LIE ALGEBRAS AND LIE GROUPS IN PHYSICS

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Lecture 1

CLASSIFICATION OF SEMISIMPLE LIE ALGEBRAS

Name	Cartan	Order, r	Rank, λ
su(n)	A_l	n^2-1	n-1
so(n), n=odd	B_l	n(n-1)/2	(n-1)/2
sp(n), n=even	C_l	n(n+1)/2	n/2
so(n), n=even	D_l	n(n-1)/2	n/2
Exceptional	G_2	14	2
	F_4	52	4
	E_6	78	6
	E_7	133	7
	E_8	248	8
u(n)	-	n^2	n

INTRODUCTION

Lie algebras and Lie groups have been used extensively in physics since their introduction at the end of the 19th Century and especially since the middle of the 20th Century when Wigner, Racah and others applied them to problems in nuclear and atomic physics.

Today, they are used in variety of fields. Three areas where Lie algebras and groups have been extensively used are:

(i) Spectroscopy

(a) Molecular physics $\sum_i \oplus u_i(n) \qquad n = 2,3,4$

Vibron model (1980)

F. Iachello and R.D. Levine, Algebraic Theory of Molecules, Oxford University Press (1995).

(b) Atomic physics $$u\left(\left(\sum_{\ell}(2\ell+1)\right)(2s+1)\right)$$

Atomic shell model (1926)

G. Racah, Group Theory and Spectroscopy, Lecture Notes, Princeton, NJ, 1951, reprinted in Springer Tracts in Modern Physics 37, 28 (1965).

(c) Nuclear physics

Nuclear shell model (1936) $$u\left(\sum_{j}(2j+1)\right)$$

G. Racah, *ibid*.

Interacting Boson Model (1974) $u(6); u(6)\oplus u(6)$

F. Iachello and A. Arima, The Interacting Boson Model, Cambridge University Press, 1987.

(d) Particle physics

Quark model (1962)

Internal degrees of freedom $su_s(2) \oplus su_f(3) \oplus su_c(3)$

M. Gell-Mann, Symmetries of baryons and mesons, Phys. Rev. 125, 1067 (1962).
F. Gürsey and L. Radicati, Spin and unitary spin dependence of strong interactions, Phys. Rev. Lett. 13, 173 (1964).

Space degrees of freedom $\Re \equiv u(3k+1)$ k=1,2,3

F. Iachello, N.C. Mukhopadhyay, and L. Zhang, Phys. Rev. D 44, 898 (1991).
R. Bijker, F. Iachello and A. Leviatan, Ann. Phys. (N.Y.) 236, 69 (1994).

(ii) Structure of space-time

(a)Lorentz so(3,1)

(b)De Sitter so(3,2), so(4,1)

(c)Conformal so(4,2)~su(2,2)

(iii) Gauge

(a)Electroweak-strong $u_{em}(1) \oplus su_{w}(2) \oplus su_{c}(3)$

(b)Grand unification su(5), so(10)

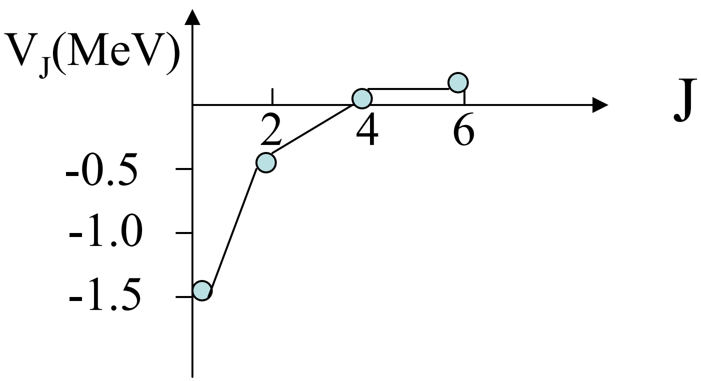
In these lectures, only some applications to the spectroscopy of nuclei and hadrons will be considered. The notation used is that of F. Iachello, Lie Algebras and Applications, 2nd ed., Springer-Verlag, Berlin, 2015.

