LIE ALGEBRAS AND LIE GROUPS IN PHYSICS

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Lecture 2

THE ALGEBRAIC QUARK MODEL

The constituents of hadrons are quarks and gluons. Quarks and gluons have internal and space degrees of freedom.
An algebraic description must involve both. The internal and space degrees of freedom may in general be mixed. However, it is usually assumed that they can be separated into

$$g = \mathfrak{R} \oplus \mathfrak{I}$$

Space ↗ ($\mathfrak{R}$) Internal ↖ ($\mathfrak{I}$)

INTERNAL QUANTUM NUMBERS OF QUARKS

Quarks are assumed to be fermions with internal degrees of freedom

Color $SU_c(3)$	blu, green, red
Spin $SU_s(2)$	↓,↑
Flavor $SU_f(6)$	light:u,d,s; heavy:c,b,t

Since c,b,t are much heavier than u,d,s, the flavor part is usually split into

$$SU_f(6) \to SU_f(3) \otimes U_{f_4}(1) \otimes U_{f_5} \otimes U_{f_6}$$

Here we will consider only the three light flavors u,d,s. The addition of the heavy flavors is trivial since U(1) is Abelian. Internal degrees of freedom considered here

$$SU_s(2) \otimes SU_f(3) \otimes SU_c(3)$$

Spin-flavor can be combined into

$$SU_{sf}(6) \supset SU_s(2) \otimes SU_f(3)$$

Gürsey-Radicati Gell-Mann

Constituents:

$$a^\dagger_{u\uparrow}, a^\dagger_{u\downarrow}, a^\dagger_{d\uparrow}, a^\dagger_{d\downarrow}, a^\dagger_{s\uparrow}, a^\dagger_{s\downarrow}$$

The bilinear products

$$G_{ij} = a^\dagger_i a_j \qquad (i, j = 1, \ldots, 6)$$

are elements of $U_{sf}(6)$. Subtracting $\sum_i a^\dagger_i a_i$, we have the 35 elements of $SU_{sf}(6)$.

In particle physics, no distinction is made between algebras and groups. Capital letters are used for both, instead of lowercase, g, for algebras and capital, G, for groups.

Further splitting of $SU_f(3)$

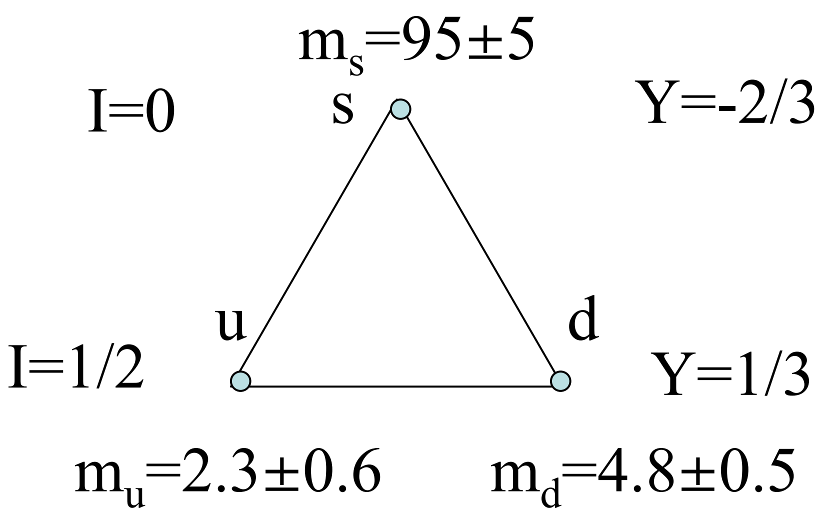
$$SU_f(3) \supset SU_I(2) \otimes U_Y(1)$$

Isospin (for $SU_I(2)$) Hypercharge (for $U_Y(1)$)

Y=B+Σ

Hypercharge=Baryon number+Strangeness

Classification of quarks and their masses



	d	u	s
B	1/3	1/3	1/3
Σ	0	0	-1
Y	1/3	1/3	-2/3

In particle physics, irreps are labeled not by the Young tableau, but by the dimension of the representation. This notation is not good as often two different representations have the same dimension. In this case a bar is put over one of them. Here both notations will be used for clarity.

Quantum number assignments

(a) Spin-flavor, $SU_{sf}(6)$

Quarks q $\qquad \square \equiv [1,0,0,0,0] \equiv 6 \equiv^2 3$

□

□

Antiquarks $\bar{q}$ $\qquad \square \equiv [1,1,1,1,1] \equiv \bar{6} \equiv^2 \bar{3}$

□

□

Complete classification scheme for multi quark-antiquark states (spin-flavor):

$$\left| \begin{array}{l} SU_{sf}(6) \supset SU_f(3) \oplus SU_s(2) \supset SU_I(2) \oplus U_Y(1) \oplus SU_s(2) \supset \\ \quad\downarrow \qquad\quad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ \quad [\lambda] \quad [\mu_1, \mu_2] \quad S \qquad\quad I \qquad\quad Y \\ \supset Spin_I(2) \oplus U_Y(1) \oplus Spin_s(2) \\ \qquad\quad \downarrow \qquad\qquad\qquad\quad \downarrow \\ \qquad\quad I_3 \qquad\qquad\qquad\quad S_3 \end{array} \right\rangle$$

(b) Color $[\gamma_1, \gamma_2]$

Quarks q $\square \equiv [1,0] \equiv 3_c$

Antiquarks $\bar{q}$ $\square \equiv [1,1] \equiv \bar{3}_c$
$\square$

Complete classification scheme for multi quark-antiquark states (color):

Hadrons are assumed to be colorless.
The only allowed representation is the one-dimensional representation

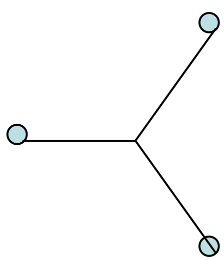
$\square$
$\square \equiv 1_c$
$\square$

