Computational Methods in Dynamical Systems and Advanced Examples

FisMat 2015

Obverse and reverse of the same coin [head and tails]

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Universidad de Sevilla

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Outline

- Lecture 1. Simulation vs Continuation.
  How do we compute the bifurcation diagrams?
  IFT + Newton

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- Lecture 4. Advanced Examples from Celestial Mechanics.

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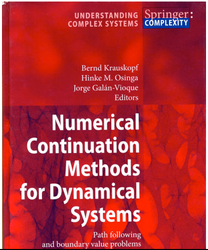
- Lecture 2. Examples and relation (translation) to Physics.
- Lecture 3. The conservative case: Pendula.
- Lecture 4. Advanced Examples from Celestial Mechanics.
- Workshop. Application to a Mean Field problem in Quantum Mechanics.

References

- Lecture Notes: Numerical Analysis of Nonlinear Equations. E. J. Doedel.
- Chapter 10: Elements of Applied Bifurcation Theory. Y.A. Kuznetsov.
- Chapter 1: Numerical Continuation Methods for Dynamical systems. Path Following and boundary value problems. B. Krauskopf, H. Osinga and J. Galán-Vioque.



Acknoledgements

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Excercises from Emilio's talk

Find all the solutions of:

- $f(x, \lambda) = \lambda + x^2$
- $f(x, \lambda) = (x - \lambda)x$
- $f(x, \lambda) = \lambda x - x^3$

for all values of x and λ.

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How do we get the answer with the computer?

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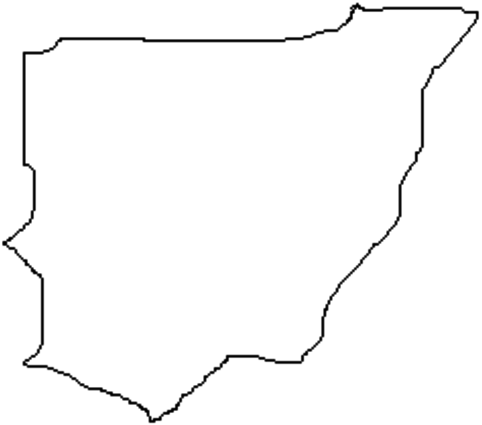
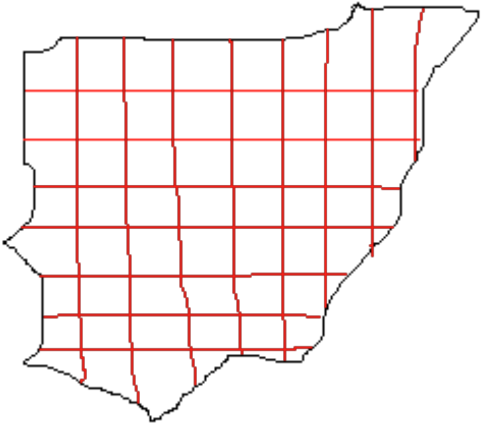
How do we get the answer with the computer?

or, how do we proceed in realistic examples?

Simulation vs continuation



Simulation vs continuation



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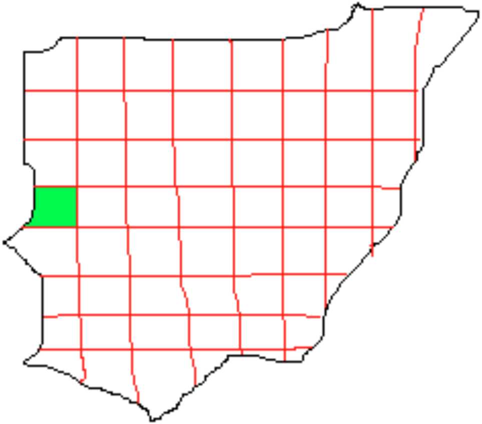

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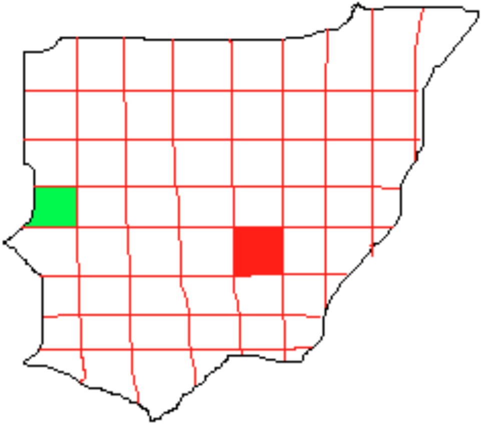

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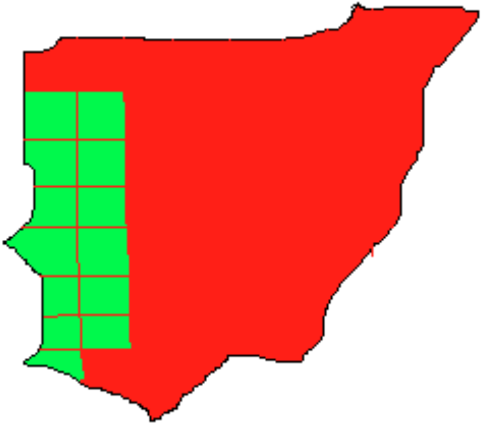

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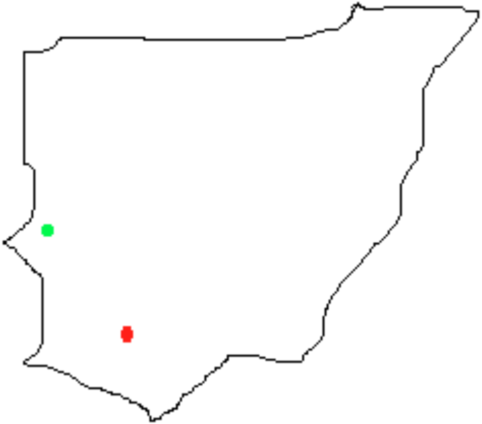

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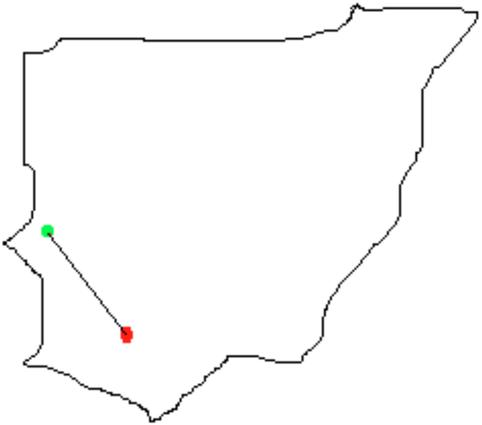

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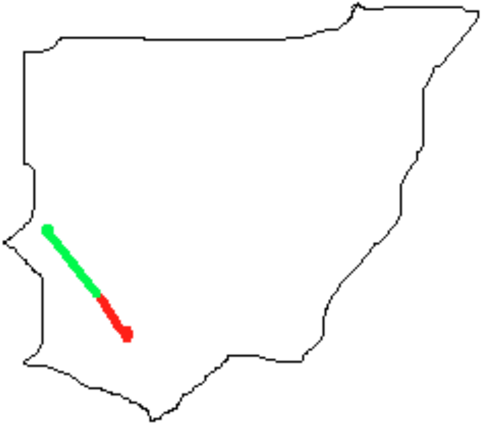

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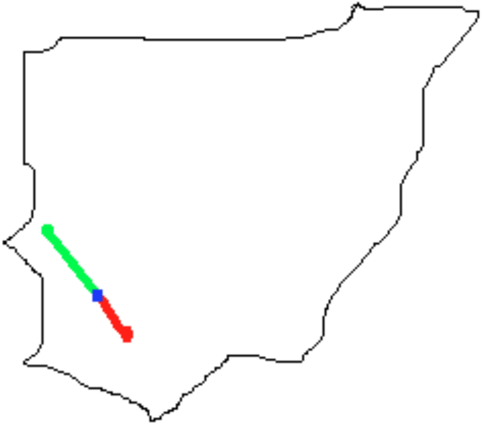

