Communication without Distortion: Soliton Propagation in Nonlinear Media

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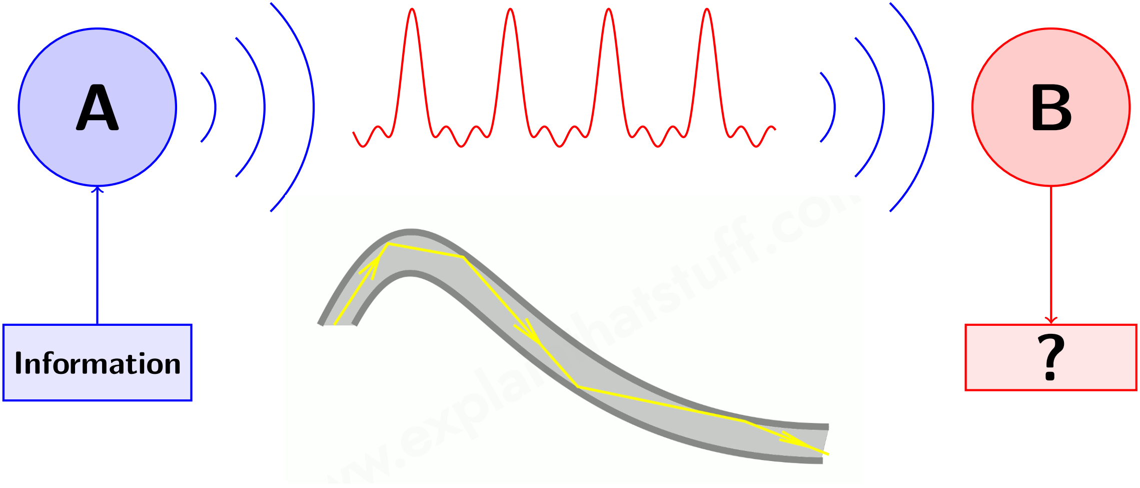
7th BYMAT Conference: Bringing Young Mathematicians Together

The communication problem

How do we transmit a signal throw a real medium with no degradation?



Waves in linear media

Waves in dispersive media

Nonlinear + dispersion effects

Soliton: Wave-packet which envelope is invariant over time.

They appear due to a cancellation of two effects: nonlinearity and dispersion.

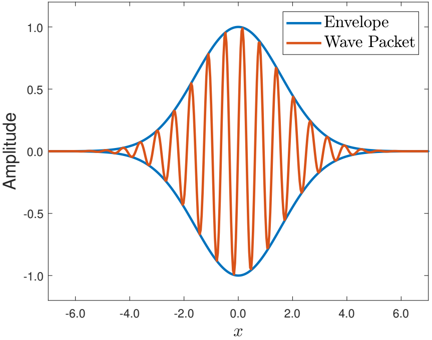


Figure: Wave-packet illustration.

Figure: Soliton in a water channel. *Laboratoire Interdisciplinaire CARNOT de Bourgogne*: `http://icb.u-bourgogne.fr`

Examples: nonlinear optics (Kerr media), waves in deep water, etc.

NLSE

We first consider the *nonlinear Schrödinger equation (NLSE)* defined by

$$\begin{cases} i\varphi_t + \varphi_{xx} + |\varphi|^{2\kappa}\varphi = 0, & \varphi : [0,T]\times\mathbb{R} \to \mathbb{C}, \\ \varphi(0,x) = \varphi_0(x), & \varphi_0 \in H^1(\mathbb{R}) \end{cases}$$

where κ is the nonlinear coefficient of the medium.

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Properties of solutions

- Local existence (*mild* solution).
- L^2-Norm and Energy conservation

$$N = \|\varphi\|_2^2, \qquad E(\varphi) = \|\varphi_x\|_2^2 - \frac{2}{p+1}\|\varphi\|_{p+1}^{p+1}.$$

Stationary states

Definition

A solution of the NLSE is ***stationary*** *when*

$$\boxed{\psi(t, x) = \phi(x)e^{i\lambda^2 t},}$$

where $\lambda \in \mathbb{R}^+$ *and* $\phi \in H^1(\mathbb{R}) \setminus \{0\}$.

