_1(x) + \ldots + \alpha_k \nabla F_k(x)$,

Proposition

Let $\mathbf{p} \in \mathbb{R}^n$ *s. t.* $\mathrm{orb}_{\varphi}(\mathbf{p})$ *be* $T-$*periodic. It holds that* $\mathrm{Im}(D\varphi_T(\mathbf{p}) - I) + \mathbb{R}f(\mathbf{p}) \subseteq \mathcal{W}_{\mathbf{p}}^{\perp}$.

General results

Definition (Normal periodic orbit)

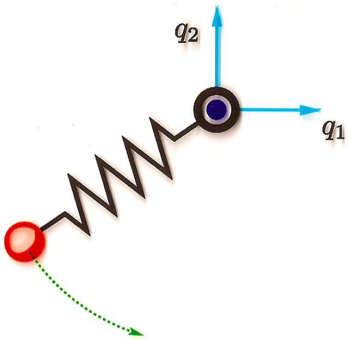
Let $\mathbf{p} \in \mathbb{R}^n$ such that the orbit $\mathrm{orb}_\varphi(\mathbf{p})$ is periodic with period $T > 0$ and $\mathbf{p}$ is not an equilibrium of $\dot{\mathbf{z}} = f(\mathbf{z})$. We say that $\mathrm{orb}_\varphi(\mathbf{p})$ is a normal periodic orbit of e $\dot{\mathbf{z}} = f(\mathbf{z})$ if

$$\mathrm{Im}(D\varphi_T(\mathbf{p}) - I) + \mathbb{R}f(\mathbf{p}) = \mathcal{W}_{\mathbf{p}}^{\perp}.$$

Theorem (Continuation with k conserved quantities)

Let $\mathbf{p} \in \mathbb{R}^n$ be a point that generates a normal periodic orbit of $\dot{\mathbf{x}} = f(\mathbf{x})$ with period $T > 0$. Then there exists a neighborhood of $T > 0$ such that the set of points that generate periodic orbits whose period is in that neighborhood of T is locally a submanifold at $\mathbf{p}$.

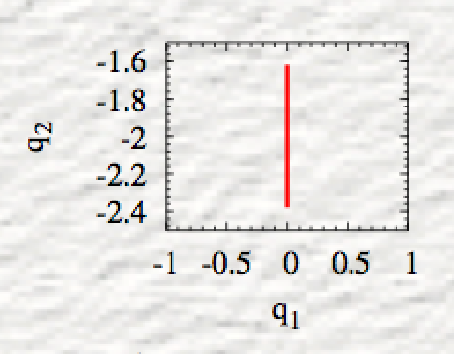
Elastic Pendulum

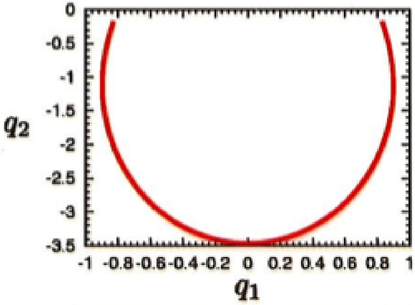


Adimensional parameter $\lambda = \frac{lk}{mg}$

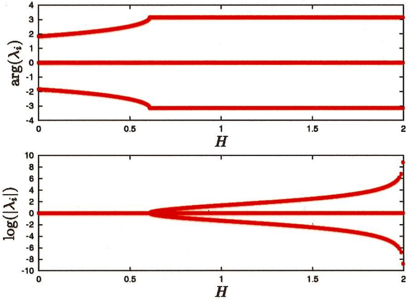
Equilibria $\begin{cases} (0, -\lambda - 1) & \text{Stable} \\ (0, \lambda - 1) & \text{Unstable } (\lambda > 1) \end{cases}$

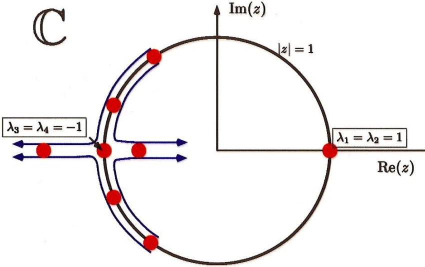
$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{1}{2}(\sqrt{q_1^2 + q_2^2} - \lambda)^2 + q_2 + \lambda + \frac{1}{2}.$$