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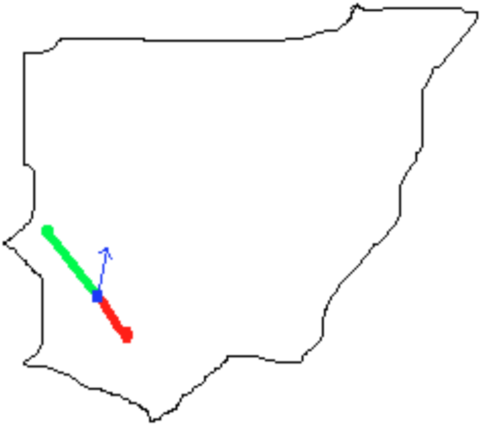

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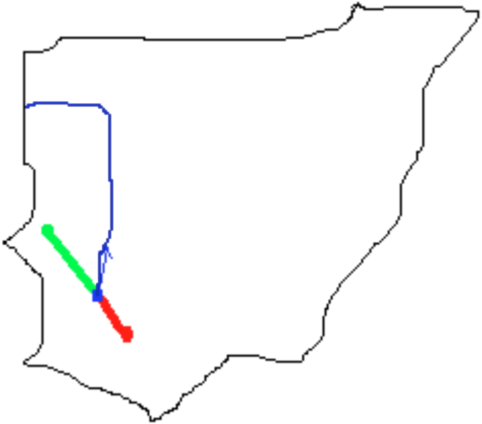

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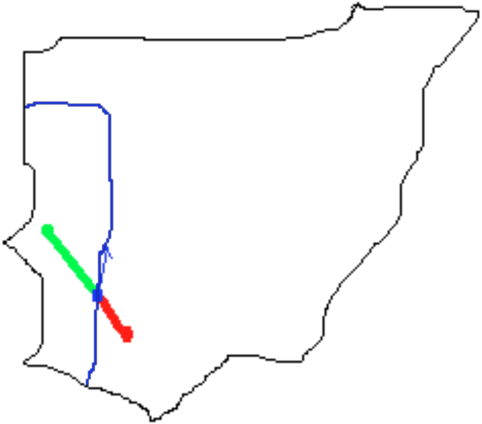

Goal of the Lecutres

Goal:

Characterize the solutions for all value of the initial conditions, parameter values and even "nearby" systems for the ODE:

$$\begin{cases} u'(t) = G(u,\lambda), & G : \Omega \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \\ u(0) = u_0, & u \in \mathbb{R}^n, \lambda \in \mathbb{R}. \end{cases}$$

- Why looking for zeros?
  - Equilibria, periodic orbits, stability, bifurcations. . .
- **Qualitative** vs **quantitative** analysis of differential equations.
- From **local** analysis to a **global** understanding of the system via the continuation of *special* solutions.

What is the best computational approach?

Skilled programmer and/or long term project

Be a man and write your own code!

or

The wimpy approach

Use a (good) black box code, but
understand what you are doing and be careful.

In this course we will follow the second path with a glance at the first. (AUTO and MATLAB).

Mathematical tools

- Taylor's theorem.
- Locating zeros: The elevator's theorem and Newton's method.
- Implicit function theorem.

Elevator's theorem

This elevator takes you to the second floor <u>without</u> passing through the first floor.

This is imposible signed: **Bolzano.**

Newton's method

Suppose u^0 is close to a zero of

$$G(u) = 0.$$

How do we compute a u^1 even closer to the zero?
Replace the left hand side by its linear part

$$G(u^1) \simeq G(u^0) + J(u^1 - u^0) \simeq 0,$$

where $J = G_u(u^0)$ is the Jacobian.

$$u^1 = u^0 - J^{-1} G(u^0).$$

In practice, solve

$$J \Delta u = -G(u^0),$$

and

$$u^1 = u^0 + \Delta u$$

and iterate up to convergence. (see Ch. 10 Kuznetsov)

The Implicit Function Theorem

Let $\mathbf{G} \; : \; \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n$ satisfy

(i) $\mathbf{G}(\mathbf{u}_0, \lambda_0) \;=\; \mathbf{0} \; , \qquad \mathbf{u}_0 \in \mathrm{R}^n \; , \quad \lambda_0 \in \mathrm{R} \; .$

(ii) $\mathbf{G}_{\mathbf{u}}(\mathbf{u}_0, \lambda_0)$ is nonsingular (*i.e.,* $\mathbf{u}_0$ is an *isolated solution*) ,

(iii) $\mathbf{G}$ and $\mathbf{G}_{\mathbf{u}}$ are smooth near $\mathbf{u}_0$.

Then there exists a unique, smooth *solution family* $\mathbf{u}(\lambda)$ such that

- $\mathbf{G}(\mathbf{u}(\lambda), \lambda) \;=\; \mathbf{0} \; , \qquad$ for all λ near λ_0 ,
- $\mathbf{u}(\lambda_0) \;=\; \mathbf{u}_0 \; .$

PROOF : See a good Analysis book $\cdots$

Persistence of solutions

Consider the equation

$$\mathbf{G}(\mathbf{u},\lambda) \;=\; \mathbf{0}\;, \qquad \mathbf{u}\;,\; \mathbf{G}(\cdot,\cdot) \in \mathrm{R}^n\;, \quad \lambda \in \mathrm{R}\;.$$

Let

$$\mathbf{x} \;\equiv\; (\mathbf{u}\;,\;\lambda)\;.$$

Then the equation can be written

$$\mathbf{G}(\mathbf{x}) \;=\; \mathbf{0}\;, \qquad \mathbf{G} \;:\; \mathrm{R}^{n+1} \;\rightarrow\; \mathrm{R}^n\;.$$

DEFINITION.

A solution $\mathbf{x}_0$ of $\mathbf{G}(\mathbf{x}) \;=\; \mathbf{0}$ is *regular* if the matrix

$$\mathbf{G}^0_{\mathbf{x}} \;\equiv\; \mathbf{G}_{\mathbf{x}}(\mathbf{x}_0)\;, \qquad \text{(with } n \text{ rows and } n+1 \text{ columns)}$$

has maximal rank, *i.e.*, if

$$\mathrm{Rank}(\mathbf{G}^0_{\mathbf{x}}) \;=\; n\;.$$

In the parameter formulation,

$$\mathbf{G}(\mathbf{u},\lambda) \;=\; \mathbf{0}\ ,$$

we have

$$\text{Rank}(\mathbf{G}_{\mathbf{x}}^0) \;=\; \text{Rank}(\mathbf{G}_{\mathbf{u}}^0 \mid \mathbf{G}_{\lambda}^0) \;=\; n \quad \Longleftrightarrow \quad \begin{cases} \text{(i)}\ \mathbf{G}_{\mathbf{u}}^0 \text{ is nonsingular,} \\ \\ \text{or} \\ \\ \text{(ii)}\ \begin{cases} \dim \mathcal{N}(\mathbf{G}_{\mathbf{u}}^0) \;=\; 1\ , \\ \text{and} \\ \mathbf{G}_{\lambda}^0 \notin \mathcal{R}(\mathbf{G}_{\mathbf{u}}^0)\ . \end{cases} \end{cases}$$

Above,

$$\mathcal{N}(\mathbf{G}_{\mathbf{u}}^0) \;\text{ denotes the } \textit{null space} \text{ of } \; \mathbf{G}_{\mathbf{u}}^0\ ,$$

and

$$\mathcal{R}(\mathbf{G}_{\mathbf{u}}^0) \text{ denotes the } \textit{range} \text{ of } \; \mathbf{G}_{\mathbf{u}}^0\ ,$$

i.e., the linear space spanned by the n columns of $\mathbf{G}_{\mathbf{u}}^0$.