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Stationary equation:

$$\begin{cases} \phi_{xx} - \lambda^2\phi + |\phi|^{2\kappa}\phi = 0, \\ \phi \in H^1(\mathbb{R}), \end{cases}$$

$$\Longrightarrow$$

Stationary solution:

$$\phi(x) = [(\kappa + 1)]^{1/(2\kappa)} \operatorname{sech}^{1/\kappa}(\lambda\kappa x)$$

The stationary state is defined **for all values of $\kappa > 0$.**

How do the perturbations affect the system?

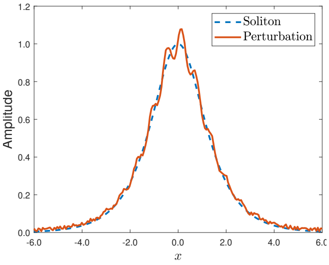

Figure: Initial soliton

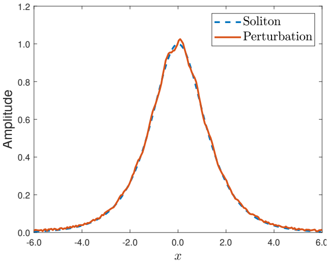

Figure: Stable soliton

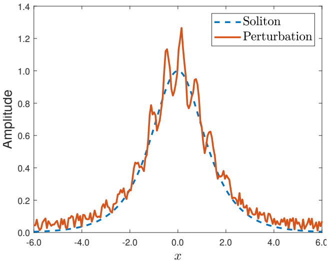

Figure: Unstable soliton

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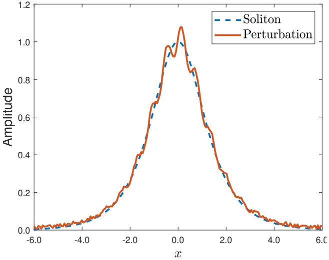

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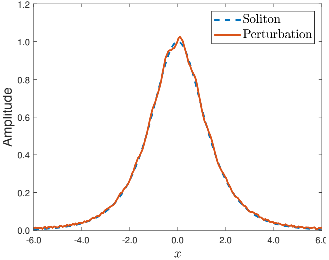

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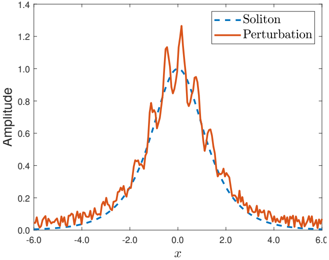

Figure: Unstable soliton

Linear Stability Analysis: Linearization of the equation around the soliton and analysis of the resulting linear problem.

$$\begin{cases} i\varphi_t + \varphi_{xx} + |\varphi|^{2\kappa}\varphi = 0, \\ \varphi(0,x) = \varphi_0(x), \end{cases}$$

$$\varphi(x,t) = [\phi(x) + \varepsilon w(x,t)]\, e^{i\lambda^2 t}, \quad \varepsilon\|w\|_{H^1} \ll \|\phi\|_{H^1}$$

The stability problem

The resulting linearized equation is

$$\boxed{iw_t + w_{xx} + [2(\kappa + 1)\phi^{2\kappa} - 1]w + 2\kappa\phi^{2\kappa} w^{\star} = 0,}$$

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which has solutions of the form

$$w(x,t) = \Re\left(e^{\Lambda t} f(x)\right) + i\Re\left(e^{\Lambda t} g(x)\right),$$

becoming the following spectral problem

$$\boxed{L_0 f = \Lambda g, \qquad L_1 g = -\Lambda f,}$$

where the operators are defined as

$$L_0 = -\partial_x^2 + 1 - (\kappa+1)\,\mathrm{sech}^2(\kappa x), \quad L_1 = -\partial_x^2 + 1 - (2\kappa+1)(\kappa+1)\,\mathrm{sech}^2(\kappa x).$$

Stability problem of the NLSE

Theorem

The stationary solution ϕ is stable if and only if the nonlinear parameter $\left(|\cdot|^{2\kappa}\right)$

$$\boxed{\kappa < 2}$$

and it is unstable due to blow-up otherwise.