-1.6
-1.8
-2
-2.2
-2.4
q2
-1
-0.5
0
0.5
1
q1

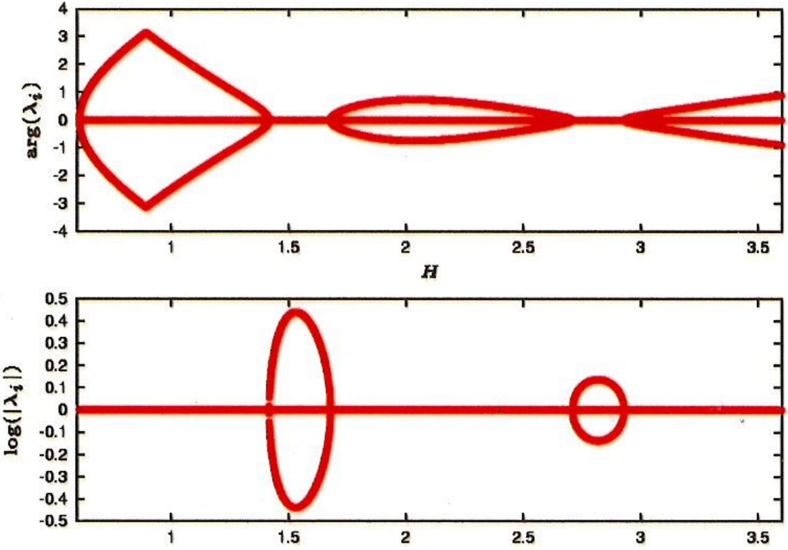

0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
q2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
q1

Vertical Nonlinear Normal Modes

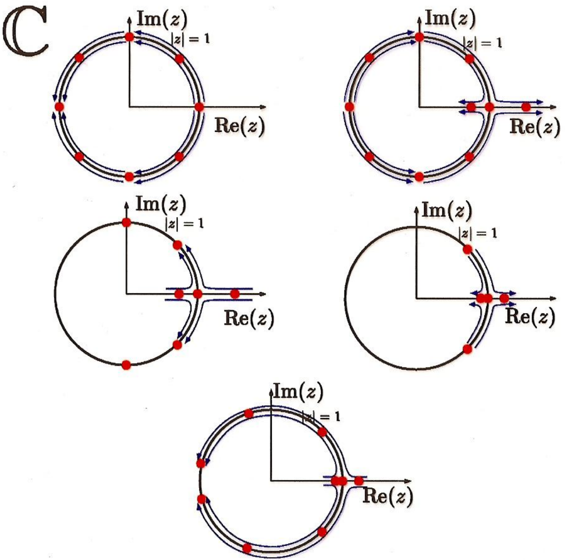




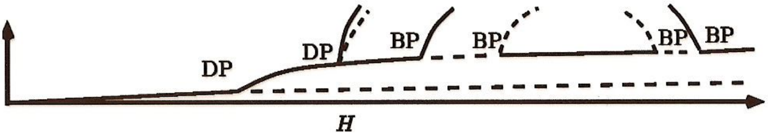
Period doubled branch



Period doubled branch



Schematic bifurcation diagram



Reversibility continuation

Definition: We say that $R \in L(\mathbb{R}^n)$ is a reversibility for the system $\dot{\mathbf{x}} = f(\mathbf{x})$, if $Rf(\mathbf{x}) = -f(R\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n$.

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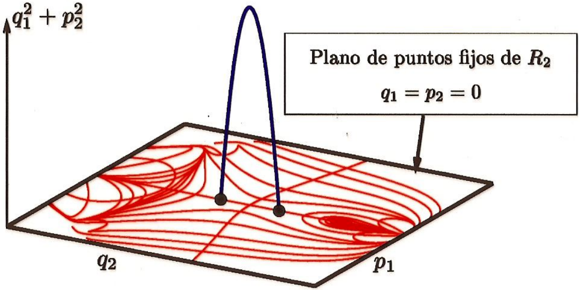

