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# CONVOLUTION AND ADDITION THEOREMS FOR $q$-ORTHOGONAL POLYNOMIALS

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**Abstract**

A short review of the use of quantum groups to addition theorems for $q$-special functions is given. We discuss a general convolution formula for Al-Salam and Chihara polynomials arising from its quantum algebra as transition coefficients. This contribution does not contain new results.

## 1 Introduction

As is very well known there is a strong connection between representation theory of groups and related structures, such as Lie groups, Lie algebras, finite groups, quantum groups, Hecke algebras, and all kinds of special functions, such as orthogonal polynomials of hypergeometric type or basic hypergeometric type in one or more variables, see e.g. Vilenkin and Klimyk [28] and references therein. The theory of special functions has a very rich tradition dating back to the great mathematicians of the 18th and 19th century such as Euler, Gauß, Heine, Riemann, Laplace, Legendre and others. The connection with group theory is much more recent and started with Gelfand and Šapiro in the 1950's who discovered the relation between Jacobi polynomials and the special linear group $SL(2,\mathbb{C})$. Since then the literature on this relationship has expanded explosively and many connection between group theory and special functions, mainly of hypergeometric type, have been established.

Very soon after the introduction of quantum groups by Jimbo, Drinfeld and Woronowicz it became clear that these 'groups' are strongly related to $q$-special functions in a way that resembles the relation discussed in the previous paragraph. In §2 we discuss a quantum group analogue of the Gelfand-Šapiro result and use it to derive an addition formula for a two-parameter family of Askey-Wilson polynomials. Classically, i.e. in the corresponding group case, this would give the addition formula for the Legendre polynomials. This section is of a descriptive nature and takes all of its results from [16].

In §3 we discuss how a very general convolution identity for the Al-Salam and Chihara polynomials which also involves Askey-Wilson polynomials. It arises from the interpretation of the Al-Salam and Chihara polynomials as transition coefficients in the representation space of the tensor product of two positive discrete series representations of the quantised universal enveloping algebra for $\mathfrak{su}(1,1)$. Since we work with a tensor product the Clebsch-Gordan coefficients will come into play. As a special case we obtain the convolution property which originally led Al-Salam and Chihara to the introduction of these polynomials. The result discussed in §3 is a special case of more general results obtained in collaboration with Joris Van der Jeugt [19] using an idea of Granovskii and Zhedanov.

# 2 Addition theorems

In this section we state some generalities on addition theorems and we discuss some of the addition theorems that can be derived using quantum groups without going through (heavy) calculations. This section doesn't contain any new results. Explicit formulas with proofs and references to the literature for other approaches to addition formulas for $q$-special functions and addition formulas for specific sets of $q$-special functions can be found in [16], see also [15], [18].

## 2.1 Generalities on addition formulas

The exponential function satisfies the addition formula $e^{x+y} = e^x e^y$. More generally, the function $f$ satisfies an *algebraic addition theorem* if $A(f(x), f(y), f(x+y))$ is identically zero for some polynomial $A(x,y,z)$. By a theorem of Weierstraß, a single valued analytic function $f(z)$ satisfying an algebraic addition theorem is either a rational function in $z$, or a rational function in $e^{\lambda z}$ for some $\lambda$, or an elliptic function, see Askey [3, Lecture 4] for more information and references. The addition formula for Legendre polynomials $P_n(x)$ (Laplace, 1872) is the prototype of an addition theorem of a more general nature. It gives the explicit Fourier-cosine expansion of $P_n(\cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta)$ in terms of $\cos m\theta$. The Fourier coefficients depending on $\phi$ and $\psi$ factorize as products of two associated Legendre functions. This addition formula was generalized by Gegenbauer (1875) to ultraspherical polynomials and more generally by Koornwinder (1972) to Jacobi polynomials. Addition theorems of this type exist for a large number of special functions of hypergeometric type. See [3, Lecture 4] and in particular [29] for several addition formulas for Bessel functions.