THEOREM. Let

$$\mathbf{x}_0 \equiv (\mathbf{u}_0 , \lambda_0)$$

be a regular solution of

$$\mathbf{G}(\mathbf{x}) = \mathbf{0} .$$

Then, near $\mathbf{x}_0$, there exists a unique one-dimensional *solution family*

$$\mathbf{x}(s) \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0 .$$

PROOF. Since

$$\text{Rank}(\mathbf{G}^0_{\mathbf{x}}) = \text{Rank}(\mathbf{G}^0_{\mathbf{u}} \mid \mathbf{G}^0_{\lambda}) = n ,$$

then either $\mathbf{G}^0_{\mathbf{u}}$ is nonsingular and by the IFT we have

$$\mathbf{u} = \mathbf{u}(\lambda) \quad \text{near} \quad \mathbf{x}_0 ,$$

or else we can interchange colums in the Jacobian $\mathbf{G}^0_{\mathbf{x}}$ to see that the solution can locally be parametrized by one of the components of $\mathbf{u}$.

Thus a unique solution family passes through a regular solution. •

NOTE:

- Such a *solution family* is sometimes also called a *solution branch* .

- Case (ii) above is that of a *simple fold* , to be discussed later.

- Thus even near a simple fold there is a unique solution family.

- However, near such a fold, the family can not be parametrized by λ.

Parameter Continuation

Here the continuation parameter is taken to be λ .

Suppose we have a solution $(\mathbf{u}_0, \lambda_0)$ of

$$\mathbf{G}(\mathbf{u}, \lambda) \ = \ \mathbf{0} \ ,$$

as well as the direction vector $\dot{\mathbf{u}}_0$.

Here

$$\dot{\mathbf{u}} \ \equiv \ \frac{d\mathbf{u}}{d\lambda} \ .$$

We want to compute the solution $\mathbf{u}_1$ at $\lambda_1 \ \equiv \ \lambda_0 \ + \ \Delta\lambda$.

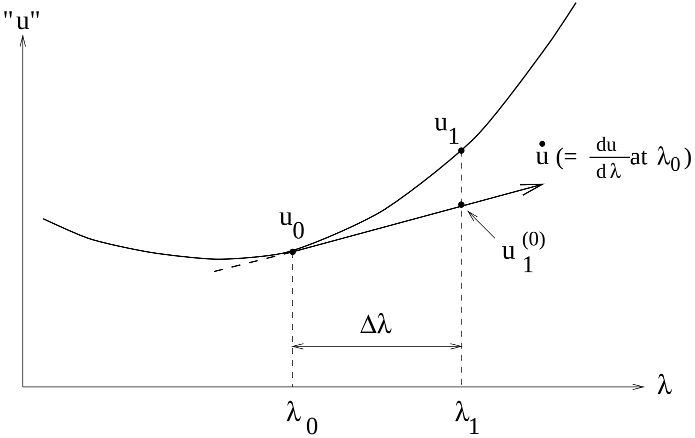

Figure 10: Graphical interpretation of parameter-continuation.

To solve the equation

$$\mathbf{G}(\mathbf{u}_1\ ,\ \lambda_1)\ =\ \mathbf{0}\ ,$$

for $\mathbf{u}_1$ (with $\lambda = \lambda_1$ fixed) we use Newton's method

$$\begin{aligned} \mathbf{G}_{\mathbf{u}}(\mathbf{u}_1^{(\nu)}, \lambda_1)\ \Delta\mathbf{u}_1^{(\nu)} &= -\ \mathbf{G}(\mathbf{u}_1^{(\nu)}, \lambda_1)\ , \\ \mathbf{u}_1^{(\nu+1)} &= \mathbf{u}_1^{(\nu)}\ +\ \Delta\mathbf{u}_1^{(\nu)}\ . \end{aligned} \qquad \nu\ =\ 0, 1, 2, \cdots\ .$$

As initial approximation use

$$\mathbf{u}_1^{(0)}\ =\ \mathbf{u}_0\ +\ \Delta\lambda\ \dot{\mathbf{u}}_0\ .$$

If

$$\mathbf{G}_{\mathbf{u}}(\mathbf{u}_1, \lambda_1)\quad \text{is nonsingular}\ ,$$

and $\Delta\lambda$ sufficiently small, then the Newton convergence theory guarantees that this iteration will converge.

After convergence, the new direction vector $\dot{\mathbf{u}}_1$ can be computed by solving

$$\mathbf{G}_{\mathbf{u}}(\mathbf{u}_1, \lambda_1)\ \dot{\mathbf{u}}_1 \ = \ -\ \mathbf{G}_{\lambda}(\mathbf{u}_1, \lambda_1)\ .$$

This equation follows from differentiating

$$\mathbf{G}(\mathbf{u}(\lambda), \lambda) \ = \ \mathbf{0}\ ,$$

with respect to λ at $\lambda = \lambda_1$.

NOTE:

- $\dot{\mathbf{u}}_1$ can be computed without another LU-factorization of $\mathbf{G}_{\mathbf{u}}(\mathbf{u}_1, \lambda_1)$.
- Thus the extra work to find $\dot{\mathbf{u}}_1$ is negligible.

Exercise

Excercise for Lecture 1

When will the parameter continuation fail?

Keller's Pseudo-Arclength Continuation

This method allows continuation of a solution family past a fold.

Suppose we have a solution $(\mathbf{u}_0, \lambda_0)$ of

$$\mathbf{G}(\ \mathbf{u}\ ,\ \lambda\ )\ =\ \mathbf{0}\ ,$$

as well as the direction vector $(\dot{\mathbf{u}}_0, \dot{\lambda}_0)$ of the solution branch.

Pseudo-arclength continuation solves the following equations for $(\mathbf{u}_1, \lambda_1)$:

$$\mathbf{G}(\mathbf{u}_1, \lambda_1)\ =\ \mathbf{0}\ ,$$

$$(\mathbf{u}_1\ -\ \mathbf{u}_0)^*\ \dot{\mathbf{u}}_0\ +\ (\lambda_1\ -\ \lambda_0)\ \dot{\lambda}_0\ -\ \Delta s\ =\ 0\ .$$

See Figure 11 for a graphical interpretation.

Pseudoarclength continuation

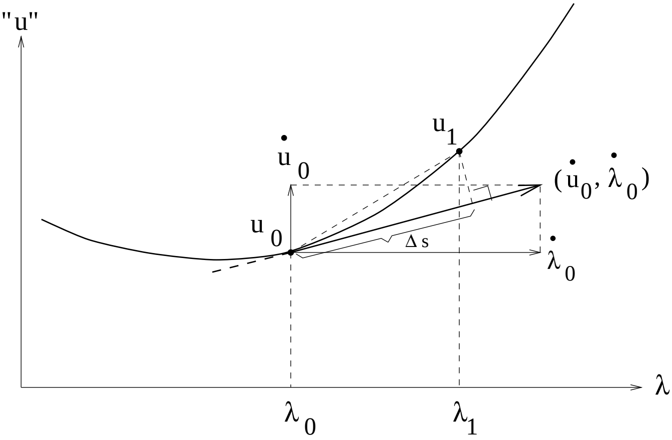

Figure 11: Graphical interpretation of pseudo-arclength continuation.