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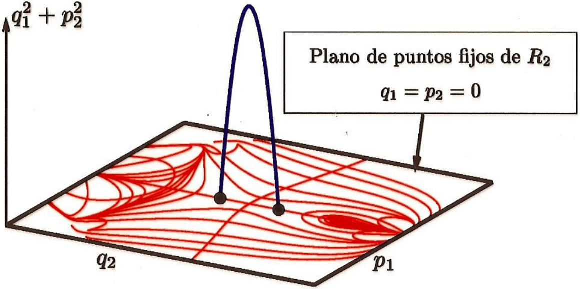
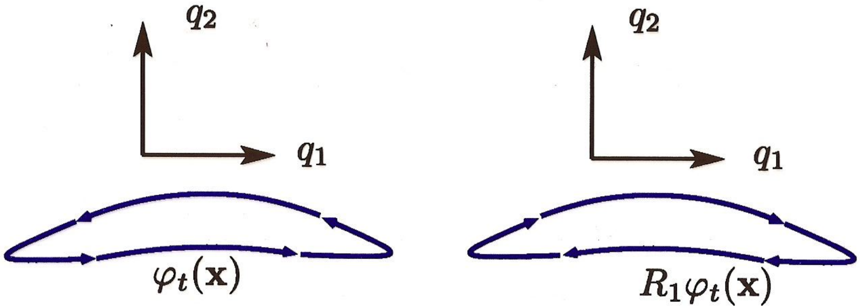


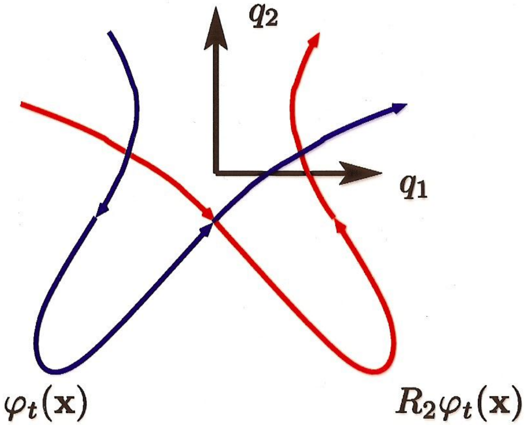
Example: in a mechanical system changing the sign to all velocities and integrate in negative time we get another solution.

Poetic definition: In an reversible system the **future** is the **past** of an alternative **present**.