SPACE DEGREES OF FREEDOM

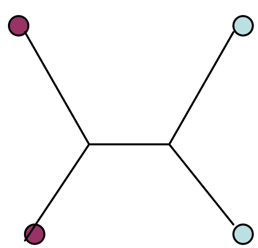
Hadrons are bound states of quarks and gluons.
In a string-like model, the lowest configurations are:

[Notation ○ q ● $\bar{q}$]

Mesons

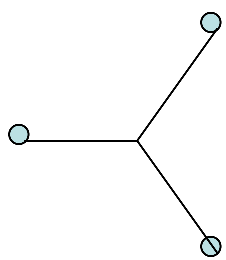
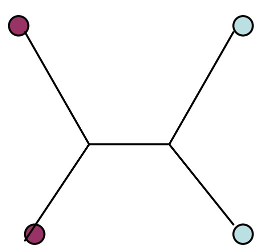


Baryons



Tetraquarks

The space algebraic structure is obtained by a bosonic quantization of the Jacobi variables in terms of representations of $U(3k-2) \equiv \Re$

		k	U(3k-2)	Discrete symmetry
	$\vec{\rho}$	2	U(4)	$C_{\infty h}$
	$\vec{\rho}, \vec{\lambda}$	3	U(7)	D_{3h}
	$\vec{\rho}, \vec{\lambda}, \vec{\eta}$	4	U(10)	D_{2h}

ALGEBRAIC STRUCTURE OF HADRONS

Total algebraic structure of light hadrons

Algebras

$$\Re \oplus su_{sf}(6) \oplus su_c(3) \supset \underset{\text{space}}{\Re} \oplus \underset{\text{spin}}{su_s(2)} \oplus \underset{\text{flavor}}{su_f(3)} \oplus \underset{\text{color}}{su_c(3)}$$

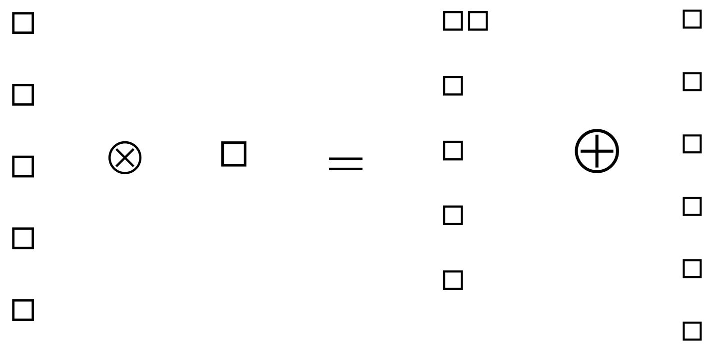
Groups

$$\Re \otimes SU_{sf}(6) \otimes SU_c(3) \supset \Re \otimes SU_s(2) \otimes SU_f(3) \otimes SU(3)$$

MESONS $q\bar{q}$

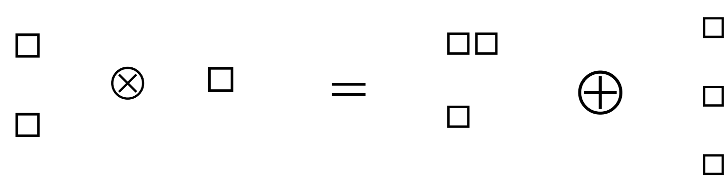
(a) Internal degrees of freedom

Spin-flavor part $SU_{sf}(6)$



$$\bar{6} \otimes 6 = 35 \oplus 1$$

Color part $SU_c(3)$



$$\bar{3}_c \otimes 3_c = 8_c \oplus 1_c$$

Only allowed irrep

Spectrum of states: SPIN-FLAVOR DYNAMIC SYMMETRY

Mass formula for the mass squared operator

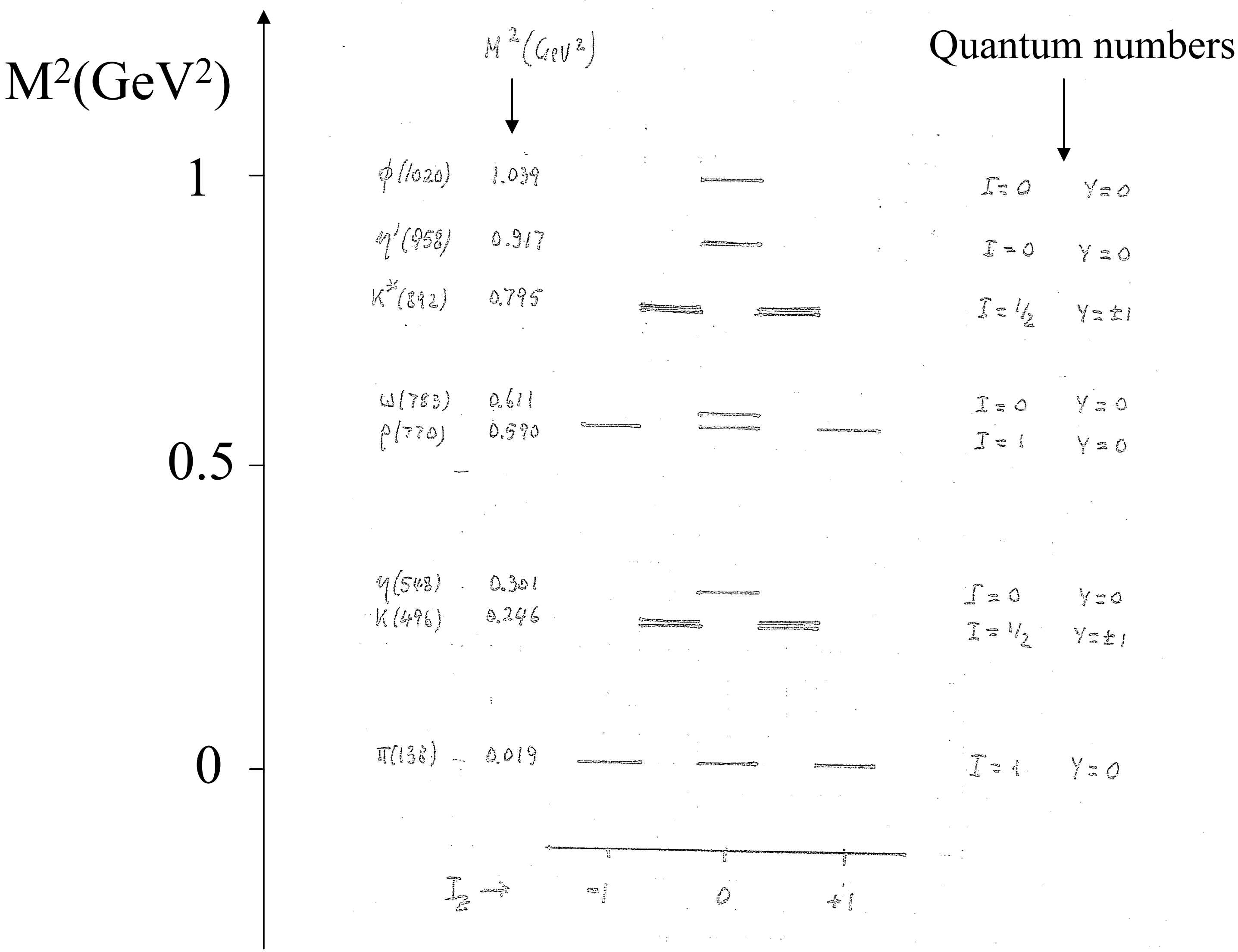
$$M^2 = M_0^2 + a' C_2\left(SU_{sf}(6)\right) + bC_2(SU_f(3)) + aC_1\left(U_Y(1)\right)$$
$$+b\left[C_2\left(SU_I(2)\right) - \frac{1}{4}C_1\left(U_Y(1)\right)^2\right] + cC_2\left(SU_s(2)\right) + dC_1\left(Spin_I(2)\right)$$