THE INTERACTING BOSON MODEL

Constituents

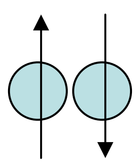
The nucleus: protons and neutrons with strong interaction.

Properties of the strong effective interaction: monopole and quadrupole pairing

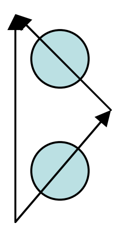


Even-even nuclei composed of nucleon pairs treated as bosons

⇒ The interacting boson model

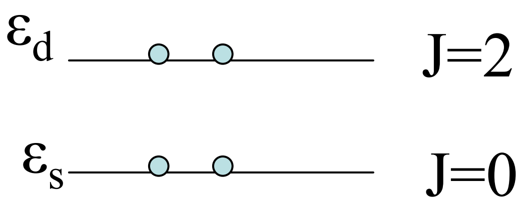


J=0 S-pairs



J=2 D-pairs

Building blocks: bosons with J=0 and 2



Boson creation and annihilation operators

$$s^\dagger, d^\dagger_\mu (\mu = 0, \pm 1, \pm 2)$$
$$s, d_\mu (\mu = 0, \pm 1, \pm 2)$$

generically denoted by $b^\dagger_\alpha, b_\alpha \left(\alpha = 1, \ldots, 6\right)$

with commutation relations $\left[b_\alpha, b^\dagger_{\alpha'}\right] = \delta_{\alpha\alpha'}; \left[b_\alpha, b_{\alpha'}\right] = \left[b^\dagger_\alpha, b^\dagger_{\alpha'}\right] = 0$

The basis B is

$$\left|N\right\rangle = \frac{1}{\sqrt{N!}} \overbrace{b^\dagger_\alpha b^\dagger_{\alpha'} \ldots}^{\text{N-times}} \left|0\right\rangle$$

where N is the number of bosons.

Hamiltonian

$$H = E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\beta\gamma\delta} \frac{1}{2} u_{\alpha\beta\gamma\delta} b_\alpha^\dagger b_\beta^\dagger b_\gamma b_\delta + \dots$$

Transition operators

$$T = t_0 + \sum_{\alpha\beta} t_{\alpha\beta} b_\alpha^\dagger b_\beta + \dots$$

Algebraic structure

$$g \doteq G_{\alpha\beta} = b_\alpha^\dagger b_\beta \qquad \alpha, \beta = 1, \dots, 6$$

Lie algebra u(6) with commutation relations

$$\left[G_{\alpha\beta}, G_{\gamma\delta} \right] = G_{\alpha\delta} \delta_{\beta\gamma} - G_{\gamma\beta} \delta_{\alpha\delta}$$

$$H = E_0 + \sum_{\alpha\beta} \varepsilon'_{\alpha\beta} G_{\alpha\beta} + \sum_{\alpha\beta\gamma\delta} \frac{1}{2} u_{\alpha\beta\gamma\delta} G_{\alpha\gamma} G_{\beta\delta} + \ldots$$

$$T = t_0 + \sum_{\alpha\beta} t_{\alpha\beta} G_{\alpha\beta} + \ldots$$

The algebra upon which the Hamiltonian and other operators are expanded is called spectrum generating algebra (SGA).

The basis B is the totally symmetric representation of u(6) with Young tableau

$$|N\rangle \equiv \underbrace{\square\square\square\ldots\square}_{\text{N-times}}$$

For rotationally invariant problems, we need to construct operators that transform as irreps of the rotation algebra so(3)

Spherical tensor boson operators

$$b^\dagger_{\ell,m} \qquad b_{\ell,m} \qquad\qquad (\ell = 0,2; -\ell \leq m \leq \ell)$$

with commutation relations

$$\left[b_{\ell,m}, b^\dagger_{\ell',m'}\right] = \delta_{\ell\ell'}\delta_{mm'}$$

$$\left[b_{\ell,m}, b_{\ell',m'}\right] = \left[b^\dagger_{\ell,m}, b^\dagger_{\ell',m'}\right] = 0$$