For a more general definition of an addition formula we follow Koornwinder [21, §4]. An *addition formula* for a family of special functions $\phi_\lambda$ expands $\phi_\lambda(\Lambda(x,y,t))$ ($\Lambda$ a fixed elementary function) explicitly and orthogonally in terms of certain special functions $\psi_\mu(t)$ with expansion coefficients of the form constant time $\phi_\lambda^\mu(x)\,\phi_\lambda^\mu(y)$. The $\phi_\lambda^\mu$ are special functions *associated with* $\phi_\lambda$. A further requirement is that $\psi_0(t) = 1$ and $\phi_\lambda^0 = \phi_\lambda$. An important corollary of any addition formula is the *product formula* giving $\phi_\lambda(x)\phi_\lambda(y)$ as an integral of $\phi_\lambda$ using the orthogonality properties of $\psi_\mu(t)$. The addition theorems for the Legendre, Gegenbauer and Jacobi polynomials fit into this definition.

For special functions of basic hypergeometric type this definition is no longer sufficiently general, although the addition theorems for the spherical functions on finite groups of Lie type, which are expressible in terms of $q$-Krawtchouk polynomials, are still of this type, see Stanton [26]. However, for most of the addition theorems derived from the interpretation of special functions on quantum groups this definition no longer suffices. Then we use the following variant that $\phi_\lambda(t)$, or even $\phi_\lambda(t)\psi_\nu^{(x,y)}(t)$, is expanded explicitly and orthogonally in terms of $\psi_\mu^{(x,y)}(t)$, with expansion coefficients of the form $\xi_\lambda^\mu(x)\,\eta_\lambda^\mu(y)$ for certain special functions $\xi_\lambda^\mu$ and $\eta_\lambda^\mu$. This type of addition formula also gives as a corollary the product formula for $\xi_\lambda^0\eta_\lambda^0(y)$. Of course, $\xi_\lambda^\mu$ and $\eta_\lambda^\mu(y)$ are related to $\phi_\lambda$ in some sense. In the next subsection we see this for the case that $\phi_\lambda$ is an Askey-Wilson polynomial and the $\xi_\lambda^\mu$ and $\eta_\lambda^\mu(y)$ are also Askey-Wilson polynomials but with different parameters.

In general addition formulas are hard to prove from scratch. Additional structure can provide a conceptual setting for such formulas. For the Jacobi polynomials and Bessel functions it is known that addition theorems follow from the representation theory of specific Lie groups. For certain $q$-special functions we can use the theory of quantum groups as sketched in the next subsections.

## 2.2 Addition theorem from Askey-Wilson polynomials

To understand the quantum group approach we recall that a quantum group $A_q(G)$ is a deformation (with deformation parameter $q$) of a function algebra $A(G)$ on the group $G$. For a compact group $G$ this would mean the polynomials on $G$ or the continuous functions on $G$, see Chari and Pressley [7], Kassel [13] or [18] and references therein. We assume that the comultiplication

$$\Delta\colon A(G) \to A(G) \otimes A(G) \cong A(G \times G), \qquad (\Delta f)(g,h) = f(gh),$$

is well-defined and we demand that $\Delta\colon A_q(G) \to A_q(G) \otimes A_q(G)$ survives the deformation unchanged (in a suitable sense). In the deformation the representations of $G$ correspond to corepresentations of $A_q(G)$ and for a suitable basis of the representation space we get matrix elements $t_{n,m} \in A_q(G)$ satisfying the cohomomorphism property

$$\Delta(t_{n,m}) = \sum_p t_{n,p} \otimes t_{p,m}, \tag{1}$$

For the undeformed case this is just $t_{n,m}(gh) = \sum_p t_{n,p}(g)t_{p,m}(h)$, $g,h \in G$, meaning that $t$ is a representation of $G$. Classically, this identity leads to addition formulas for special functions, see [28] for numerous examples.

If the matrix elements can be expressed in terms of $q$-special functions we consider (1) as an (implicit) addition formula. Usually $A_q(G)$ is a non-commutative algebra and thus we need to investigate the representations of $A_q(G)$ in order to convert (1) into an addition formula involving $q$-special functions in commuting variables. Usually, we normalise $n = m = 0$ in (1) and $t_{0,0}$ is considered as an analogue of a spherical function, i.e. as a function which is left and right $K$-invariant for a certain subgroup $K \subset G$.

In some specific cases we can fill all the details of this program in order to derive very general addition theorems for $q$-special functions. As an example we discuss the addition formula for a two-parameter family of Askey-Wilson polynomials arising from the interpretation of Askey-Wilson polynomials as generalised matrix elements on the quantum $SU(2)$ group.