Figure: Evolution of the stationary state + noise for $\lambda = 1$, $\kappa = 1$ (left) and $\kappa = 3$ (right).

Real media: dissipation

Figure: Evolution of the stationary state in non-dissipative and dissipative media.

NLSE with dissipation and parametric force

We consider the following modified NLS equation:

$$\boxed{i\varphi_t + \varphi_{xx} + 2|\varphi|^{2\kappa}\varphi = re^{2it}\varphi^{\star} - i\rho\varphi,}$$

where ρ is the dissipation and r is the force's amplitude.

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Which value of r produces a stable soliton for a fixed value of ρ?

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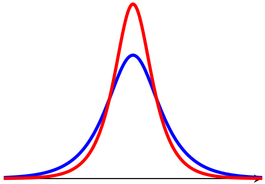
There are two solitons

$$\boxed{\phi_\pm(x,t) = \left[\left(\frac{\omega_\pm(\kappa+1)}{2}\right)^{1/2\kappa} \operatorname{sech}^{1/\kappa}\left(\kappa\sqrt{\omega_\pm}x\right)\right] e^{it - i\Theta_\pm/2}}$$



where $\omega_\pm = 1 \pm \sqrt{r^2 - \rho^2}$ and $\Theta_+ = \arcsin(\rho/r)\,, \Theta_- = \pi - \arcsin(\rho/r)\,.$

Soliton ϕ_+

- $r \geq \rho$

Soliton ϕ_-

- $r \geq \rho$, - $1 + \rho^2 > r^2$

Existence condition

Soliton ϕ_+

- ❑ $r \geq \rho$

Figure: Existence zone of ϕ_+ in the r-ρ plot.

Soliton ϕ_-

- ❑ $r \geq \rho$, ❑ $1 + \rho^2 > r^2$

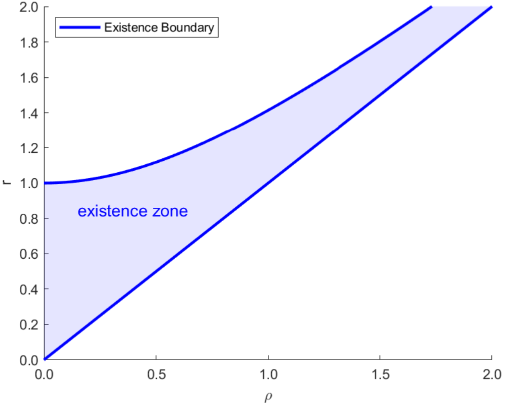


Figure: Existence zone of ϕ_- in the r-ρ plot.

Stability system

The spectral problem associated (up to a change of variables) is

$$\begin{cases} \left(L_0 - \varepsilon_{\pm}\right) g = \Lambda f, \\ \qquad\quad L_1 f = -\Lambda g. \end{cases} \qquad\qquad L_0 = -\partial_x^2 + 1 - (\kappa + 1)\,\mathrm{sech}^2(\kappa x), \quad L_1 = -\partial_x^2 + 1 - (2\kappa + 1)(\kappa + 1)\,\mathrm{sech}^2(\kappa x).$$

where

$$\boxed{\varepsilon_{\pm} = 2\frac{\sqrt{r^2 - \rho^2}}{1 \pm \sqrt{r^2 - \rho^2}}} \qquad \varepsilon_{-} \in (-\infty, 0], \quad \varepsilon_{+} \in [0, 2).$$

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The associated spectral problem resembles that without dissipation!