Complication: If $b^\dagger_{\ell,m}$ transforms as a spherical tensor under rotations, $b_{\ell,m}$ does not. An operator that transforms as a spherical tensor is

$$\tilde{b}_{\ell,m} = (-)^{\ell-m} b_{\ell,-m}$$

The Lie algebra g in Racah form is obtained by taking tensor products

$$G_{\kappa}^{(k)}(\ell,\ell') = \left[b_{\ell}^{\dagger} \times \tilde{b}_{\ell'}\right]_{\kappa}^{(k)} \qquad (\ell,\ell' = 0,2)$$

Tensor products of two operators are defined as

$$\left[T^{(k_1)} \times U^{(k_2)}\right] = \sum_{\kappa_1,\kappa_2} \langle k_1\kappa_1 k_2\kappa_2 | k\kappa\rangle T_{\kappa_1}^{(k_1)} U_{\kappa_2}^{(k_2)}$$

Clebsch-Gordan coefficients of so(3)

THE BRANCHING PROBLEM

Representations of g' contained in a given representation of g.

$$g' \subset g$$

Solved in mathematics by use of canonical chains

$$u(n) \supset u(n-1) \supset ... \supset u(1)$$
$$so(n) \supset so(n-1) \supset ... \supset so(2)$$

(Gel'fand method)

In physics, we need branching with conditions. For finite systems, we need states to be representations of $so(3) \supset so(2)$

The branching must be done in Racah's form.

Subalgebras of the interacting boson model u(6) in Racah form and their labeling¶

Branching I

$$\left| \begin{array}{ccccccccc} u(6) & \supset & u(5) & \supset & so(5) & \supset & so(3) & \supset & so(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N & & n_d & & v, n_\Delta & & L & & M_L \end{array} \right\rangle$$

Branching II

$$\left| \begin{array}{ccccccc} u(6) & \supset & su(3) & \supset & so(3) & \supset & so(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N & & (\lambda,\mu)K & & L & & M_L \end{array} \right\rangle$$

¶ A. Arima and F. Iachello, Ann. Phys. (N.Y.) 99, 253 (1976); *ibid.* 111, 201 (1978); *ibid.* 123, 468 (1979).

Branching III

$$\left|\begin{array}{ccccc} u(6)\supset & so(6)\supset & so(5)\supset & so(3)\supset & so(2) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ N & \sigma & \tau,\nu_\Delta & L & M_L \end{array}\right\rangle$$

Lattice of algebras

$$\begin{tikzcd}
 & & u(6) \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] & \\
u(5) \arrow[dr, dash] & & so(6) \arrow[dl, dash] & su(3) \arrow[ddl, dash] \\
 & so(5) \arrow[dr, dash] & & \\
 & & so(3) \arrow[d, dash] & \\
 & & so(2) &
\end{tikzcd}$$

DYNAMIC SYMMETRIES

$$H = \sum_{\alpha} \varepsilon_{\alpha\alpha} G_{\alpha\alpha} + \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} u_{\alpha\alpha'\beta\beta'} G_{\alpha\alpha'} G_{\beta\beta'} + \ldots$$

$$G_{\alpha\alpha'} \in g$$

g=spectrum generating algebra (SGA)

Special case: H is a function only of Casimir operators of a chain

$$g \supset g' \supset g'' \supset \ldots$$

$$H = \alpha C(g) + \alpha' C(g') + \alpha'' C(g'') + \ldots$$

The eigenvalue problem for H is then analytically solvable

$$E = \alpha \langle C(g) \rangle + \alpha' \langle C(g') \rangle + \alpha'' \langle C(g'') \rangle + \ldots$$

This situation is called a dynamic symmetry (DS).