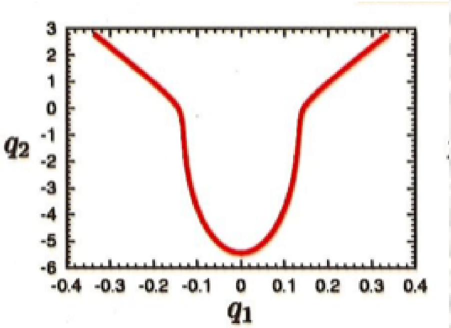
Reversibility continuation: R1

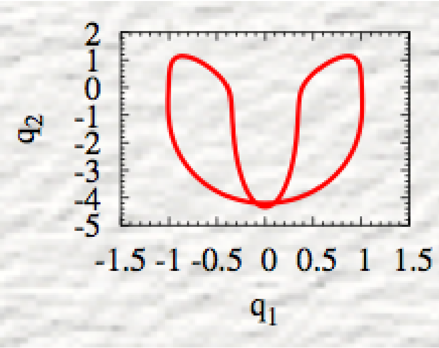


Reversibility continuation: R2

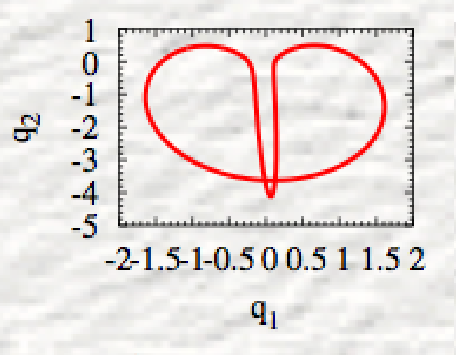


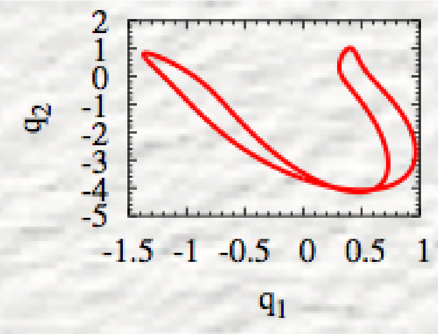
Reversibility continuation: orbits



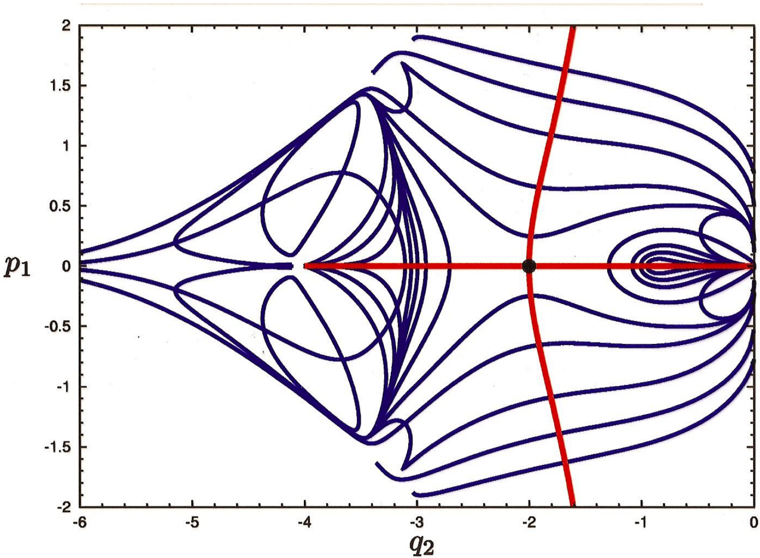


Reversibility continuation: orbits

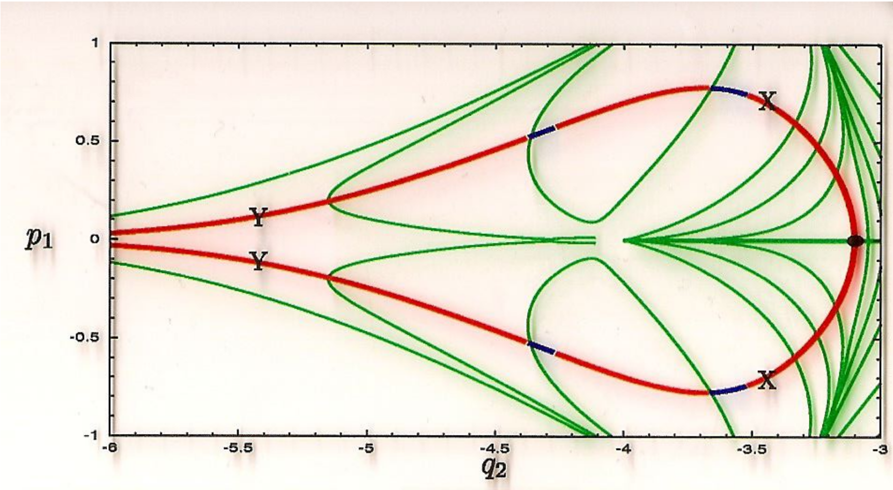




Reversibility continuation: results



Reversibility continuation: results



References

- B. Krauskopf, H. Osinga, J. Galan, Numerical Continuation Methods for Dynamical Systems.
- Meyer, K. R. and Hall, G. R., Offin, D. An introduction to Hamiltonian Dynamical Systems

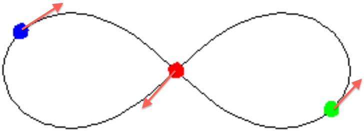
Content

Lecture 4: Advanced Applications in Celestial Mechanics

- Lagrange, Euler and the figure eight.
- Horseshoe periodic orbits of the 3BP and 5BP.
- Periodic orbits in the Sitnikov Problem

Aplication to the figure 8 of the TBP

- In 1999 A. Chenciner & R. Montgomery [Ann. Math. **152** 881 (2000)] proved with variatonal techniques the existence of a new solution of the TBP.
  A remarkable periodic solution of the three body problem
- C. Simó computed it numerically, coined the name choreography, determined its stability and showed that it belonged to a one parameter family.



$$\begin{aligned}
\ddot{\mathbf{x}}_1 &= -m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - m_3 \frac{\mathbf{x}_1 - \mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_3|^3} \,, \\
\ddot{\mathbf{x}}_2 &= -m_1 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - m_3 \frac{\mathbf{x}_2 - \mathbf{x}_3}{|\mathbf{x}_2 - \mathbf{x}_3|^3} \,, \\
\ddot{\mathbf{x}}_3 &= -m_2 \frac{\mathbf{x}_3 - \mathbf{x}_2}{|\mathbf{x}_3 - \mathbf{x}_2|^3} - m_1 \frac{\mathbf{x}_3 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_3|^3} \,,
\end{aligned}$$

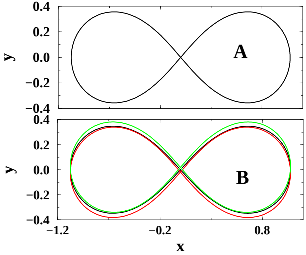
7 first integrals H, **P** and **J**, permutations (if $m_2 = m_3$)