Eigenvalues

$$M^2\left([\lambda],[\mu_1,\mu_2];I,Y,S;M_T,M_S\right) = M_0^2 + a'\left\langle C_2\left(SU_{sf}(6)\right)\right\rangle +$$
$$b'\left\langle C_2\left(SU_f(3)\right)\right\rangle + aY + b\left[I(I+1) - \frac{1}{4}Y^2\right] + cS(S+1) + dM_T$$

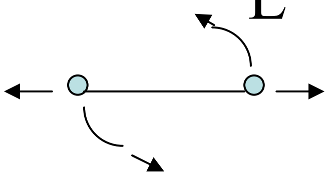
[For mesons, a=0. Also the electromagnetic splittings between different charge states are small, d~0.]

OBSERVED MASS SPECTRUM OF MESONS (SPIN-FLAVOR)



(b) Space degrees of freedom

$$\Re \equiv u(4)$$



Breaking of u(4)

$$U(4) \supset SO(4) \supset SO(3) \supset SO(2) \quad \text{(I)}$$

$$U(4) \supset U(3) \supset SO(3) \supset SO(2) \quad \text{(II)}$$

Only breaking (I) is considered.

Classification of states

$$\left| N, v, L, M_L \right\rangle$$

SPACE DYNAMIC SYMMETRY ¶

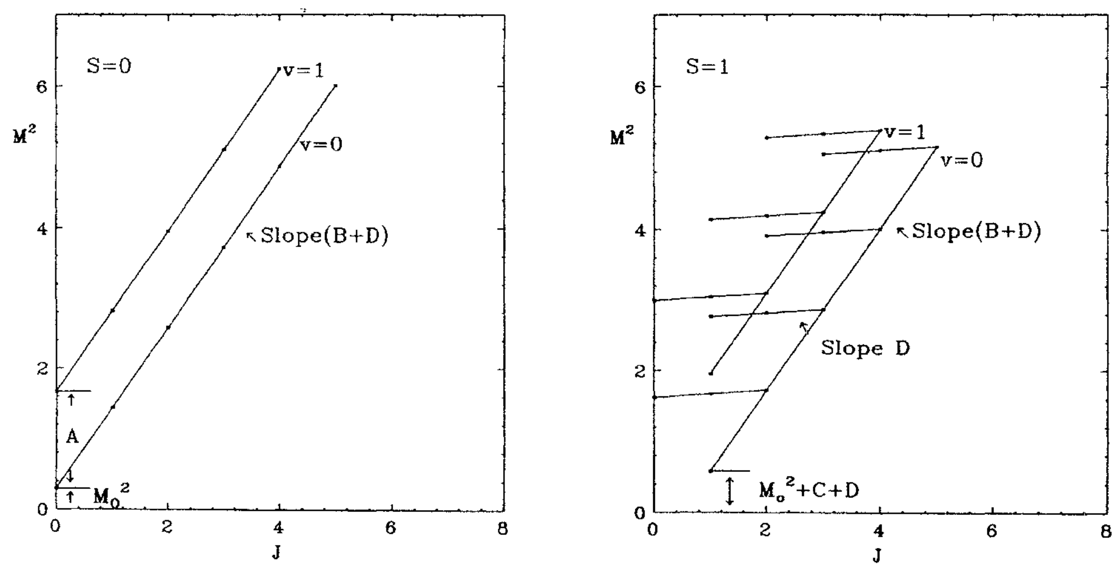
$$M^2 = M_0^2 + A'\left[C_2(SO(4) - N(N+2)\right] + B\left[\left[C_2(SO(3) + \frac{1}{4}\right]^{1/2} - \frac{1}{2}\right]$$