Pseudoarclength continuation

Solve the equations

$$\mathbf{G}(\mathbf{u}_1, \lambda_1) \;=\; \mathbf{0} \;,$$

$$(\mathbf{u}_1 \;-\; \mathbf{u}_0)^* \;\dot{\mathbf{u}}_0 \;+\; (\lambda_1 \;-\; \lambda_0)\;\dot{\lambda}_0 \;-\; \Delta s \;=\; 0 \;.$$

for $(\mathbf{u}_1, \lambda_1)$ by Newton's method:

$$\begin{pmatrix} (\mathbf{G}^1_{\mathbf{u}})^{(\nu)} & (\mathbf{G}^1_{\lambda})^{(\nu)} \\ \dot{\mathbf{u}}_0^* & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u}_1^{(\nu)} \\ \Delta \lambda_1^{(\nu)} \end{pmatrix} = - \begin{pmatrix} \mathbf{G}(\mathbf{u}_1^{(\nu)}, \lambda_1^{(\nu)}) \\ (\mathbf{u}_1^{(\nu)} - \mathbf{u}_0)^* \dot{\mathbf{u}}_0 + (\lambda_1^{(\nu)} - \lambda_0)\dot{\lambda}_0 - \Delta s \end{pmatrix} .$$

Next direction vector :

$$\begin{pmatrix} \mathbf{G}^1_{\mathbf{u}} & \mathbf{G}^1_{\lambda} \\ \dot{\mathbf{u}}_0^* & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}}_1 \\ \dot{\lambda}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} .$$

Pseudoarclength continuation

NOTE:

- In practice $(\dot{\mathbf{u}}_1, \dot{\lambda}_1)$ can be computed with one extra backsubstitution.
- The orientation of the branch is preserved if Δs is sufficiently small.
- The direction vector must be rescaled, so that indeed $\| \dot{\mathbf{u}}_1 \|^2 + \dot{\lambda}_1^2 = 1$.

Pseudoarclength continuation

THEOREM.

The *Jacobian* of the pseudo-arclength system is *nonsingular* at a *regular* solution point.

PROOF. Let

$$\mathbf{x} \equiv (\mathbf{u}, \lambda) \in \mathrm{R}^{n+1} .$$

Then pseudo-arclength continuation can be written as

$$\mathbf{G}(\mathbf{x}_1) = 0 ,$$

$$(\mathbf{x}_1 - \mathbf{x}_0)^* \dot{\mathbf{x}}_0 - \Delta s = 0 , \qquad (\| \dot{\mathbf{x}}_0 \| = 1) .$$

(See Figure 12 for a graphical interpretation.)

Pseudoarclength continuation

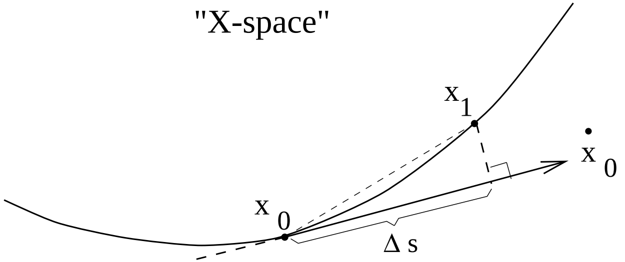

Figure 12: Parameter-independent pseudo-arclength continuation.

Pseudoarclength continuation

The matrix in Newton's method at $\Delta s = 0$ is

$$\begin{pmatrix} \mathbf{G}_{\mathbf{x}}^{0} \\ \dot{\mathbf{x}}_{0}^{*} \end{pmatrix} .$$

At a regular solution we have

$$\mathcal{N}(\mathbf{G}_{\mathbf{x}}^{0}) = \text{Span}\{\dot{\mathbf{x}}_{0}\} .$$

We must show that

$$\begin{pmatrix} \mathbf{G}_{\mathbf{x}}^{0} \\ \dot{\mathbf{x}}_{0}^{*} \end{pmatrix}$$

is nonsingular at a regular solution.

Pseudoarclength continuation

If on the contrary

$$\begin{pmatrix} \mathbf{G}_{\mathbf{x}}^0 \\ \dot{\mathbf{x}}_0^* \end{pmatrix}$$

is singular then

$$\mathbf{G}_{\mathbf{x}}^0 \, \mathbf{z} \; = \; 0 \qquad \text{and} \qquad \dot{\mathbf{x}}_0^* \, \mathbf{z} \; = \; 0 \; ,$$

for some vector $\mathbf{z} \neq \mathbf{0}$.

Thus

$$\mathbf{z} \; = \; c \, \dot{\mathbf{x}}_0 \; , \qquad \text{for some constant} \;\; c \; .$$

But then

$$0 \; = \; \dot{\mathbf{x}}_0^* \, \mathbf{z} \; = \; c \, \dot{\mathbf{x}}_0^* \, \dot{\mathbf{x}}_0 \; = \; c \, \| \, \dot{\mathbf{x}}_0 \, \|^2 \; = \; c \; ,$$

so that $\mathbf{z} = \mathbf{0}$, which is a contradiction. •

Recall the ingredients

The building blocks for the continuation of solutions are:

- Newton's method of the properly chosen function $G(x)$.
- Pseudoarclength continuation.
- Convergence, step control and accuracy.
- Appropriate test function.
- Data handling and representation.

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All these in an efficient way.

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All these in an efficient way.
Extensions:

- detect and identify bifurcation points
- branch switching
- homo- and heteroclinic orbits

Ready to solve the excercises

Compute the bifurcation diagram of

- $f(x, \lambda) = \lambda + x^2$
- $f(x, \lambda) = (x - \lambda)x$
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for all values of x and λ.

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for all values of x and λ.

Exercise Continue the perturbed pitchfork case. (add a $+\epsilon$ term, and continue in ϵ.

A Predator-Prey Model

(`AUTO` demo `pp2`.)

$$\begin{cases} u_1' = 3u_1(1-u_1) - u_1u_2 - \lambda(1-e^{-5u_1}) \ , \\ u_2' = -u_2 + 3u_1u_2 \ . \end{cases}$$

Here u_1 may be thought of as "fish" and u_2 as "sharks", while the term

$$\lambda\ (1-e^{-5u_1}) \ ,$$

represents "fishing", with "fishing-quota" λ .

When $\lambda = 0$ the *stationary solutions* are

$$\left.\begin{array}{rl} 3u_1(1-u_1) - u_1u_2 & = 0 \\ -u_2 + 3u_1u_2 & = 0 \end{array}\right\} \Rightarrow (u_1,u_2) = (0,0) \ , \ (1,0) \ , \ (\frac{1}{3},2) \ .$$

The Jacobian matrix is

$$\mathbf{G_u} = \begin{pmatrix} 3 - 6u_1 \; - \; u_2 \; - \; 5\lambda e^{-5u_1} & -u_1 \\ 3u_2 & -1 + 3u_1 \end{pmatrix} \; = \; \mathbf{G_u}(u_1, u_2; \lambda) \; .$$

$$\mathbf{G_u}(0,0;0) \; = \; \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}; \; \text{eigenvalues 3,-1} \quad \text{(unstable)} \; .$$

$$\mathbf{G_u}(1,0;0) \; = \; \begin{pmatrix} -3 & -1 \\ 0 & 2 \end{pmatrix}; \; \text{eigenvalues -3,2} \quad \text{(unstable)} \; .$$

$$\mathbf{G_u}(\frac{1}{3},2;0) \; = \; \begin{pmatrix} -1 & -\frac{1}{3} \\ 6 & 0 \end{pmatrix}; \; \text{eigenvalues} \; \begin{cases} (-1-\mu)(-\mu)+2 \; = \; 0 \\ \mu^2 \; + \; \mu + 2 \; = \; 0 \\ \mu_\pm \; = \; \frac{-1\pm\sqrt{-7}}{2} \\ \text{Re}(\mu_\pm) < 0 \quad \text{(stable)} \; . \end{cases}$$

All three Jacobians at $\lambda = 0$ are nonsingular.

Thus, by the IFT, all three stationary points persist for (small) $\lambda \neq 0$.