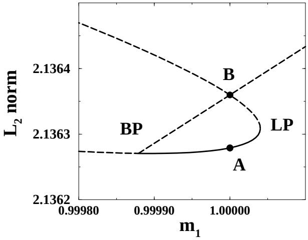
$$(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \mapsto (\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2).$$

Orbital symmetry (scaling)

$$(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \mapsto (\lambda^{-2}\mathbf{q}_1, \lambda^{-2}\mathbf{q}_2, \lambda^{-2}\mathbf{q}_3, \lambda\mathbf{p}_1, \lambda\mathbf{p}_2, \lambda\mathbf{p}_3)$$

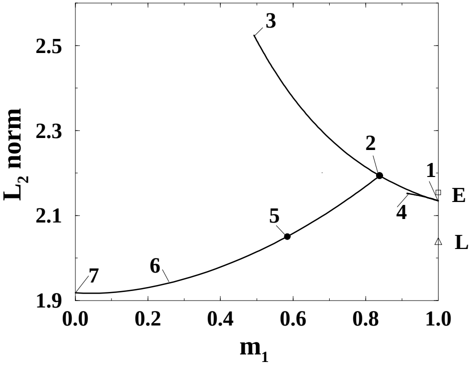
Local continuation





Physical Review Letters **24** 2002.

Global continuation: from TBP to RTBP



Application to the RTBP

- An easy to state but challenging problem with a single conserved quantity and tons of known (and unknown) families of PO.

Application to the RTBP

- An easy to state but challenging problem with a single conserved quantity and tons of known (and unknown) families of PO.

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x},\\ \ddot{y} - 2\dot{x} &= \frac{\partial \Omega}{\partial y},\end{aligned}$$

$$\Omega(x,y) = \frac{1}{2}(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu),$$

$$r_1^2 = (x+\mu)^2 + y^2,$$

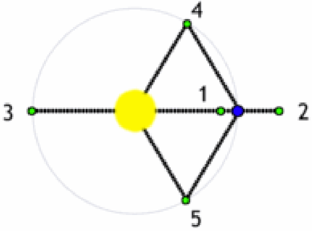
$$r_2^2 = (x+\mu-1)^2 + y^2.$$

Jacobi constant: $C = 2\Omega(x,y) - \dot{x}^2 - \dot{y}^2.$

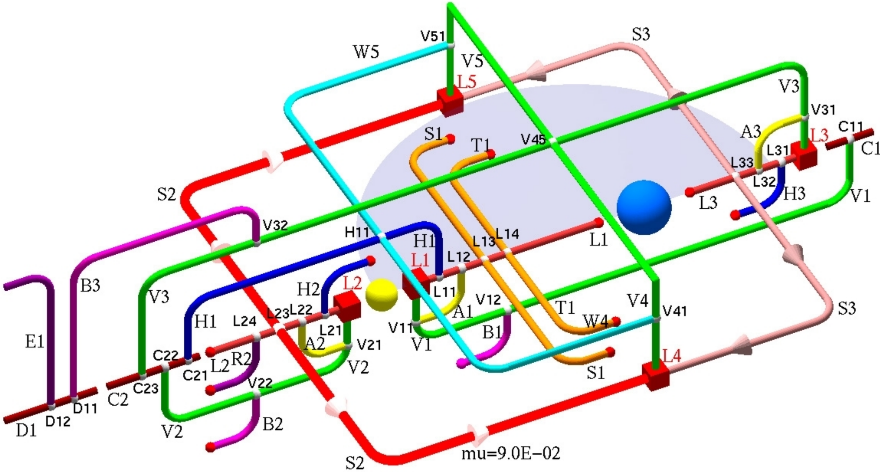
- The RTBP has five equilibrium points L_i that depend on μ .
- The Jacobi constant on them ($C_i = C(L_i)$) fulfills the relation

$$3 = C_4 = C_5 < C_3 < C_1 < C_2,$$

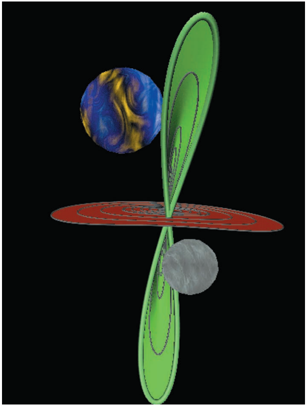
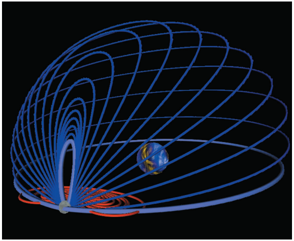
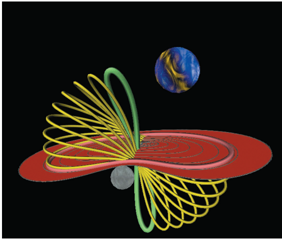
and $C_3 = C_1$ for $\mu = \frac{1}{2}$.