CASIMIR OPERATORS

The Casimir (or invariant) operators of an algebra g are the operators which commute with all the elements of the algebra

$$\left[C_p, X_\rho\right] = 0$$

$$X_\rho \in g$$

The number of independent Casimir operators and their eigenvalues are known for any Lie algebra g. This number is equal to the rank of the algebra.

The Casimir operators lie in the embedding algebra of g, T(g)

$$C_p = \sum_{\alpha_1\alpha_2\ldots\alpha_p} f^{\alpha_1\alpha_2\ldots\alpha_p} X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_p}$$

p= order of C

DYNAMIC SYMMETRIES OF THE INTERACTING BOSON MODEL

$$u(6) \begin{array}{l} \nearrow \; u(5) \supset so(5) \supset so(3) \supset so(2) \qquad \text{(I)} \\ - \; su(3) \supset so(3) \supset so(2) \qquad \text{(II)} \\ \searrow \; so(6) \supset so(5) \supset so(3) \supset so(2) \qquad \text{(III)} \end{array}$$

Dynamic symmetry I

$$H^{(I)} = E_0 + \varepsilon C_1\left(u(5)\right) + \alpha C_2\left(u(5)\right) + \beta C_2\left(so(5)\right) + \gamma C_2\left(so(3)\right)$$

Eigenvalues

$$E^{(I)}\left(N, n_d, v, n_\Delta, L, M_L\right) = E_0 + \varepsilon n_d + \varepsilon n_d (n_d + 4) + \beta v(v+3) + \gamma L(L+1)$$

Usually, hierarchy of couplings $\quad |\varepsilon| \gg |\alpha| \gg |\beta| \gg |\gamma|$

Dynamic symmetry II

$$H^{(II)} = E_0 + \kappa C_2\left(su(3)\right) + \kappa' C_2\left(so(3)\right)$$

Eigenvalues

$$E^{(II)}\left(N,\lambda,\mu,K,L,M_L\right) = E_0 + \kappa\left(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu\right) + \kappa' L(L+1)$$

Usually

$$\left|\kappa\right| \gg \left|\kappa'\right|$$

Dynamic symmetry III

$$H^{(III)} = E_0 + AC_2\left(so(6)\right) + BC_2\left(so(5)\right) + CC_2\left(so(3)\right)$$