For the quantum $SU(2)$ group we have a generalisation of (1) by varying the subgroup $K$. Then all the matrix elements of the finite dimensional irreducible representations, which can be labelled by the spin $l \in \frac{1}{2}\mathbb{Z}_+$ such that the dimension is $2l+1$, are explicitly known in terms of Askey-Wilson polynomials [5]. This set of orthogonal polynomials has four degrees of freedom (apart form the $q$) and these correspond in this interpretation with $n$ and $m$, i.e. discrete parameters corresponding to the labeling of the vectors in the representation space, and two parameters, say $\sigma$ and $\tau$, labeling the 'quantum subgroups' under which these matrix elements behave nicely for left and right multiplication. This is due to Noumi and Mimachi following Koornwinder's introduction of 'quantum subgroups', see references in [15], [18]. In this case, taking $n = m = 0$ in the appropriate analogue of (1), we get

$$a^l_{0,0}(\tau,\sigma) = \sum_{p=-l}^{l} a^l_{0,p}(\tau,\mu) \otimes a^l_{p,0}(\mu,\sigma), \qquad l \in \mathbb{Z}_+, \tag{2}$$

as an identity in $A_q(SU(2)) \otimes A_q(SU(2))$. In general the (generalised) matrix elements $a^l_{n,m}(\tau,\sigma)$ are an explicitly known element of $A_q(SU(2))$ times $p_{l-k}^{(|n-m|,|n+m|)}(\rho; s,t|q^2)$, with $n,m \in \{-l,\dots,l\}$, $k = \max(|n|,|m|)$, labeling the vectors, $s,t \in \{q^{\pm\sigma}, q^{\pm\tau}\}$ with $\sigma$, $\tau$ labeling the quantum subgroups and $\rho \in A_q(SU(2))$ some explicitly known quadratic element. This analogue of the Jacobi polynomial is

$$p_n^{(\alpha,\beta)}(x;s,t|q) = p_n(x; q^{1/2}t/s, q^{1/2+\alpha}s/t, -q^{1/2}/(st), -stq^{1/2+\beta}|q),$$

where the left hand side denotes a standard Askey-Wilson polynomial. For $s = t = 1$ this corresponds to Rahman's continuous $q$-Jacobi polynomials, see [5].

So (2) can be considered as the first stage of an addition formula. To obtain the second stage we use the representation theory of the non-commutative algebra $A_q(SU(2))$. Its irreducible $*$-representations are either one-dimensional or infinite dimensional acting in $\ell^2(\mathbb{Z}_+)$. Applying a one dimensional representation on the first leg and an infinite dimensional representation on the second leg of (2) we obtain an operator identity in the space of bounded linear operators of $\ell^2(\mathbb{Z}_+)$. The spectral decompostion of the operator corresponding to $\rho$, which is a five-term recurrence operator, can be determined by splitting the five-term recurrence in two three-term recurrences, which each can be solved using orthogonal polynomials, namely Al-Salam and Carlitz polynomials and Al-Salam and Chihara polynomials, see [20] for the explicit calculations. Having these (generalised) eigenvectors at hand we can let the operator identity on such an eigenvector and take inner product with the appropriate eigenvectors for the right hand side. This leads to a very general formula, which we call an addition theorem for a two-parameter family of Askey-Wilson polynomial, see [16, Thm. 4.1]. However, in general this addition formula doesn't fit the general definition of §2.1, since the functions $\psi_\mu^{(x,y)}(t)$ are not (yet) known to be orthogonal in a suitable sense. In this case the $\psi_\mu^{(x,y)}(t)$ are explicitly known in terms of very-well-poised ${}_8\varphi_7$-series, see Gasper and Rahman [11], which is related to the non-symmetric Poisson kernel for the Al-Salam and Chihara polynomials, see [4]. It's an open problem to find suitable orthogonality relations for the functions $\psi_\mu^{(x,y)}(t)$ in this case.

Taking a suitable limit $\mu \to \infty$ for the free parameter $\mu$ in the right hand side of (2) we can derive an addition formula for a two-parameter Askey-Wilson polynomial which fits into the general definition of §2.1, see [16, §5], [18, Lecture 6]. Then the Askey-Wilson polynomials in the right hand side of (2) go over in big $q$-Jacobi polynomials, so the functions $\xi_\lambda^\mu$ and $\eta_\lambda^\mu$ of §2.1 are expressible in terms of big $q$-Jacobi polynomials. For this case the functions $\psi_\mu^{(x,y)}(t)$ are $q$-analogues of the Laguerre polynomials which can be obtained by letting $\beta \to \infty$ in $p_n^{(\alpha,\beta)}(\cdot; s, t|q)$. This addition formula can also be given a direct proof, cf. [18, Lecture 6], and this was presented in one of my lectures at IWOP'96.