Eigenvalues

$$M^2(v, L) = M_0^2 + Av + BL$$

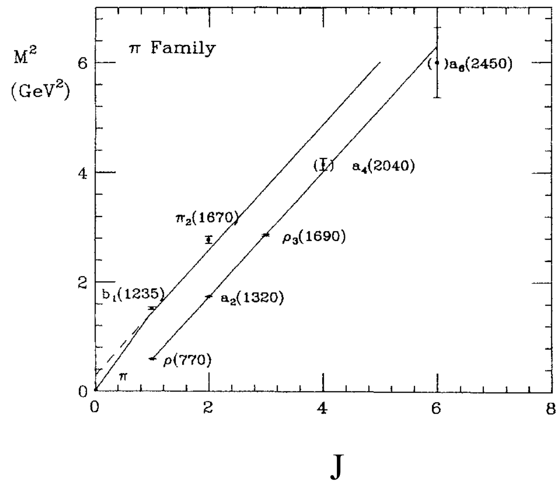
$$A = -4(N+1)A'$$

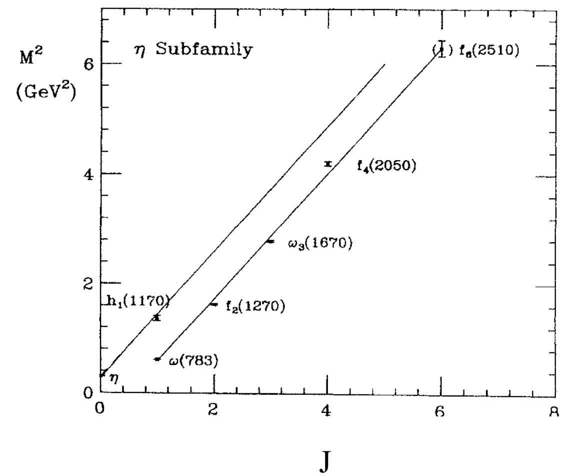
Linear Regge trajectories →

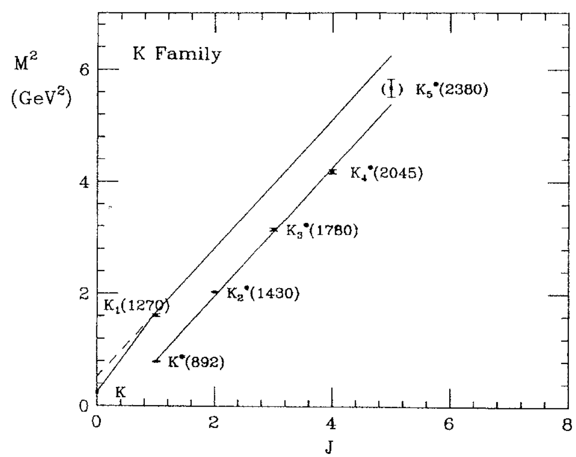


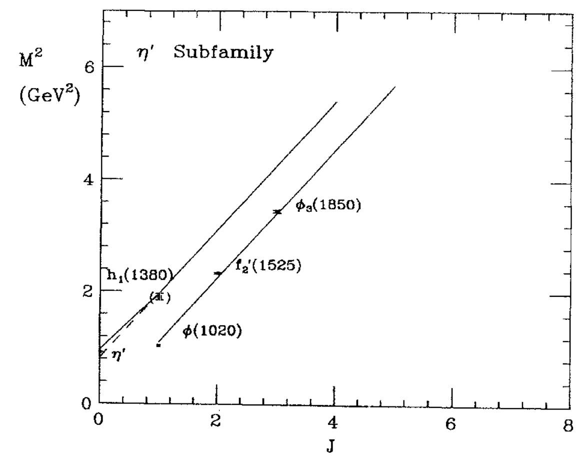
¶ F. Iachello, N.C. Mukhopadyay, and L. Zhang, Phys. Rev. D44, 898 (1991).

OBSERVED SPECTRUM OF MESONS (SPACE)









BARYONS q^3

(a) Internal degrees of freedom

Spin-flavor part

$$6 \otimes 6 \otimes 6 = 56_S \oplus 70_M \oplus 70_M \oplus 20_A$$

BRANCHING: Breaking of $SU_{sf}(6)$ into $SU_s(2) \oplus SU_f(3)$

$$56 = {}^4 10 \oplus {}^2 8$$

$$70 = {}^2 10 \oplus {}^4 8 \oplus {}^2 8 \oplus {}^2 1$$

$$20 = {}^2 8 \oplus {}^4 1$$

Color part

$$3_c \otimes 3_c \otimes 3_c = \left(10_S \oplus 8_M \oplus 8_M \oplus 1_A\right)_c$$

Only 1_c allowed (hadrons are colorless)

SPIN-FLAVOR DYNAMIC SYMMETRY ¶

Mass squared operator

$$M^2 = M_0^2 + a' C_2\left(SU_{sf}(6)\right) + b' C_2\left(SU_f(3)\right) + a C_1\left(U_Y(1)\right)$$
$$+ b\left[C_2\left(SU_I(2)\right) - \frac{1}{4}C_1\left(U_Y(1)\right)^2\right] + c C_2\left(SU_s(2)\right) + d C_1\left(Spin_T(2)\right)$$