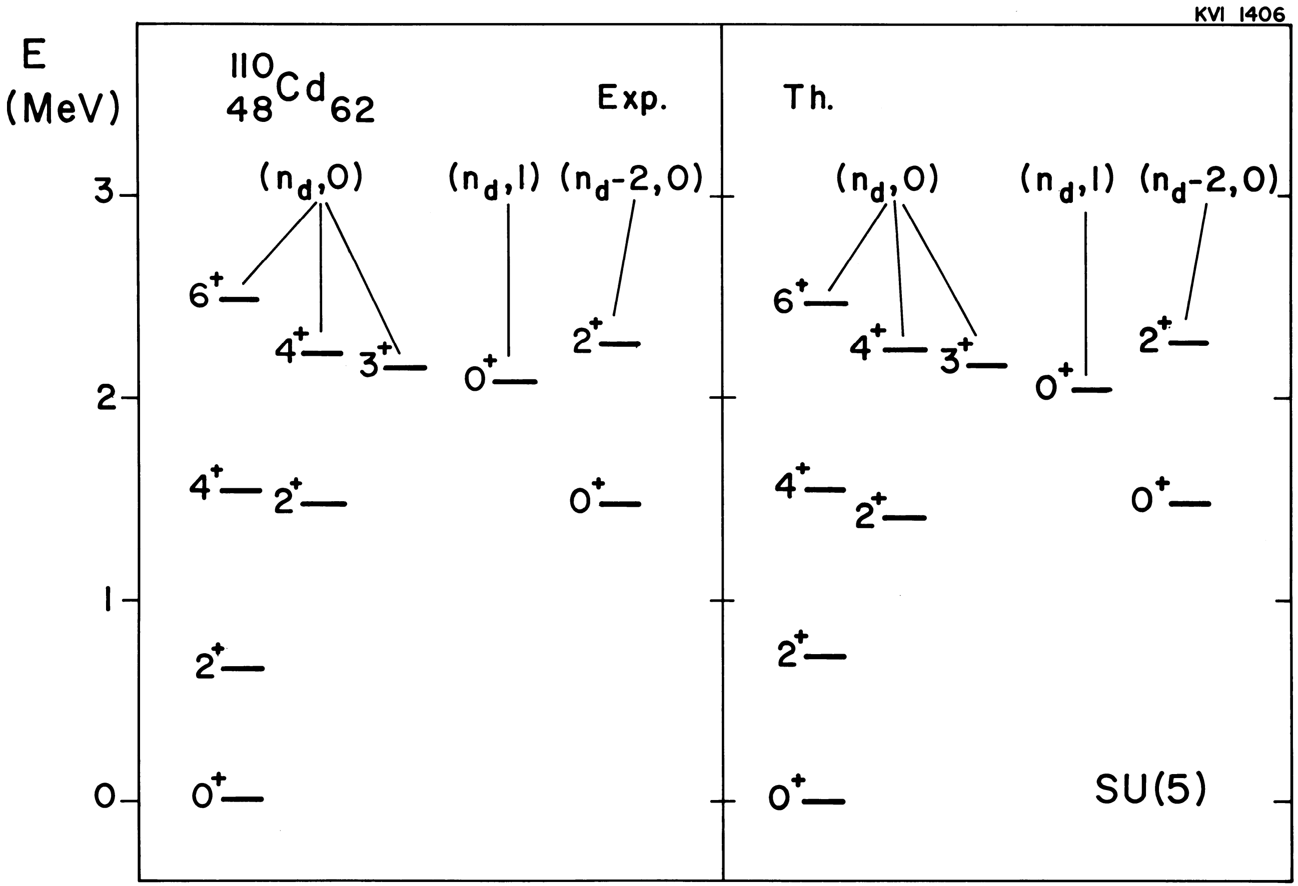
Eigenvalues

$$E^{(III)}\left(N,\sigma,\tau,\nu_\Delta,L,M_L\right) = E_0 + A\sigma(\sigma+4) + B\tau(\tau+3) + CL(L+1)$$