RTBP: Lagrange Families [IJBC **17** (2007)]



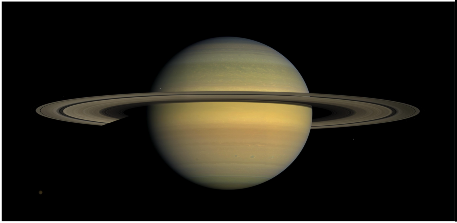
RTBP: Lagrange Families [IJBC **17** (2007)]

Why the name Horseshoe?

Two co-orbital satellites of Saturn

- Janus and Epimetheus are satellites of Saturn with coplanar orbits that are very close to each other.
- $m_S = 5.69 \times 10^{26} kg$, $R_S = 60268 Km$,
  $m_J = 1.98 \times 10^{18} Kg, 196 \times 192 \times 150$ and
  $m_E = 5.50 \times 10^{17} Kg, 144 \times 108 \times 98$.

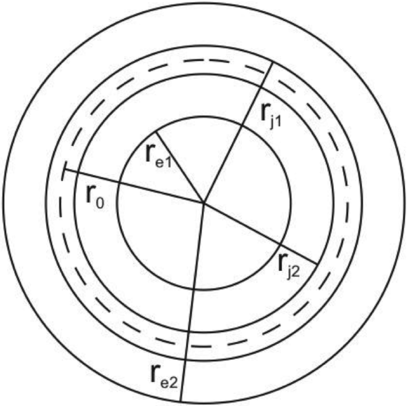


The orbits

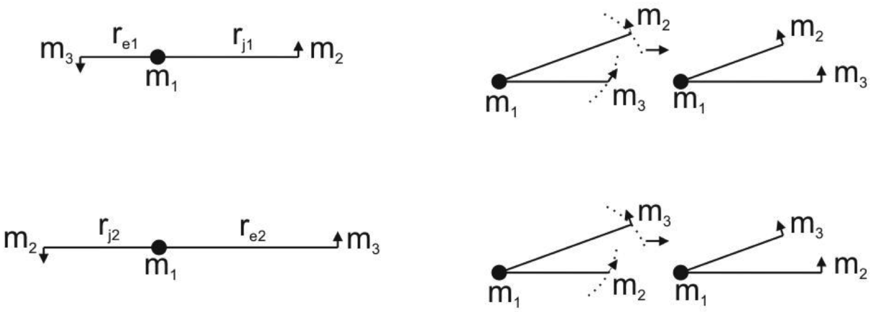
- The motion of both satellites occurs in the same plane.
- Most of the time the satellites do not feel each other (two body Kepler solution).
- According to Kepler's laws the inner satellite goes faster than the outer one and eventually after some full revolutions will catch it (encounter).
- Only when they are close to each other they feel the mutual gravitational attraction (encounter).

The orbits

- $$r = \frac{a(1-e^2)}{1-e\cos\theta}$$
- $a_J = 151460, e_J = 0.0068$
- $a_E = 151410, e_E = 0.0098$