In this problem we can *explicitly* find all solutions (see Figure 1) :

Branch I :

$$(u_1, u_2) \; = \; (0, 0) \; .$$

Branch II :

$$u_2 \; = \; 0 \; , \qquad \lambda \; = \; \frac{3u_1(1 - u_1)}{1 - e^{-5u_1}} \; .$$

$$\text{(Note that } \lim_{u_1 \to 0} \lambda \; = \; \lim_{u_1 \to 0} \frac{3(1 - 2u_1)}{5e^{-5u_1}} \; = \; \frac{3}{5} \; .)$$

Branch III :

$$u_1 \; = \; \frac{1}{3}, \qquad \frac{2}{3} - \frac{1}{3} u_2 - \lambda(1 - e^{-5/3}) \; = \; 0 \; \Rightarrow \; u_2 \; = \; 2 - 3\lambda(1 - e^{-5/3}) \; .$$

These solution families intersect at two *branch points*, one of which is

$$(u_1, u_2, \lambda) \; = \; (0, 0, 3/5) \; .$$

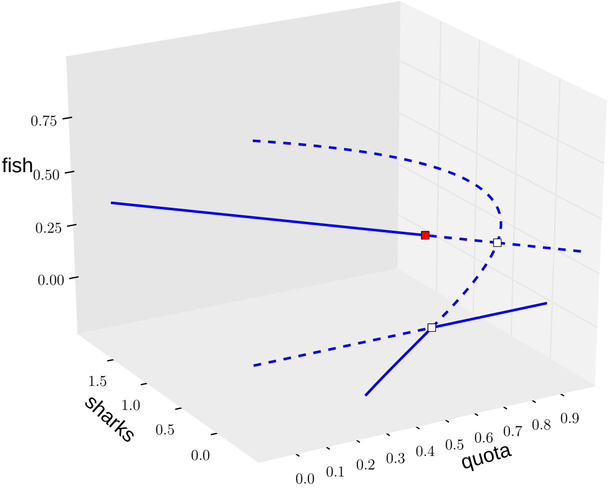


Figure 1: Stationary solution families of the predator-prey model. Solid/dashed lines denote stable/unstable solutions. Note the *fold* , the *bifurcations* (open squares), and the *Hopf bifurcation* (red square).

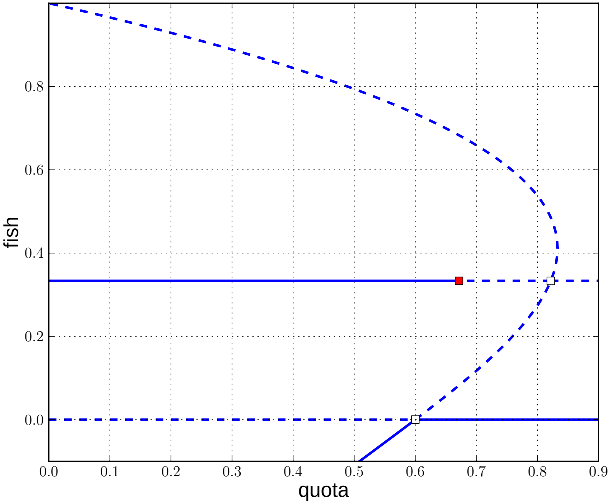


Figure 2: Stationary solution families of the predator-prey model, showing *fish* versus *quota*. Solid/dashed lines denote stable/unstable solutions.

- Stability of branch I :

$$\mathbf{G_u}((0,0);\lambda) \;=\; \begin{pmatrix} 3-5\lambda & 0 \\ 0 & -1 \end{pmatrix}; \quad \text{eigenvalues} \;\; 3-5\lambda, \;\; -1 \; .$$

Hence the trivial solution is :

$$\text{unstable if } \lambda < 3/5 \; ,$$

and

$$\text{stable if } \lambda > 3/5 \; ,$$

as indicated in Figure 2.

- Stability of branch II :

This family has no stable positive solutions.

∘ Stability of branch III :

At

$$\lambda_H \approx 0.67 ,$$

(the red square in Figure 2) the complex eigenvalues cross the imaginary axis.

This crossing is a *Hopf bifurcation*, a topic to be discussed later.

Beyond λ_H there are *periodic solutions* whose period T increases as λ increases. (See Figure 4 for some representative periodic orbits.)

The period becomes infinite at $\lambda = \lambda_\infty \approx 0.70$.

This final orbit is called a *heteroclinic cycle*.

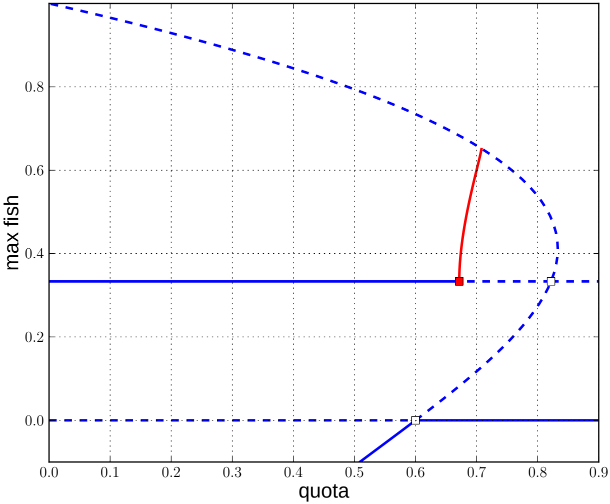


Figure 3: Stationary (blue) and periodic (red) solution families of the predator-prey model.

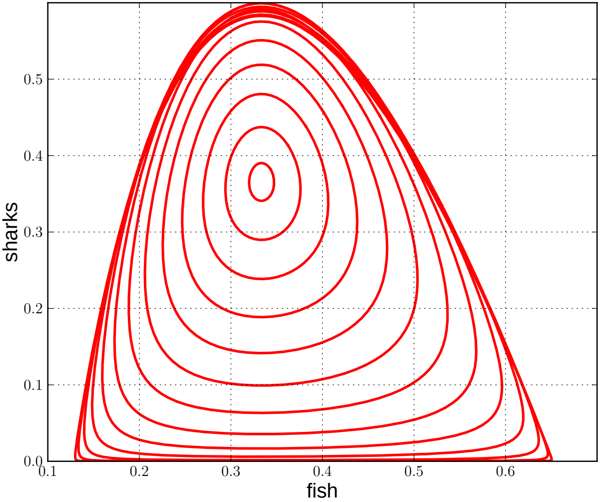


Figure 4: Some periodic solutions of the predator-prey model. The largest orbits are very close to a *heteroclinic cycle*.

From Figure 3 we can deduce the solution behavior for (slowly) increasing λ :

- Branch III is followed until $\lambda_H \approx 0.67$.

- Periodic solutions of increasing period until $\lambda = \lambda_\infty \approx 0.70$.

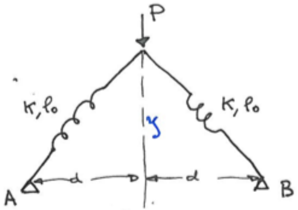
- Collapse to trivial solution (Branch I).

EXERCISE.

Use `AUTO` to repeat the numerical calculations (demo `pp2`) .

Sketch *phase plane diagrams* for $\lambda = 0,\ 0.5,\ 0.68,\ 0.70,\ 0.71$.

Nontrivial example II: equilibria of the loaded arch



$$\frac{\partial \tilde{V}}{\partial \tilde{y}} = (\tilde{\ell}(y)-1)\cdot\frac{d\tilde{\ell}}{d\tilde{y}} + \mu = (\tilde{\ell}(y)-1)\frac{\tilde{y}}{\tilde{\ell}} + \mu = \tilde{y} - \frac{\tilde{y}}{\sqrt{\rho^2+\tilde{y}^2}} + \mu$$

$$y'' = -y + \frac{y}{\sqrt{\rho^2 + y^2}} + \mu$$

Analyze

$$\begin{aligned} \frac{du_1}{dt} &= \alpha u_1 - u_2 - \beta u_1(u_1^2 + u_2^2) \\ \frac{du_2}{dt} &= u_1 + \alpha u_2 - \beta u_2(u_1^2 + u_2^2) \end{aligned} \tag{1}$$

The Hopf Bifurcation Theorem

THEOREM. Suppose that along a stationary solution family $(\mathbf{u}(\lambda), \lambda)$, of

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, \lambda) ,$$

a complex conjugate pair of eigenvalues

$$\alpha(\lambda) \pm i\,\beta(\lambda) ,$$

of $f_{\mathbf{u}}(\mathbf{u}(\lambda), \lambda)$ crosses the imaginary axis transversally, *i.e.*, for some λ_0 ,

$$\alpha(\lambda_0) = 0 , \qquad \beta(\lambda_0) \neq 0 , \quad \text{and} \quad \dot{\alpha}(\lambda_0) \neq 0 .$$

Also assume that there are no other eigenvalues on the imaginary axis.