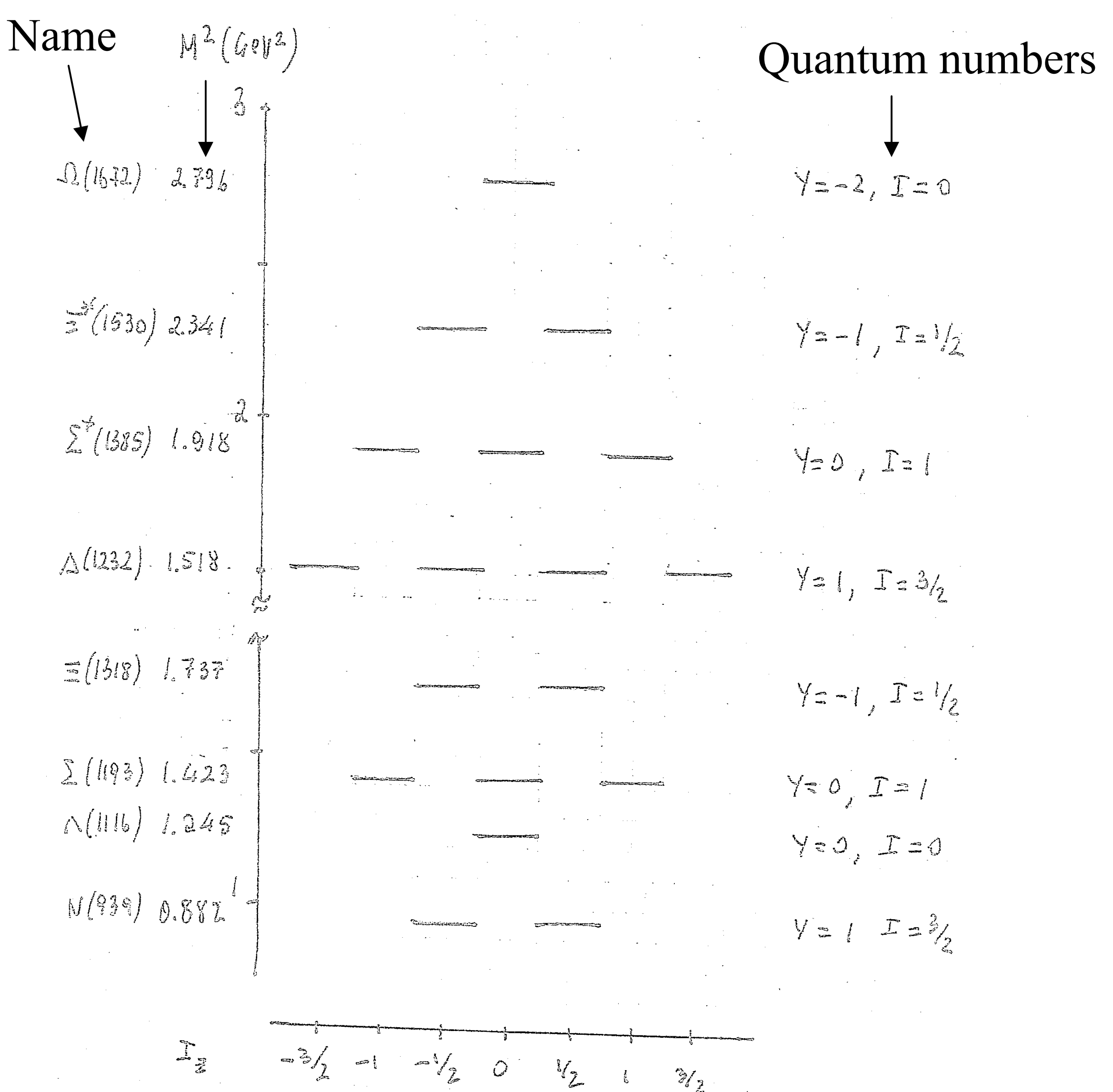
Eigenvalues

$$M^2\left([\lambda],[\mu_1,\mu_2];I,Y,S;M_S,M_I\right) = M_0^2 + a'\left\langle C_2\left(SU_{sf}(6)\right)\right\rangle$$
$$+ b'\left\langle C_2\left(SU_f(3)\right)\right\rangle + aY + b\left[I(I+1) - \frac{1}{4}Y^2\right] + cS(S+1) + dM_I$$

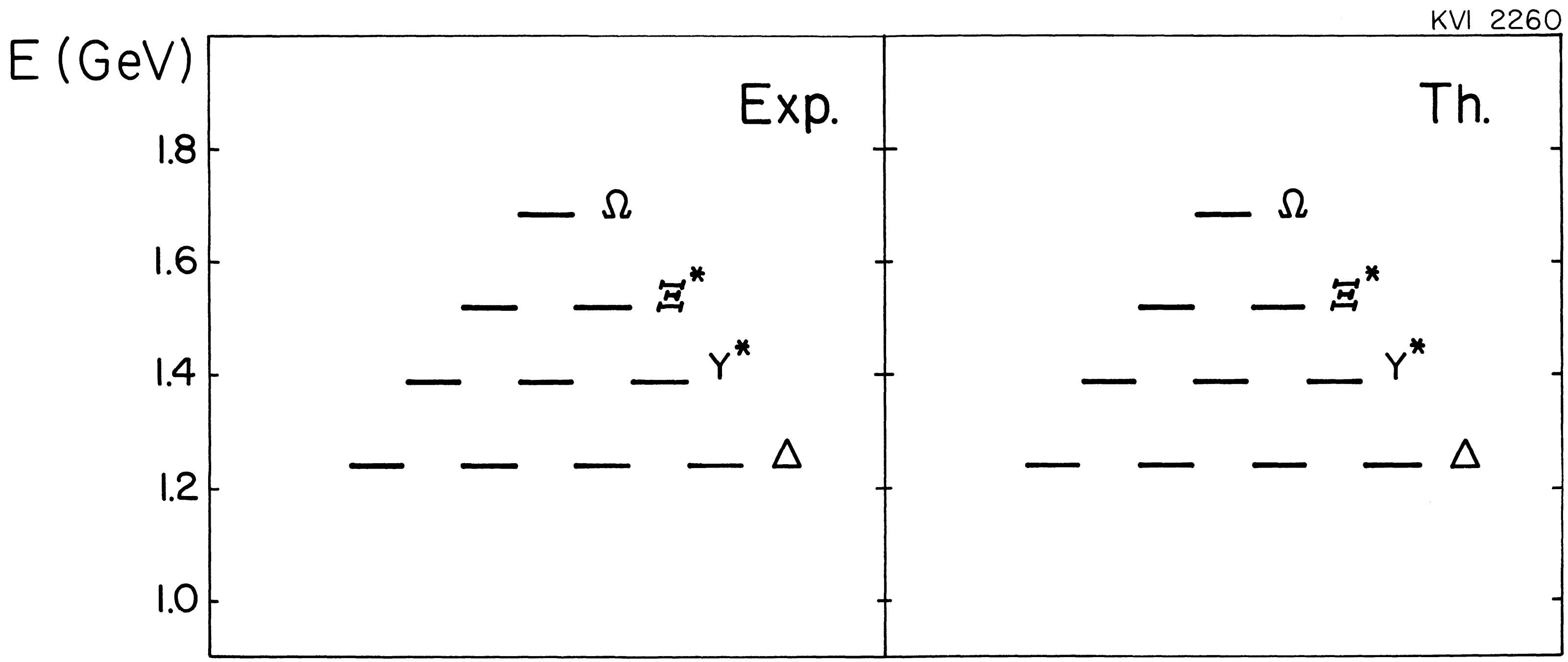
[Electromagnetic splittings between different charge states are small, d~0.]