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where

$$\boxed{\varepsilon_\pm = 2\frac{\sqrt{r^2-\rho^2}}{1 \pm \sqrt{r^2-\rho^2}}} \qquad \varepsilon_- \in (-\infty, 0], \quad \varepsilon_+ \in [0, 2).$$

Consequences:

❑ The soliton ϕ_- is always unstable (existence of a real eigenvalue).

What happens to the soliton ϕ_+?

- **Existence**:

$$r \geq \rho$$

Spectrum of the eigenvalue problem:

$$\boxed{(L_0 - \varepsilon_+)g = \Lambda f, \quad L_1 f = -\Lambda g.}$$

Figure: Stability diagram $r - \rho$, for $\kappa = 1$.

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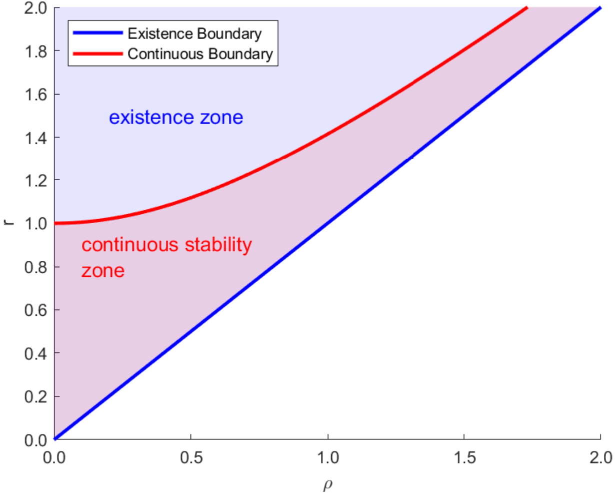
❑ **Continuous spectrum**:

$$r \leq \sqrt{1 + \rho^2}$$



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- **Stability curve** (Numerical):

$$(\rho(\varepsilon_+), r(\varepsilon_+))$$

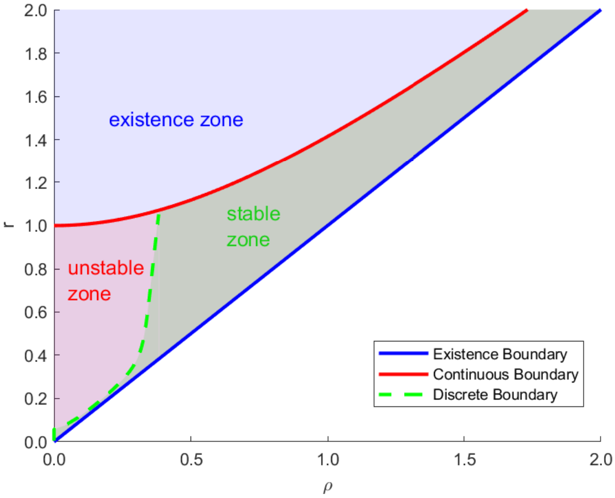
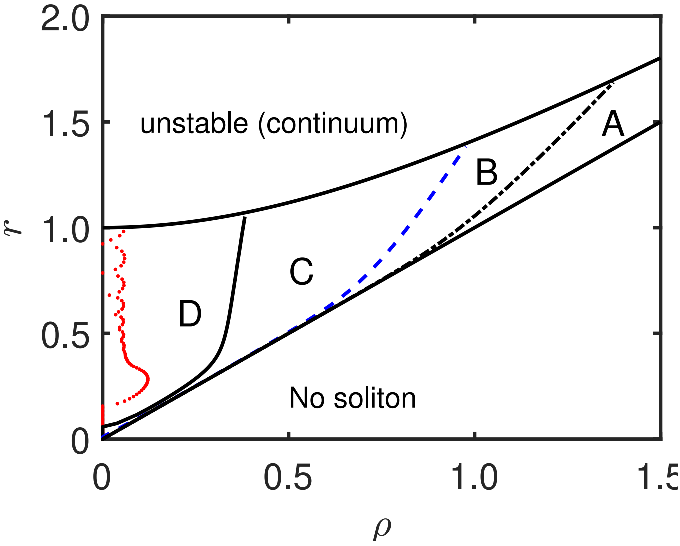


Figure: Stability diagram $r - \rho$, for $\kappa = 1$.