Then there is a *Hopf bifurcation*, *i.e.*, a family of periodic solutions bifurcates from the stationary solution at $(\mathbf{u}_0, \lambda_0)$. $\circ$

NOTE: The assumptions also imply that $\mathbf{f}_{\mathbf{u}}^0$ is nonsingular, so that the stationary solution family can indeed be parametrized locally using λ .

The BVP Approach.

Consider

$$\mathbf{u}'(t) = \mathbf{f}(\, \mathbf{u}(t) \,,\, \lambda \,) \,, \qquad \mathbf{u}(\cdot) \,,\, \mathbf{f}(\cdot) \in \mathrm{R}^n \,, \qquad \lambda \in \mathrm{R} \,.$$

Fix the interval of periodicity by the transformation

$$t \rightarrow \frac{t}{T} \,.$$

Then the equation becomes

$$\boxed{\mathbf{u}'(t) = T\, \mathbf{f}(\, \mathbf{u}(t) \,,\, \lambda \,)} \,, \qquad \mathbf{u}(\cdot) \,,\, \mathbf{f}(\cdot) \in \mathrm{R}^n \,, \qquad T \,,\, \lambda \in \mathrm{R} \,.$$

and we seek solutions of period 1 , *i.e.*,

$$\boxed{\mathbf{u}(0) = \mathbf{u}(1)} \,.$$

Note that the period T is one of the unknowns.

The above equations do not uniquely specify $\mathbf{u}$ and T :

Assume that we have computed

$$(\mathbf{u}_{k-1}(\cdot) \ , \ T_{k-1} \ , \ \lambda_{k-1} \ ) \ ,$$

and we want to compute the next solution

$$(\mathbf{u}_{k}(\cdot) \ , \ T_{k} \ , \ \lambda_{k} \ ) \ .$$

Specifically, $\mathbf{u}_k(t)$ can be translated freely in time:

If $\mathbf{u}_k(t)$ is a periodic solution, then so is

$$\mathbf{u}_k(t+\sigma) \ ,$$

for any σ .

Thus, a "*phase condition*" is needed.

An example is the Poincaré orthogonality condition

$$(\mathbf{u}_k(0) \; - \; \mathbf{u}_{k-1}(0))^* \; \mathbf{u}_{k-1}^{'}(0) \; = \; 0 \; .$$

(Below we derive a numerically more suitable phase condition.)

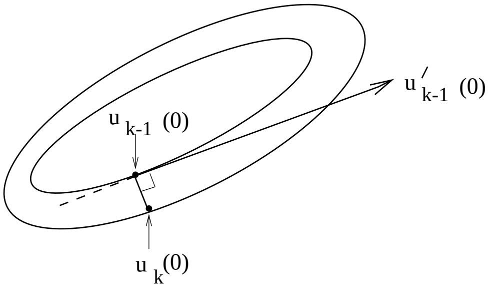


Figure 48: Graphical interpretation of the Poincaré phase condition.

Integral Phase Condition

If $\tilde{\mathbf{u}}_k(t)$ is a solution then so is

$$\tilde{\mathbf{u}}_k(t+\sigma) ,$$

for any σ .

We want the solution that minimizes

$$D(\sigma) \equiv \int_0^1 \| \tilde{\mathbf{u}}_k(t+\sigma) - \mathbf{u}_{k-1}(t) \|_2^2 \, dt .$$

The optimal solution

$$\tilde{\mathbf{u}}_k(t+\hat{\sigma}) ,$$

must satisfy the necessary condition

$$D'(\hat{\sigma}) = 0 .$$

Differentiation gives the necessary condition

$$\int_0^1 (\ \tilde{\mathbf{u}}_k(t+\hat{\sigma}) \ - \ \mathbf{u}_{k-1}(t)\ )^* \ \tilde{\mathbf{u}}_k'(t+\hat{\sigma})\ dt \ = \ 0 \ .$$

Writing

$$\mathbf{u}_k(t) \ \equiv \ \tilde{\mathbf{u}}_k(t+\hat{\sigma}) \ ,$$

gives

$$\int_0^1 (\ \mathbf{u}_k(t) \ - \ \mathbf{u}_{k-1}(t)\ )^* \ \mathbf{u}_k'(t)\ dt \ = \ 0 \ .$$

Integration by parts, using periodicity, gives

$$\boxed{\int_0^1 \ \mathbf{u}_k(t)^* \ \mathbf{u}_{k-1}'(t)\ dt \ = \ 0} \ .$$

This is the *integral phase condition.*

Pseudo-Arclength Continuation

We use pseudo-arclength continuation to follow a family of periodic solutions.

This allows calculation past folds along a family of periodic solutions.

It also allows calculation of a "vertical family" of periodic solutions.

For periodic solutions the pseudo-arclength equation is

$$\int_0^1 (\mathbf{u}_k(t) - \mathbf{u}_{k-1}(t))^* \dot{\mathbf{u}}_{k-1}(t)\ dt \ + \ (T_k - T_{k-1})\dot{T}_{k-1} \ + \ (\lambda_k - \lambda_{k-1})\dot{\lambda}_{k-1} = \Delta s \ .$$

In summary, we have the following equations for continuing periodic solutions:

$$\mathbf{u}_k'(t) \;=\; T\;\mathbf{f}(\;\mathbf{u}_k(t)\;,\;\lambda_k\;)\;,$$

$$\mathbf{u}_k(0) \;=\; \mathbf{u}_k(1)\;,$$

$$\int_0^1 \mathbf{u}_k(t)^*\;\mathbf{u}_{k-1}^{'}(t)\;dt \;=\; 0\;,$$

with pseudo-arclength continuation equation

$$\int_0^1 (\mathbf{u}_k(t) - \mathbf{u}_{k-1}(t))^* \dot{\mathbf{u}}_{k-1}(t)\;dt \;+\; (T_k - T_{k-1})\dot{T}_{k-1} \;+\; (\lambda_k - \lambda_{k-1})\dot{\lambda}_{k-1} = \Delta s\;.$$

Here

$$\mathbf{u}(\cdot)\;,\;\mathbf{f}(\cdot)\;\in\;\mathrm{R}^n\;,\qquad \lambda\;,T\;\in\;\mathrm{R}\;.$$

Starting at a Hopf Bifurcation

Let

$$(\mathbf{u}_0 \ , \ \lambda_0) \ ,$$

be a Hopf bifurcation point, *i.e.*,

$$\mathbf{f}_{\mathbf{u}}(\ \mathbf{u}_0 \ , \ \lambda_0 \ ) \ ,$$

has a simple conjugate pair of purely imaginary eigenvalues

$$\pm \ i \ \omega_0 \ , \qquad \omega_0 \ \neq \ 0 \ ,$$

and no other eigenvalues on the imaginary axis.

Also, the pair crosses the imaginary axis transversally with respect to λ .

By the Hopf Bifurcation Theorem, a family of periodic solutions bifurcates.

Hopf theorem

Asymptotic estimates for periodic solutions near the Hopf bifurcation :

$$\begin{aligned} \mathbf{u}(\ t\ ;\ \epsilon\ ) &= \mathbf{u}_0 \ + \ \epsilon\ \boldsymbol{\phi}(t) \ + \ \mathcal{O}(\epsilon^2)\ , \\ T(\epsilon) &= T_0 \ + \ \mathcal{O}(\epsilon^2)\ , \\ \lambda(\epsilon) &= \lambda_0 \ + \ \mathcal{O}(\epsilon^2)\ . \end{aligned}$$

Here ϵ locally parametrizes the family of periodic solutions.