If we let also $\sigma = \tau \to \infty$ we obtain the addition formula for little $q$-Legendre polynomials. In the definition of §2.1 we have $\phi_\lambda$ as little $q$-Legendre polynomials, $\xi_\lambda^\mu$ and $\eta_\lambda^\mu$ are expressed in terms of little $q$-ultraspherical polynomials, and the $\psi_\mu^{(x,y)}(t)$ are Wall polynomials. This addition formula is due to Koornwinder [22] and is the first example of an addition formula derived from the relation between quantum groups and $q$-special functions.

A decade before this general addition formula Rahman and Verma [25] proved an addition formula for the continuous $q$-ultraspherical polynomials by analytic means. This addition formula has the same structure as the addition formula for the ultraspherical (or Gegenbauer) polynomials. For the case $\sigma = \tau = \mu = 0$ the Rahman-Verma addition formula for the continuous $q$-Legendre polynomials should be recovered from the general formula described here. This is indeed the case, see [16, §5] and [14, §4] for another derivation of the Rahman-Verma result starting from (2) using only one-dimensional representations of $A_q(SU(2))$.

## 2.3 Addition formulas from quantum groups

In the previous subsection we discussed the quantum group theoretic derivation of an addition theorem for a two-parameter subclass of Askey-Wilson polynomials. This method also works for other groups, notably the quantum group of plane motions, and analogues of the space $U(n)/U(n-1)$, see references in [16].

Instead of working with deformed function algebras it is also possible to work with the dual object (in a suitable sense) of deformed universal enveloping algebras. In this approach (see also the next section) it's also possible to derive addition formulas for special functions. In a certain sense this method is more general since it needs less structure. See the references to the papers by Kalnins, Miller and coworkers, and by Floreanini and Vinet in [16].

We may consider also (1) or (2) as addition formulas in non-commuting variables. Even at this level these formulas turn out to be useful. See e.g. Floris [10], where non-negativity of linearisation coefficients for so-called disk polynomials involving little $q$-Jacobi polynomials is proved from a non-commutative addition formula. Its proof is similar to the proof in [21]. This proof also can be used to prove non-negativity of the linearisation coefficients for $p_l^{(0,0)}(\cdot; s, s|q)$ from (2). See [15, §8.3], [23, §7] for different proofs of this fact and [19, §5], cf. Remark 3.5(iv), for an explicit formula for the linearisation coefficients for the more general case $p_l^{(0,0)}(\cdot; s, t|q)$. See Koornwinder [24] for more information on $q$-special functions in non-commuting variables.

Addition theorems for $q$-Krawtchouk polynomials for three specific choices of the parameter have been derived by Stanton [26] using its interpretation as spherical functions on certain finite groups of Lie type. This interpretation can be generalised to an interpretation of $q$-Krawtchouk polynomials on the Hecke algebra of type $B$, see [17], as explained in my other lecture at IWOP'96. It's my hope that this may lead to an addition formula for $q$-Krawtchouk polynomials.

# 3 Convolution theorems

The results in this section have been obtained in collaboration with Joris Van der Jeugt (Universiteit Gent, Belgium). Full details for this section as well as references and the discussion for the case of the continuous Hahn, Hahn, Meixner-Pollaczek, Meixner, Laguerre and Hermite polynomials and the Askey-Wilson polynomials can be found in [19]. For the orthogonal polynomials of hypergeometric type this follows using the Lie algebra $\mathfrak{su}(1,1)$, whereas for the Al-Salam and Chihara and the Askey-Wilson polynomials we have to use the quantised universal enveloping algebra $U_q(\mathfrak{su}(1,1))$.