¶ M. Gell'Mann, Phys. Rev. 125, 1067 (1962); F. Gürsey and L. Radicati, Phys. Rev. Lett. 13, 173 (1964).

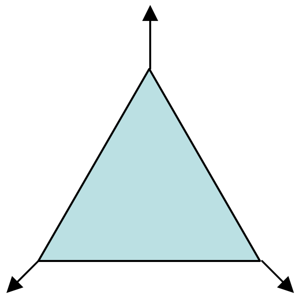
OBSERVED MASS SPECTRUM OF BARYONS (SPIN-FLAVOR)



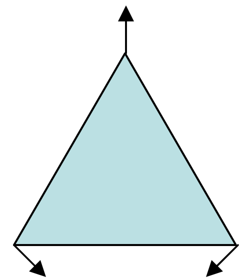
Comparison between experiment and theory for the baryon decuplet, $^4 10$



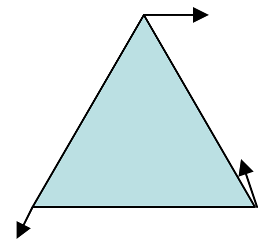
(b) Space degrees of freedom $\quad \Re \equiv u(7)$



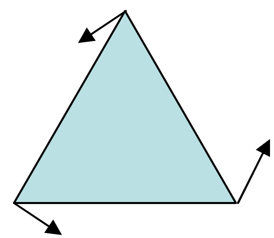
u-vibration



v-vibration



w-vibration



rotation

Breaking of u(7)

$$u(7) \supset u(6) \supset so(6) \supset so_{\rho}(3) \oplus so_{\lambda}(3) \supset so(3) \supset so(2)$$

$$u(7) \supset so(7) \supset so(6) \supset so_{\rho}(3) \oplus so_{\lambda}(3) \supset so(3) \supset so(2)$$

A modification of the concept of dynamic symmetry is needed in this case.
Analytic solutions for situations other that those in which H is a function of Casimir operators can be obtained in large N limit: asymptotic dynamic symmetry.

Mass formula of the rigid oblate top with D_{3h} symmetry [¶]

$$M^2 = M_0^2 + \kappa_1 n_u + \kappa_2 (n_v + n_w) + \alpha L$$

vibrations (↑ $\kappa_1 n_u$, ↗ $\kappa_2 (n_v + n_w)$) rotations (↑ αL)

⇨ Linear Regge trajectories both for vibrations and rotations

[¶] R. Bijker, F. Iachello and A. Leviatan, Ann. Phys. (N.Y.) 236, 69 (1994).

Quarks are fermions and therefore their total wave function must be antisymmetric.

Hadrons are colorless and therefore their color wave function is antisymmetric (color singlet).

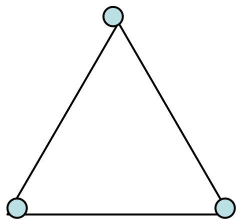
Hence the space-spin-flavor wave function must be symmetric.

The space wave-function must be combined with the spin-flavor part to give symmetric wave functions, i.e. the symmetry of the space wave functions must be the same as the symmetry of the spin-flavor part.
The parity of the states is given by complicated rules.

Combination of space and spin-flavor (labels of the representations)

Space D_{3h}	Spin-flavor $SU_{sf}(6)$	Young tableau
A_1	56	□□□≡ *S*
A_2	20	□≡ *A* □ □
E	70	□□≡ *M* □