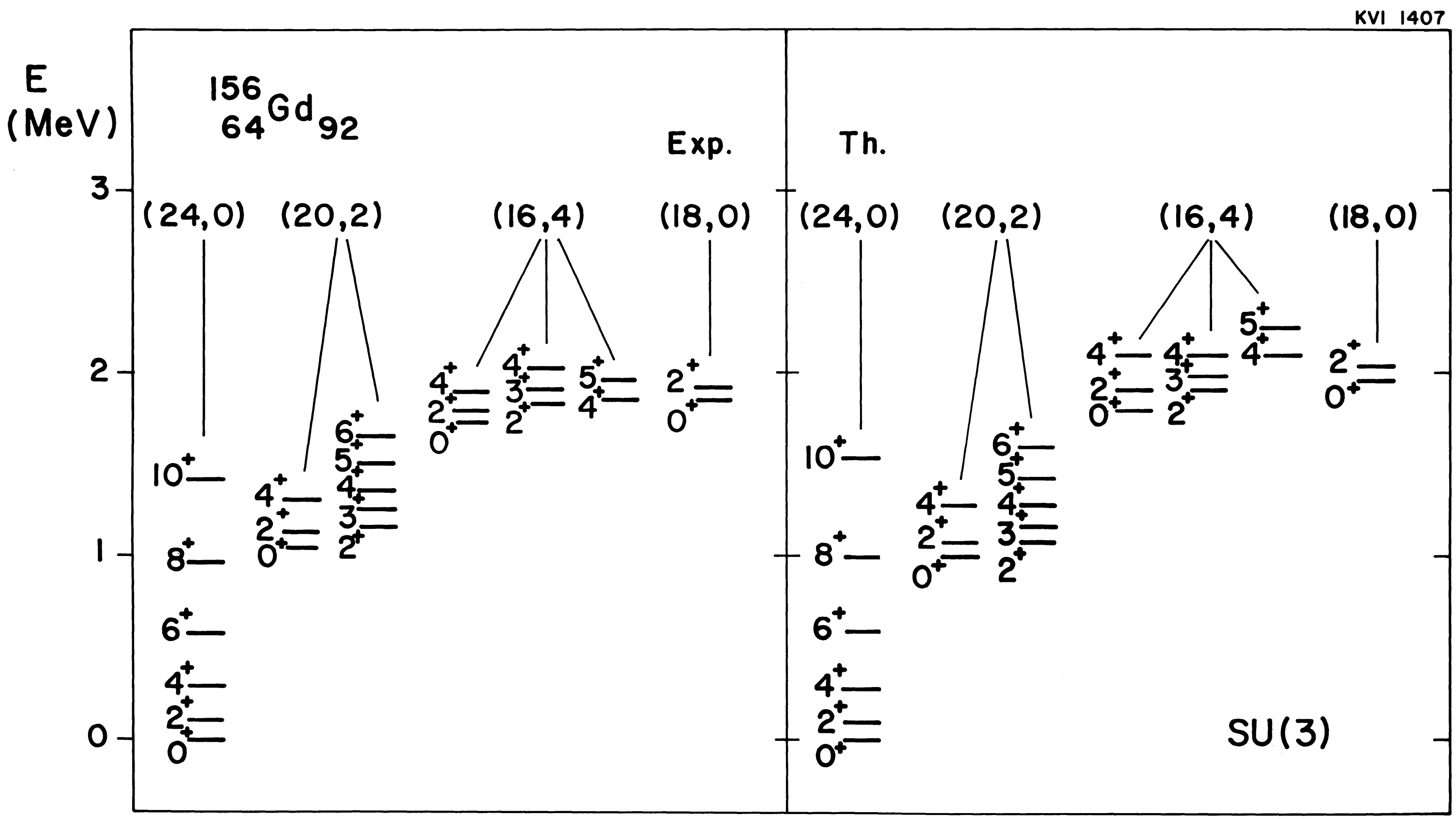
Usually

$$|A| \gg |B| \gg |C|$$

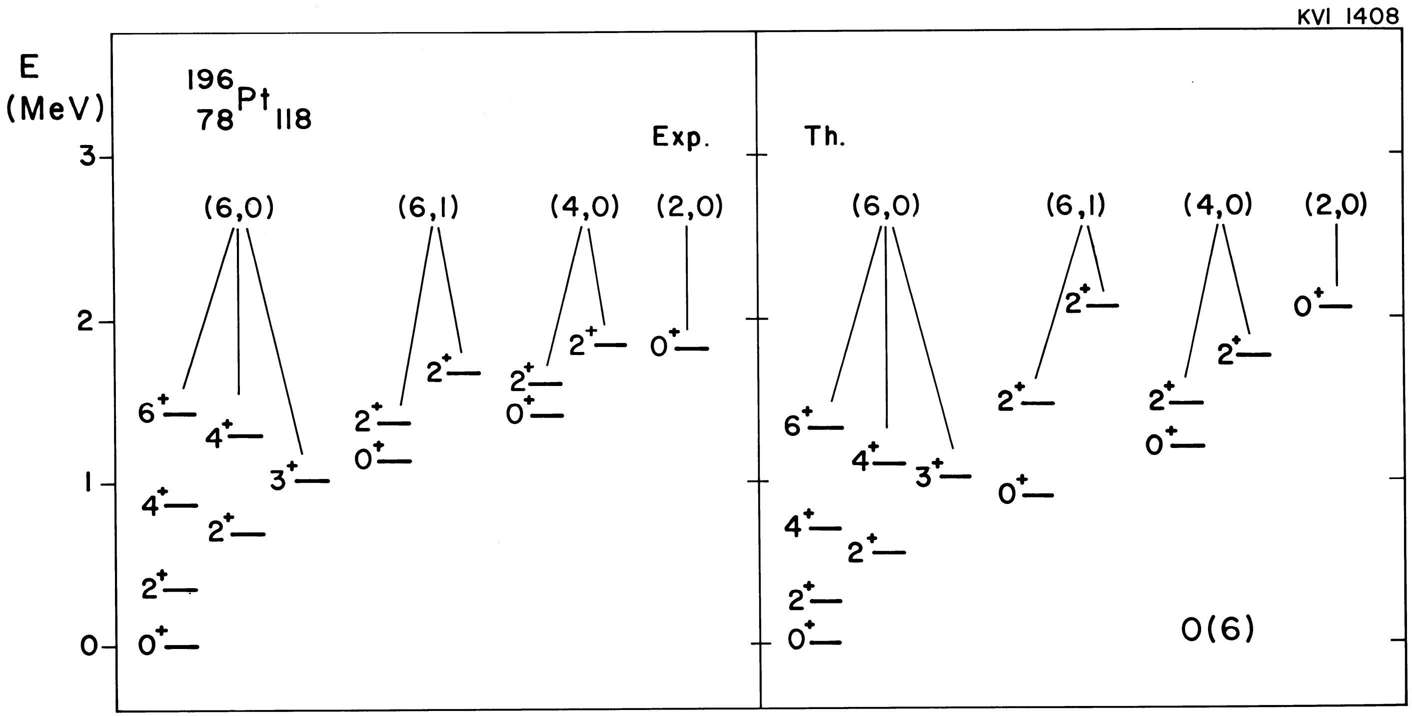
Example of dynamic symmetry (I): U(5)