## 3.1 Al-Salam and Chihara polynomials

The convolution of two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ is the sequence $(c_n)_{n=0}^{\infty}$ with $c_n = \sum_{k+l=n} a_k b_l$. This arises for instance from multiplying the formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ by $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and then the $c_n$'s are the coefficients in the resulting power series. In case $a_n = p_n(x)$, $b_n = r_n(y)$ for a certain set of orthogonal polynomials, we have $c_n = Q_n(x,y)$ for some polynomial in two variables of total degree $n$. Al-Salam and Chihara [2], see also [1, §8], posed and solved the question for which orthogonal polynomials $p_n$ and $r_n$ the polynomial $Q_n(x,y)$ is orthogonal as a polynomial in $x$. This led to the introduction of the Al-Salam and Chihara polynomials, which was later identified as a two-parameter subset of the Askey-Wilson polynomials.

To put this convolution property in a wider perspective let us introduce the orthogonality measures $\mu$ and $\nu$ for $p_n$ and $r_n$. Then the two-variable polynomial $R_{n,m}(x,y) = p_n(x)r_m(y)$ is orthogonal with respect to the product measure of $\mu$ and $\nu$. From the convolution property $Q_n(x,y) = \sum_{k=0}^{n} R_{k,n-k}(x,y)$ we get

$$\iint Q_n(x,y)Q_m(x,y)\, d\mu(x)d\nu(y) = 0, \qquad n \neq m.$$

Now a natural question arises: is $Q_n(x,y)$ part of a set of two-variable orthogonal (with respect to the product measure of $\mu$ and $\nu$) polynomials $P_{k,l}(x,y)$? If so, can we calculate the connection coefficients between $R_{n,m}(x,y)$ and $P_{k,l}(x,y)$ in general? In the rest of this section we indicate how the quantised universal enveloping algebra for $\mathfrak{su}(1,1)$ can be used to obtain a formula which answers this question for the Al-Salam and Chihara polynomials.

## 3.2 Positive discrete series representations

From now on $q$ is assumed to satisfy $0 < q < 1$. Let $U_q(\mathfrak{su}(1,1))$ be the complex unital associative $*$-algebra generated by $A$, $B$, $C$, $D$ subject to the relations

$$AD = 1 = DA, \quad AB = qBA, \quad AC = q^{-1}CA, \quad BC - CB = \frac{A^2 - D^2}{q - q^{-1}}.$$

The $*$-structure on $U_q(\mathfrak{su}(1,1))$ is given by $A^* = A$, $B^* = -C$, $D^* = D$.

The positive discrete series representations $\pi_k$ of are $*$-representations of $U_q(\mathfrak{su}(1,1))$ labelled by $k > 0$. They act in $\ell^2(\mathbb{Z}_+)$ and the action of the generators is given by

$$\begin{aligned}
\pi_k(A)\, e_n^k &= q^{k+n}\, e_n^k, \qquad \pi_k(D)\, e_n^k = q^{-k-n} \\
\pi_k(C)\, e_n^k &= q^{1/2-k-n} \frac{\sqrt{(1-q^{2n})(1-q^{4k+2n-2})}}{q-q^{-1}}\, e_{n-1}^k, \\
\pi_k(B)\, e_n^k &= q^{-1/2-k-n} \frac{\sqrt{(1-q^{2n+2})(1-q^{4k+2n})}}{q^{-1}-q}\, e_{n+1}^k.
\end{aligned}$$

Note that $\pi_k(D)$ is an unbounded operator, but that $\pi_k(A)$, $\pi_k(B)$ and $\pi_k(C)$ are bounded. We consider only bounded operators.

Recall that $U_q(\mathfrak{su}(1,1))$ is actually a Hopf algebra, so that we can define the tensor product of two representations by use of the comultiplication. The tensor product of two positive discrete series representation decomposes as for the Lie algebra $\mathfrak{su}(1,1)$ in positive discrete series representations; $\pi_{k_1} \otimes \pi_{k_2} \cong \bigoplus_{j=0}^{\infty} \pi_{k_1+k_2+j}$. So there exists a unitary matrix mapping the orthogonal basis $e_{n_1}^{k_1} \otimes e_{n_2}^{k_2}$ onto $e_n^{k_1+k_2+j}$ intertwining the action of $U_q(\mathfrak{su}(1,1))$. The matrix elements of this unitary mapping are the Clebsch-Gordan coefficients;

$$e_n^k = \sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2,n}^{k_1,k_2,k}\, e_{n_1}^{k_1} \otimes e_{n_2}^{k_2}, \tag{3}$$

where $k = k_1 + k_2 + j$ for $j \in \mathbb{Z}_+$. The sum is finite; the Clebsch-Gordan coefficients are non-zero only for $n_1 + n_2 = n + j$ as follows from considering the action of the group-like element $A$ on both sides of (3). The Clebsch-Gordan coefficients are normalised by $\langle e_0^k, e_0^{k_1} \otimes e_j^{k_2} \rangle > 0$.