Spectrum of the oblate top with D_{3h} symmetry



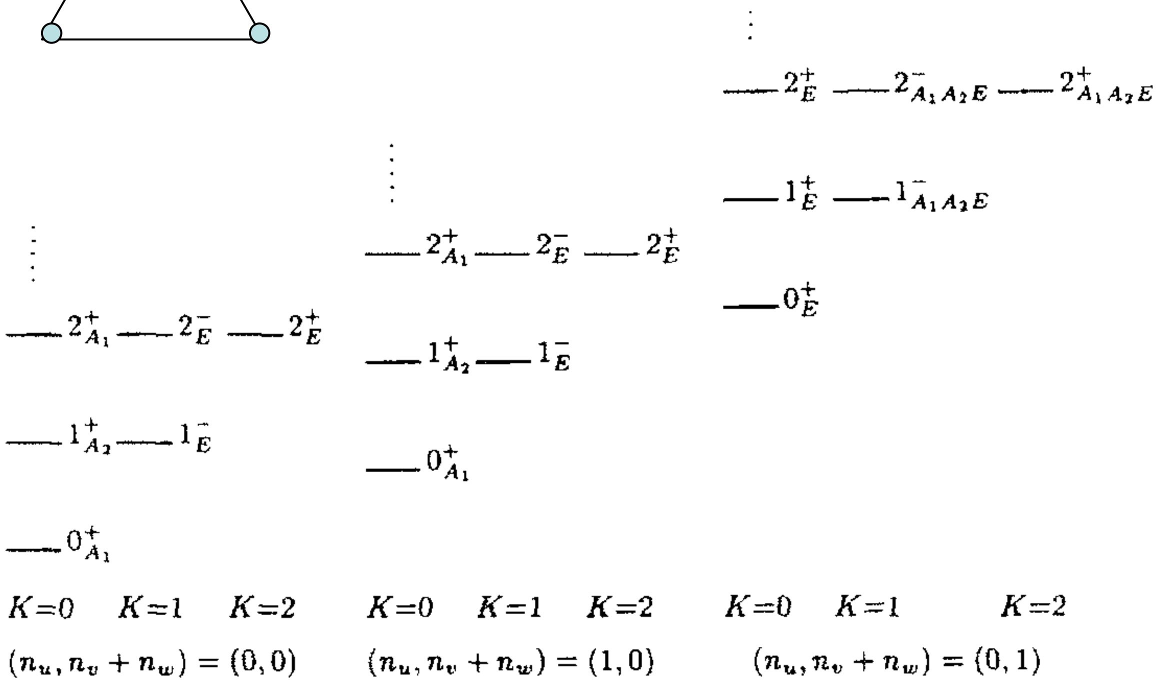
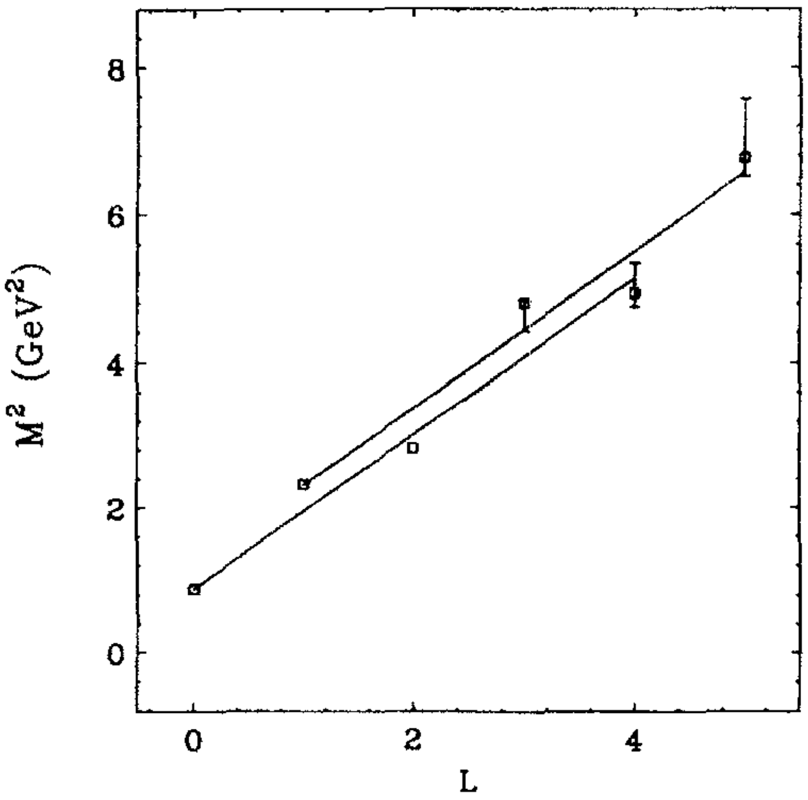


FIG. 4. Schematic representation of the vibrational and rotational excitations of the string-like configuration of Fig. 1 with three identical constituent parts. The vibrational excitations are labeled by $(n_u, n_v + n_w)$ and the rotational levels by K, L^π_t, where K is the projection of the angular momentum L, π denotes the parity and t is the overall (vibrational plus rotational) transformation property under the point group D_3. Each E state is doubly degenerate.

OBSERVED MASS SPECTRUM OF BARYONS (SPACE)

N-family



TRANSITIONS

Transition operators can be written as tensors in the space

$$SU_{sf}(6) \supset SU_f(3) \otimes SU_s(2)$$

$$f(\Re) T^{SU_{sf}(6)}_{SU_s(2) \otimes SU_f(3)}$$

All matrix elements, diagonal and non-diagonal, can then be calculated from

$$\left\langle R'';[\lambda''],[\mu''],S'',I'',Y'',I_3'',S_3'' \right| f(R) T^{SU_{sf}(6)}_{SU_s(2) \otimes SU_f(3)} \left| R';[\lambda'],[\mu'],S',I',Y',I_3',S_3' \right\rangle$$

The calculation involves the evaluation of the space part

$$\left\langle R'' \middle| f(R) \middle| R' \right\rangle$$

and the $SU_{sf}(6)$ part

$$\left\langle [\lambda''],[\mu''],S'',I'',Y'',I_3'',S_3'' \middle| T^{SU_{sf}(6)}_{SU_s(2)\otimes SU_f(3)} \middle| [\lambda'],[\mu'],S',I',Y',I_3',S_3' \right\rangle$$

The latter is given in terms of the reduced matrix elements and of the Clebsch-Gordan coefficients of

$$SU_{sf}(6) \supset SU_s(2) \otimes SU_f(3) \supset SU_s(2) \otimes SU_I(2) \otimes U_Y(1)$$

Example:
Calculation of the magnetic moment of baryons in the 56 representation

The magnetic moment operator can be written as

$$\vec{\mu} = \frac{e}{2m} Q \vec{\sigma}$$

$$Q = I_3 + \frac{Y}{2}$$

This operator is a generator of $SU_{sf}(6)$ belonging to the representation 35 and component $^2 8$ of $SU_s(2) \otimes SU_f(3)$. The magnetic moments of baryons in the representation $56 = {}^4 10 \oplus {}^2 8$ are thus proportional to the matrix elements

$$\langle 56 | 35 | 56 \rangle$$

Consider now the product

$$35 \otimes 56 = 700 \oplus 1134 \oplus 70 \oplus 56 = 1960$$

Since 56 is contained only once in the product, then all matrix elements are given in terms of Clebsch-Gordan coefficients and the reduced matrix elements

$$\langle 56 \| 35 \| 56 \rangle$$

As a result

$$\mu_p = \mu$$

$$\mu_n = -\frac{2}{3}\mu$$

$$\frac{\mu_p}{\mu_n} = -\frac{3}{2} = -1.5$$

Experiment:

$$\frac{\mu_p}{\mu_n} = -1.46 \pm 0.02$$

Considered one of the greatest successes of algebraic methods in physics.