$T(\epsilon)$ denotes the period, and

$$T_0 = \frac{2\pi}{\omega_0}\ .$$

The function $\boldsymbol{\phi}(t)$ is the normalized nonzero periodic solution of the linearized, constant coefficient problem

$$\boldsymbol{\phi}'(t) = \mathbf{f}_{\mathbf{u}}(\mathbf{u}_0, \lambda_0)\ \boldsymbol{\phi}(t)\ .$$

Hopf theorem

To compute a first periodic solution

$$(\mathbf{u}_1(\cdot) \ , \ T_1 \ , \ \lambda_1 \ ) \ ,$$

near a Hopf bifurcation $(\mathbf{u}_0, \lambda_0)$, we still have

$$\boxed{\mathbf{u}_1'(t) \ = \ T \ \mathbf{f}(\ \mathbf{u}_1(t) \ , \ \lambda_1 \ )} \ , \qquad (10)$$

$$\boxed{\mathbf{u}_1(0) \ = \ \mathbf{u}_1(1)} \ . \qquad (11)$$

Initial estimates for Newton's method are

$$\mathbf{u}_1^{(0)}(t) \ = \ \mathbf{u}_0 \ + \ \Delta s \ \boldsymbol{\phi}(t) \ , \qquad T_1^{(0)} \ = \ T_0 \ , \qquad \lambda_1^{(0)} \ = \ \lambda_0 \ .$$

Above, $\boldsymbol{\phi}(t)$ is a nonzero solution of the time-scaled, linearized equations

$$\boldsymbol{\phi}'(t) = T_0 \ \mathbf{f_u}(\mathbf{u}_0, \lambda_0) \ \boldsymbol{\phi}(t) \ , \qquad \boldsymbol{\phi}(0) = \boldsymbol{\phi}(1) \ ,$$

namely,

$$\boldsymbol{\phi}(t) = \sin(2\pi t) \ \mathbf{w}_s + \cos(2\pi t) \ \mathbf{w}_c \ ,$$

where

$$(\ \mathbf{w}_s \ , \ \mathbf{w}_c \ ) \ ,$$

is a null vector in

$$\begin{pmatrix} -\omega_0 \ I & \mathbf{f_u}(\mathbf{u}_0, \lambda_0) \\ \mathbf{f_u}(\mathbf{u}_0, \lambda_0) & \omega_0 \ I \end{pmatrix} \begin{pmatrix} \mathbf{w}_s \\ \mathbf{w}_c \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} , \qquad \omega_0 = \frac{2\pi}{T_0} \ .$$

The nullspace is generically two-dimensional since

$$\begin{pmatrix} -\mathbf{w}_c \\ \mathbf{w}_s \end{pmatrix} ,$$

is also a null vector.

Hopf theorem

For the phase equation we "align" $\mathbf{u}_1$ with $\boldsymbol{\phi}(t)$, *i.e.*,

$$\boxed{\int_0^1 \mathbf{u}_1(t)^* \, \boldsymbol{\phi}'(t) \, dt \; = \; 0} \, .$$

Since

$$\dot{\lambda}_0 \; = \; \dot{T}_0 \; = \; 0 \, ,$$

the pseudo-arclength equation for the first step reduces to

$$\boxed{\int_0^1 \; (\; \mathbf{u}_1(t) - \mathbf{u}_0(t) \;)^* \; \boldsymbol{\phi}(t) \; dt \; = \; \Delta s} \, .$$

Numerical details

- AUTO solves BVP with orthogonal collocation with adaptative mesh selection.
- Floquet multipliers are computed for free.
- The code is partially parallelized (openmp and mpi).
- AUTO can solve in a efficient way system of moderate to large dimensions.

A difficult example: Bogdanov-Takens bifurcation

Analyze

$$\begin{aligned}\frac{du_1}{dt} &= u_2\\ \frac{du_2}{dt} &= -n + bu_2 + u_1^2 + u_1 u_2\end{aligned}$$

Some results on homoclinic and heteroclinic connections in planar systems, A. Gasull, H. Giacomini and J. Torregrosa (Nonlinearity)
See also Kuznetsov's book.

Unusual applications of continuation.

Zeros, continuation and bifurcations.

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Any problem that may be formulated as $G(u, \lambda) = 0$ is suitable for continuation.

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- What about computing eigenvalues?

Unusual applications of continuation.

Zeros, continuation and bifurcations.

Any problem that may be formulated as $G(u, \lambda) = 0$ is suitable for continuation.

- What about computing eigenvalues?
- and initial value problems?

Computing eigenvalues with AUTO

What are the eigenvaues of

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}?$$

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with Matlab,

eig(A)= [15.0000 4.89990 -4.89990]

with AUTO

$$(A - \lambda I)v = 0$$

Example: The Lorenz Equations

(`AUTO` demos `lor`, `lrz`, `man`.)

$$
\begin{aligned}
x' &= \sigma\ (y - x)\ , \\
y' &= \rho\ x\ -\ y\ -\ x\ z\ , \\
z' &= x\ y\ -\ \beta\ z\ ,
\end{aligned}
$$

where

$$\sigma\ =\ 10 \qquad \text{and} \qquad \beta\ =\ 8/3\ .$$

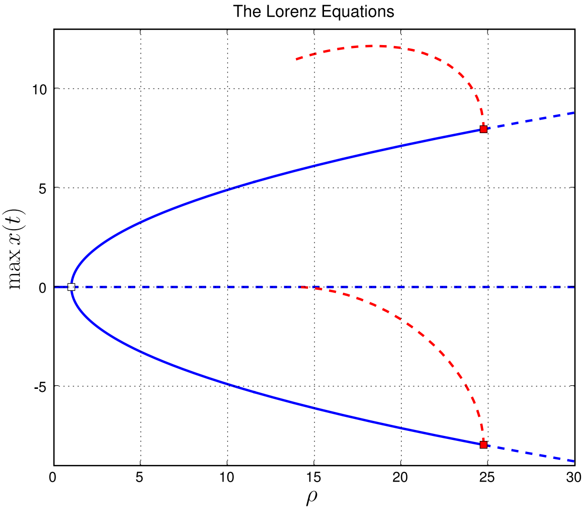

Figure 68: Bifurcation diagram of the Lorenz equations.

NOTE:

- The zero solution is unstable for $\rho > 1$.
- Two nonzero stationary solutions bifurcate at $\rho = 1$.
- The nonzero stationary solutions become unstable for $\rho > \rho_H$.
- At ρ_H ($\rho_H \approx 24.7$) there are Hopf bifurcations.
- Unstable periodic solutions emanate from each Hopf bifurcation.
- These families end in *homoclinic orbits* (infinite period) at $\rho \approx 13.9$.
- For $\rho > \rho_H$ there is the famous *Lorenz attractor*.

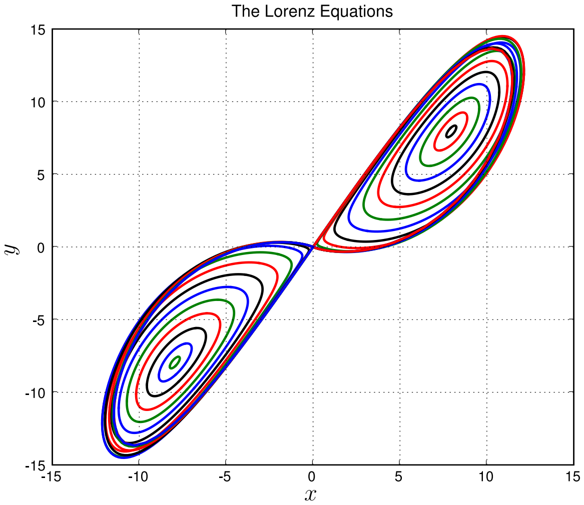

Figure 69: Unstable periodic orbits of the Lorenz equations.