Example of dynamic symmetry (II): SU(3)



Example of dynamic symmetry (III): SO(6)



TRANSITIONS

When a dynamic symmetry occurs, all matrix elements can be calculated in explicit analytic form.

Introducing tensor operators of g⊃g', and representations, the matrix elements are given by

$$T_{\lambda}^{\Lambda}$$

$$\left|\Lambda_1, \lambda_1\right\rangle$$

$$\left\langle \Lambda_1, \lambda_1 \middle| T_{\lambda}^{\Lambda} \middle| \Lambda_2, \lambda_2 \right\rangle = \sum_{a} \left\langle a\Lambda_1\lambda_1 \middle| \Lambda\lambda\Lambda_2\lambda_2 \right\rangle^{*} \left\langle a\Lambda_1 \middle\| T^{\Lambda} \middle\| \Lambda_2 \right\rangle$$

(multiplicity label)

To do this calculation, one needs the Clebsch-Gordan coefficients (isoscalar factors) of g⊃g' and the reduced matrix elements.

[For nested chains, one can use Racah's factorization lemma.]

NUMERICAL STUDIES

In many cases, spectra of nuclei cannot be described by a dynamic symmetry. For these cases, H must be diagonalized numerically.

Lie algebras are useful here to construct the basis B in which the diagonalization is done.

For the interacting boson model, the basis used in the diagonalization is usually the basis
$u(6) \supset u(5) \supset so(5) \supset so(3) \supset so(2)$

GEOMETRY

Associated to any algebra g, there is a geometry (homogeneous Riemann space). The spaces associated with this geometry are called coset spaces.

Coset spaces can be constructed by writing the algebra g as

$$g = h \oplus p$$

where h is a subalgebra of g and p is the remainder (not closed with respect to commutation).

The coset space is then constructed as

$$\left|\alpha_k\right\rangle = e^{\sum_k \alpha_k p_k} \left|\Lambda_{ext}\right\rangle$$

where Λ_{ext} is an extremal state.