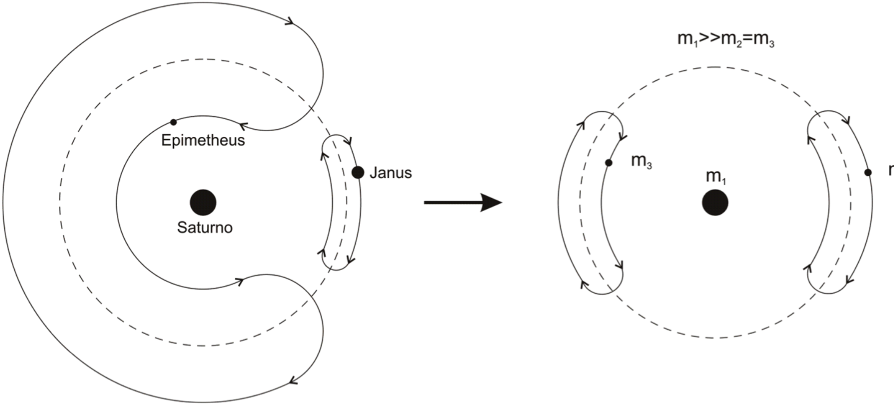


A closer look at the encounter

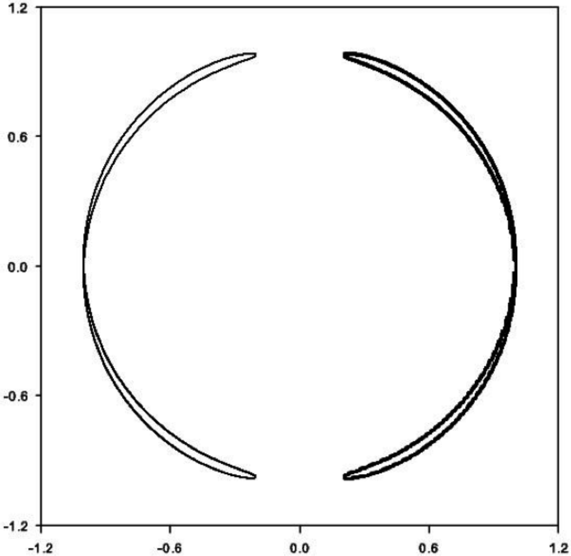


- The inner body **does not overtake** the outer one but they interchange orbits; the inner becomes outer and vice-versa.
- The defining property of a horseshoe orbit is this no overtaking condition.

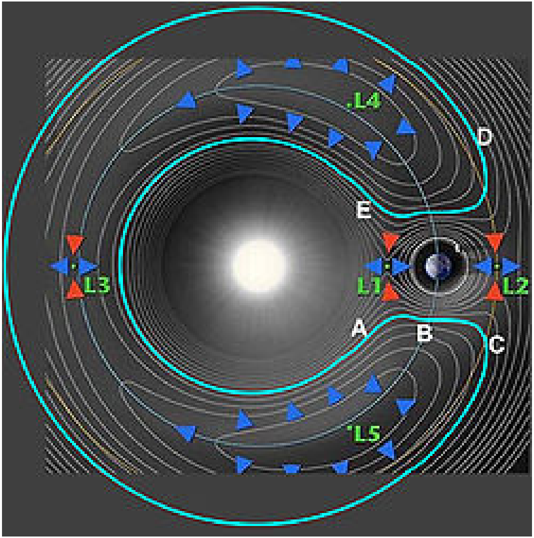
Horseshoe in a rotating frame; scheme



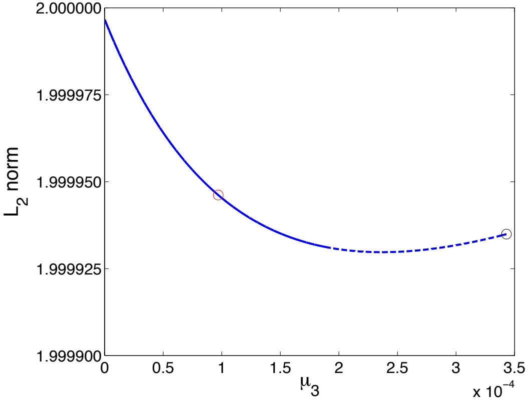
Horseshoe in a rotating frame; calculated



Stability of the Lagrange points: Horseshoe orbit



Stability of the Lagrange points: Horseshoe orbit



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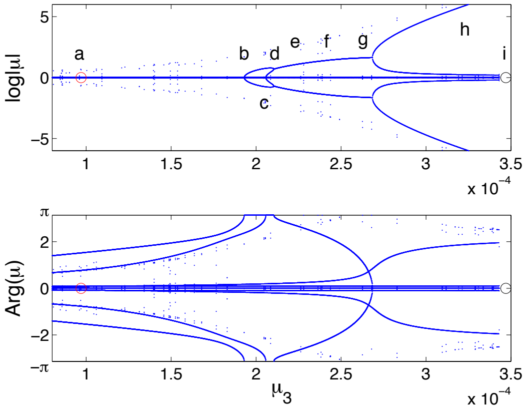