The Lorenz Manifold

- For $\rho > 1$ the origin is a *saddle point* .
- The Jacobian has two negative eigenvalues and one positive eigenvalue.
- The two negative eigenvalues give rise to a 2D *stable manifold* .
- This manifold is known as as the *Lorenz Manifold* .
- The Lorenz Manifold helps us understand the *Lorenz attractor* .

Discrete and Continuous Dynamical Systems, 2010; (to appear).

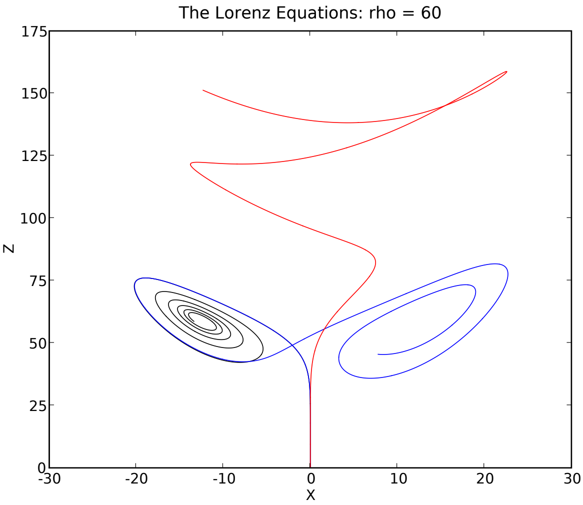

Figure 70: Three orbits whose initial conditions agree to >11 decimal places !

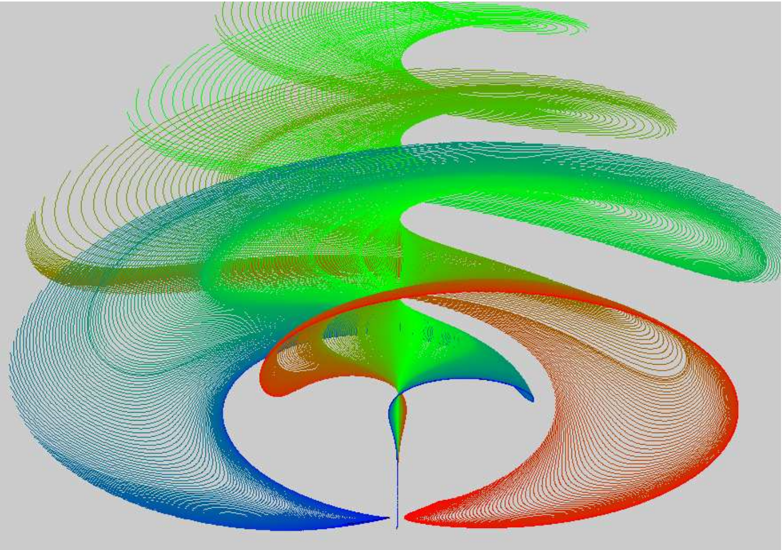

Figure 71: A small portion of a Lorenz Manifold ···

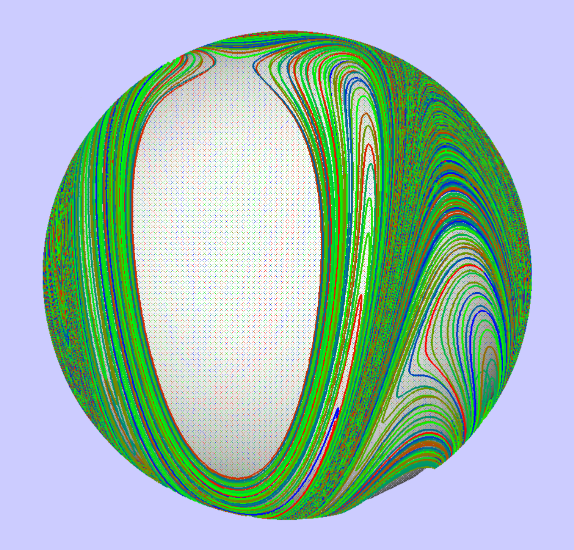

Figure 72: Intersection of a Lorenz Manifold with a sphere ($\rho = 35$, $R = 100$).

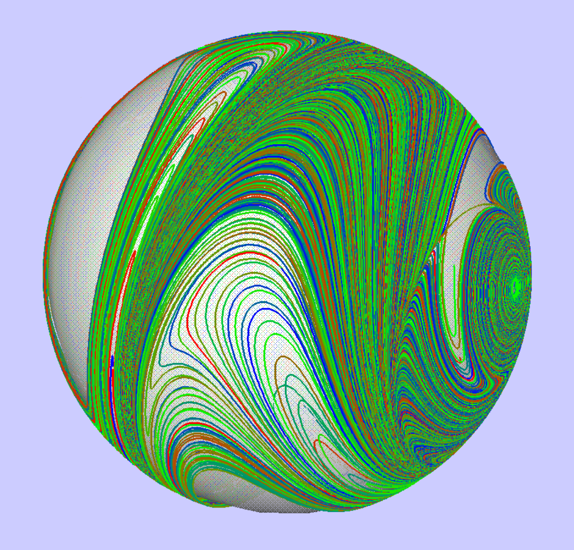

Figure 73: Intersection of a Lorenz Manifold with a sphere ($\rho = 35$, $R = 100$).

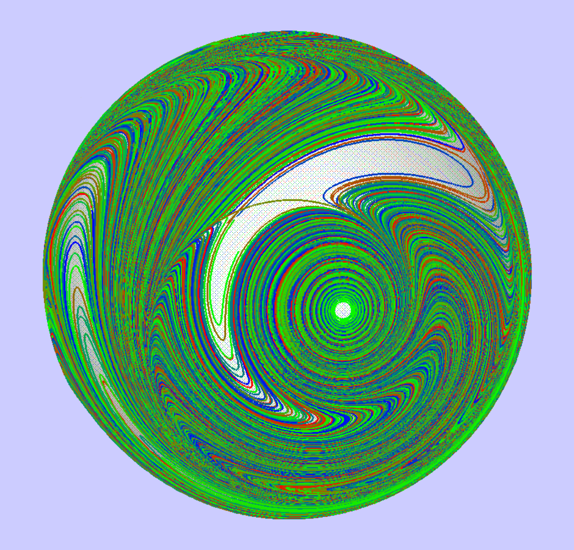

Figure 74: Intersection of a Lorenz Manifold with a sphere ($\rho = 35$, $R = 100$).

NOTE:

- As shown, crossings of the Lorenz manifold with *a sphere* can be located.
- Crossings of the Lorenz manifold with *the plane* $z = \rho - 1$ can be located.
- Connections between the origin and the nonzero equilibria can be located.
- There are subtle variations on the algorithm !

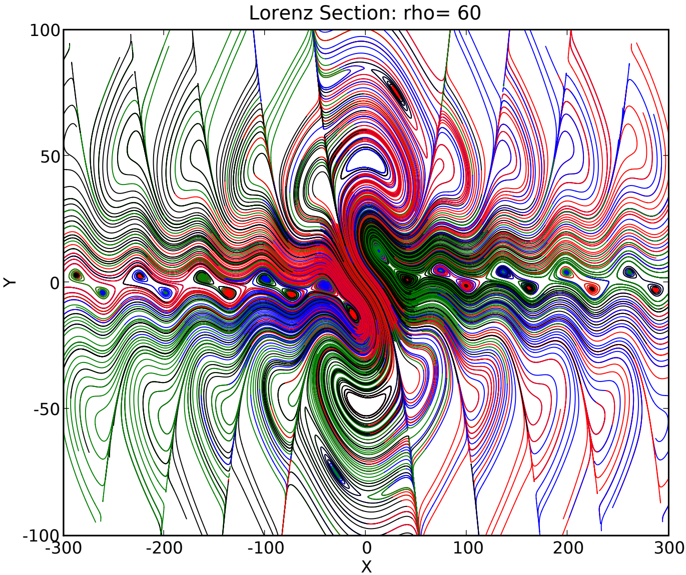

Figure 75: Crossings of the Lorenz Manifold with the plane $z = \rho - 1$