CLASSIFICATION OF RIEMANNIAN SPACES

Name	Rank	Dimension
$SU(n)/SO(n)$	$n-1$	$(n-1)(n+2)/2$
$SU(2n)/Sp(n)$	$n-1$	$(n-1)(2n+1)$
$SU(p+q)/S(U(p)\otimes U(q))$	$\min(p,q)$	$2pq$
$SO(p+q)/SO(p)\otimes SO(q)$	$\min(p,q)$	pq
$SO(2n)/U(n)$	$[n/2]$	$n(n-1)$
$Sp(2n)/U(n)$	n	$n(n+1)$
$Sp(p+q)/Sp(p)\otimes Sp(q)$	$\min(p,q)$	$4pq$

For the case of the interacting boson model

$$g \doteq u(6) = b_\alpha^\dagger b_\beta \qquad \alpha, \beta = 1, \ldots, 6$$

$$h = u(5) \oplus u(1) \qquad h = b_1^\dagger b_1, b_\alpha^\dagger b_\beta \qquad \alpha, \beta = 2, \ldots, 6$$

$$p = b_1^\dagger b_\alpha, b_\alpha^\dagger b_1 \qquad \alpha = 2, \ldots, 6$$

$$\left|\Lambda_{ext}\right\rangle = \frac{1}{\sqrt{N!}}\left(b_1^\dagger\right)^N \left|0\right\rangle$$

The Riemannian space for this case is

$$U(6)/U(5) \otimes U(1)$$

The coset space so constructed is a five dimensional complex space. The real and imaginary parts are the coordinates and momenta

$$\alpha_\mu, \pi_\mu \qquad\qquad (\mu = 0, \pm 1, \pm 2)$$

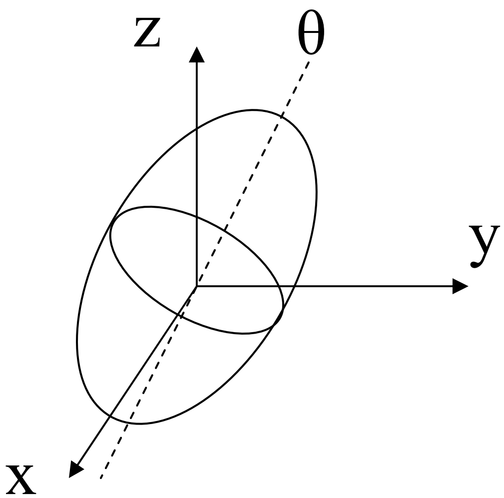
The intrinsic (or coherent) state in the coset space, for fixed number of bosons N, can also be written as

$$|N; \alpha_\mu\rangle = \left(s^\dagger + \sum_\mu \alpha_\mu d_\mu^\dagger \right)^N |0\rangle$$

Instead of α_μ one can use 3 Euler angles θ_1, θ_2 , θ_3 and two intrinsic variables β,γ (Bohr variables).

The intrinsic space of IBM is then plotted as a surface with radius

$$R = R_0\left[1+\sum_{\mu}\alpha_{\mu}Y_{2\mu}(\theta,\phi)\right]$$



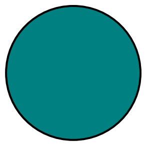
The nucleus is then sometimes referred as a "liquid drop".

The Interacting Boson Model can be viewed as a quantization of the Bohr Hamiltonian in the Riemannian space α_μ

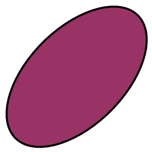
$$H = T\left(\beta,\gamma;\theta_1,\theta_2,\theta_3\right) + V(\beta,\gamma)$$

Shapes corresponding to

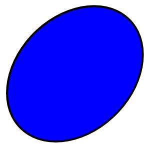
Symmetry I: Sphere



Symmetry II: Ellispoidal shapes with axial symmetry



Symmetry III: Deformed with potential independent of γ (γ-unstable)



CONCLUSIONS

In this lecture, a brief description of Lie algebraic methods in the study of collective states in nuclei has been given. In this application, Interacting Boson Model (IBM), Lie algebraic methods have been exploited in full.

- Construction of the algebra
- Construction of its representations
- Construction of its Clebsch-Gordan coefficients
- Construction of the associated Riemannian space

IBM is a textbook example of Lie algebraic methods applied to physics.