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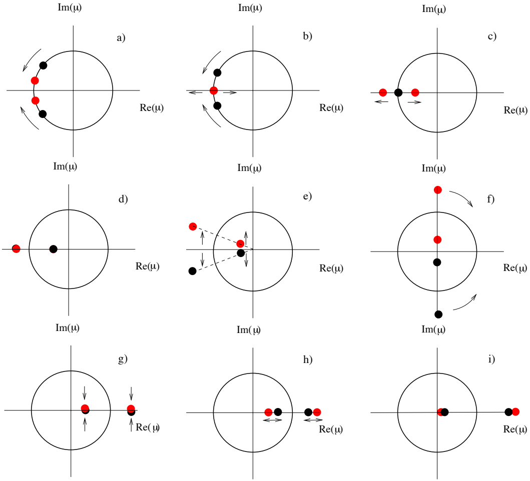
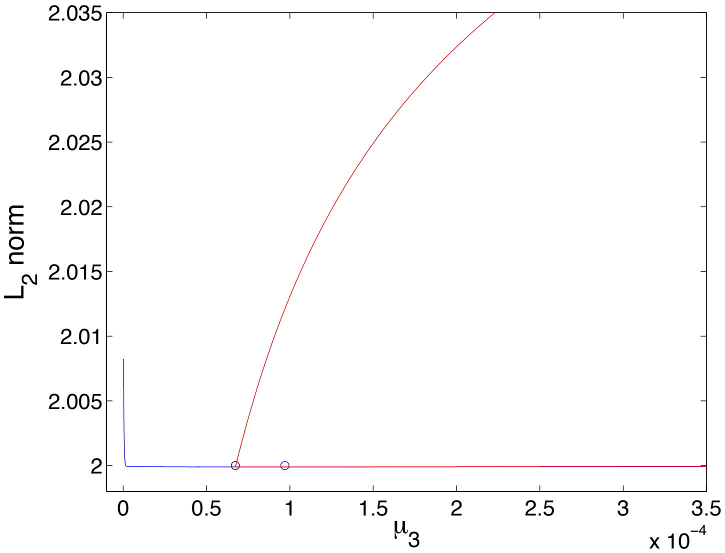
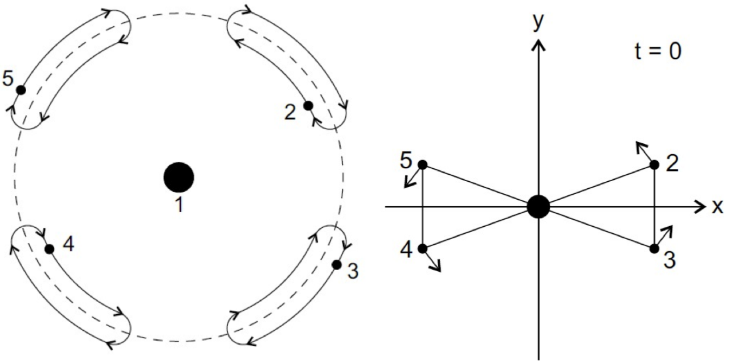


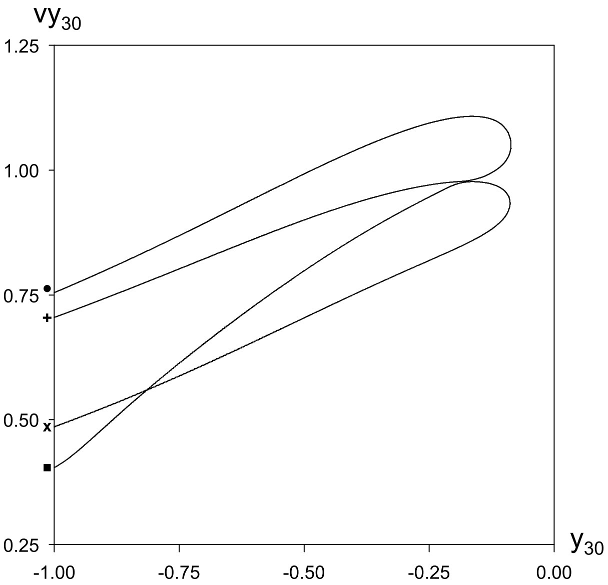
Bifurcation diagram



2k+1 Horseshoe solution



5 body Horseshoe connected to Lagrange



Conclusions and open problems

- January 2016: **DANCE RTNS** Kam Theorem by A. Celleti.
- Google for *whooping solution three body*.
- Jgv talk at the workshop.