Stability diagram for $\kappa < 2$

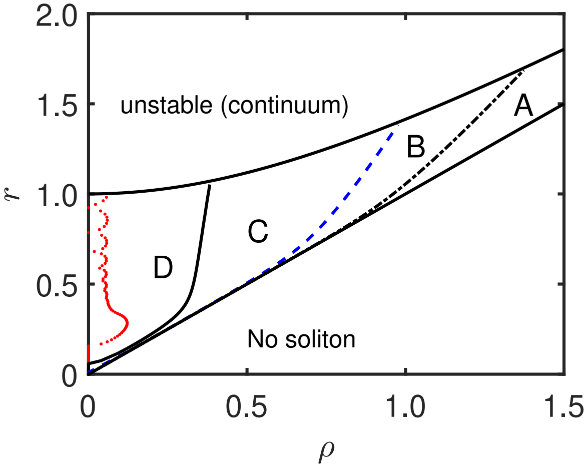
Stability zone:

- ❑ $\kappa = 0.50 \rightarrow A + B + C + D$
- ❑ $\kappa = 1.00 \rightarrow A + B + C$
- ❑ $\kappa = 1.50 \rightarrow A + B$
- ❑ $\kappa = 1.75 \rightarrow A$



Stability diagram for other values of κ

- In the NLSE ($\rho = 0, r = 0$), the solitons are always stable for $\kappa < 2$.
- The parametric force and dissipation produce instability zones.



Stability diagram for $\kappa \geq 2$

- In the NLSE ($\rho = 0, r = 0$), the solitons are always unstable for $\kappa \geq 2$.
- The parametric force and dissipation produce stability zones.

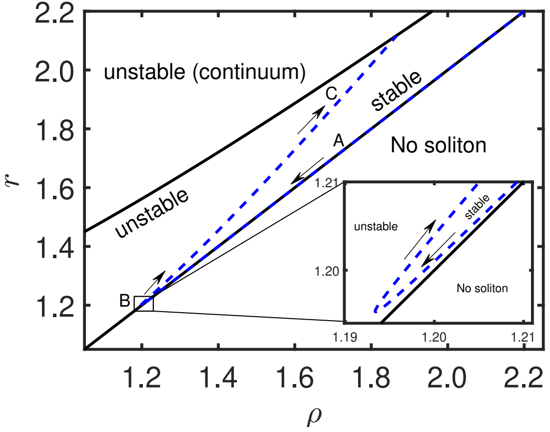


Figure: Stability diagram for $\kappa = 2.0$

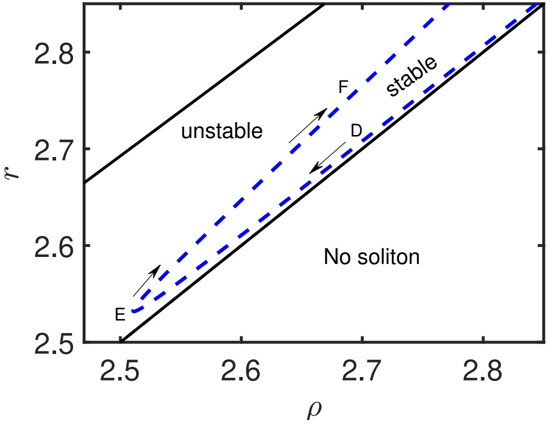


Figure: Stability diagram for $\kappa = 2.5$

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