[For magnetic moments, there is no space part, and thus the calculation is just $\langle R'' | 1 | R' \rangle = \delta_{R'R''}$].

CONCLUSIONS

For the algebraic quark model, AQM, the application of Lie algebraic methods is very intricate due to the combination of space and internal degrees of freedom and of the many conditions imposed on the wave functions.

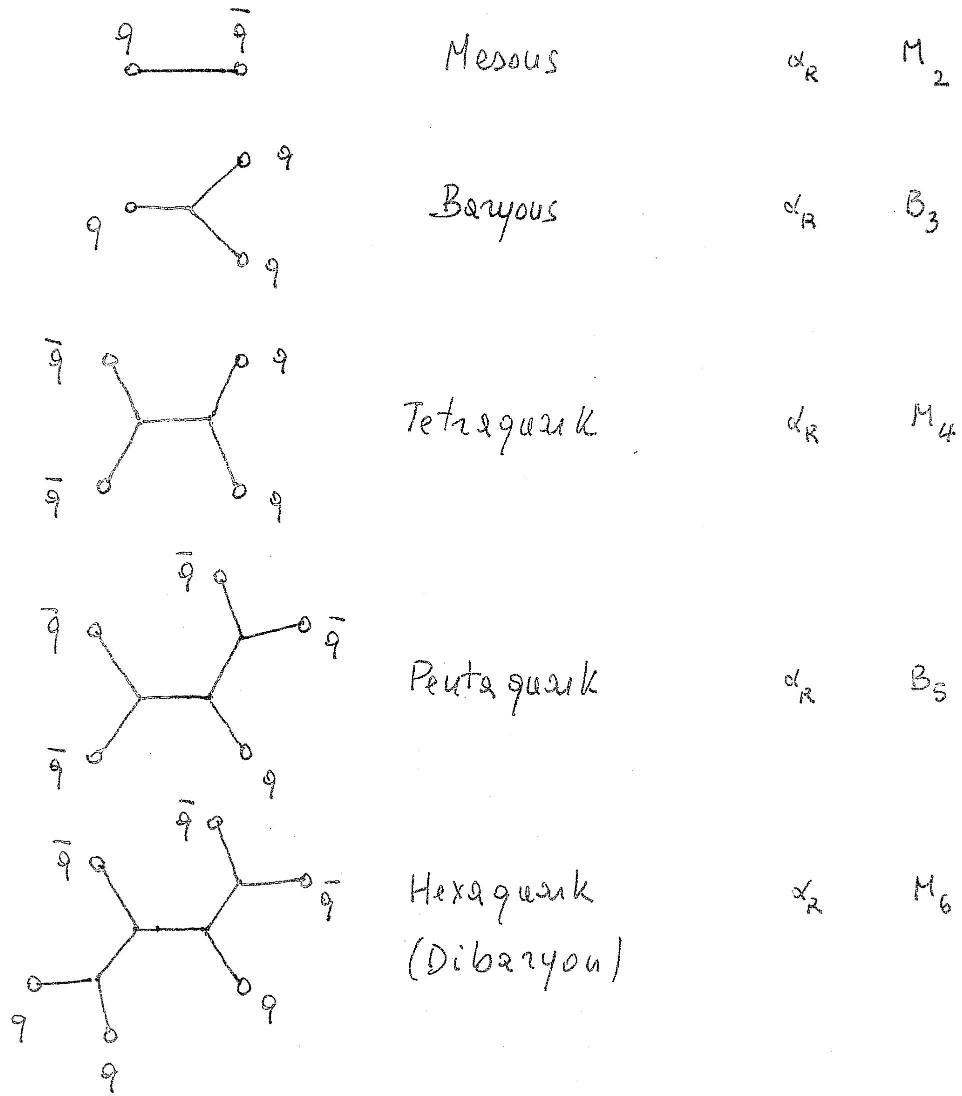
What is required for the internal degrees of freedom is

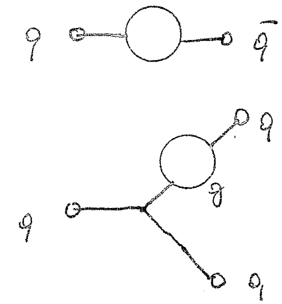
- Construction of the representations
- Branching of the representations

For the space degrees of freedom one needs

- Construction of the algebra
- A generalization of the concept of DS

Hadrons in a string-like model



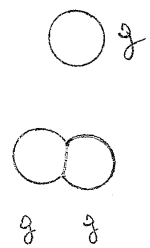


Hybrids

$\frac{1}{2}\alpha_R$ M_2^g

$\frac{1}{2}\alpha_R$ B_3^g

. . .

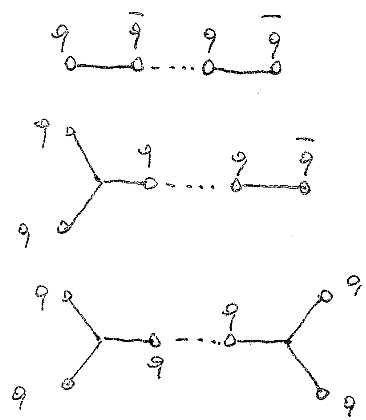


Glueloops
(Glueballs)

$\frac{1}{3}\alpha_R$ M_0^g

$\frac{1}{3}\alpha_R$ M_0^{gg}

. . .



Hadronic Molecules