These results can be found in Burban and Klimyk [6] and Kalnins, Manocha and Miller [12]. See Chari and Pressley [7] and Kassel [13] for general information on quantised universal enveloping algebras.

### 3.3 Spectral analysis of a self-adjoint element

We now define

$$Y_s = q^{1/2} B - q^{-1/2} C + \frac{s^{-1}+s}{q^{-1}-q}(A - D) \in U_q(\mathfrak{su}(1,1)).$$

Then $Y_s A$ is a self-adjoint element in $U_q(\mathfrak{su}(1,1))$ for $s \in \mathbb{R}\backslash\{0\}$, or $s \in \mathbb{T}$. We want to calculate the action of $Y_s A$ in $\pi_{k_1} \otimes \pi_{k_2}$ and in $\bigoplus_{j=0}^{\infty} \pi_{k_1+k_2+j}$, and then match the results.

In order to do so we first find the spectrum of $\pi_k(Y_s A)$. We need the orthonormal Al-Salam and Chihara polynomials $S_n(x; a, b|q)$, which satisfy the three-term recurrence relation

$$2x\, S_n(x) = a_{n+1}\, S_{n+1}(x) + q^n (a+b)\, S_n(x) + a_n\, S_{n-1}(x),$$

$a_n = \sqrt{(1-abq^{n-1})(1-q^n)}$. Let $dm(\cdot; a, b|q)$ denote the corresponding normalised orthogonality measure, see [5] with $c = d = 0$. Let $M_g$ denote multiplication by the function $g$, i.e. $M_g f(x) = g(x) f(x)$. We use the notation $\mu(x) = (x + x^{-1})/2 = \mu(x^{-1})$ for $x \neq 0$.

**Proposition 3.1** *Let $\Lambda\colon \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}, dm(\cdot; q^{2k}s, q^{2k}/s|q^2))$ be the unitary mapping defined by $\Lambda\colon e_n^k \mapsto S_n(\cdot; q^{2k}s, q^{2k}/s|q^2)$, then $\Lambda$ intertwines $\pi_k(Y_s A)$ acting in $\ell^2(\mathbb{Z}_+)$ with $2(M_x - \mu(s))/(q^{-1}-q)$.*

Since $\Delta(Y_s A) = A^2 \otimes Y_s A + Y_s A \otimes 1$ we can also calculate the spectrum of $Y_s A$ in $\pi_{k_1} \otimes \pi_{k_2}$ using Proposition 3.1.

**Proposition 3.2** *Define* $\Upsilon\colon \ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)\to L^2(\mathbb{R}^2, dm(x_1,x_2))$, *where*

$$dm(x_1,x_2)=dm(x_1;q^{2k_1}w_2,q^{2k_1}/w_2|q^2)\,dm(x_2;q^{2k_2}s,q^{2k_2}/s|q^2),\qquad x_2=\mu(w_2),$$

*by*

$$\Upsilon\colon e^{k_1}_{n_1}\otimes e^{k_2}_{n_2}\mapsto S_{n_1}(x_1;q^{2k_1}w_2,q^{2k_1}/w_2|q^2)\,S_{n_2}(x_2;q^{2k_2}s,q^{2k_2}/s|q^2)$$

*then* $\Upsilon$ *is a unitary mapping intertwining* $\pi_{k_1}\otimes\pi_{k_2}(\Delta(Y_sA))$ *with* $2(M_{x_1}-\mu(s))/(q^{-1}-q)$ *in* $L^2(dm(x_1,x_2))$.

The proof of both propositions consists of comparing three-term recurrence relations arising from the operator under consideration with the three-term recurrence relations for the Al-Salam and Chihara polynomials. For Proposition 3.2 we use the result of Proposition 3.1 on the second leg of the tensor product. See [19] for details.

It's important to observe that $\Upsilon e^{k_1}_{n_1}\otimes e^{k_2}_{n_2}$ gives a complete set of orthonormal polynomials in $L^2(\mathbb{R}^2, dm(x_1,x_2))$. Here we use that the Al-Salam and Chihara polynomial is symmetric in its parameters $a$ and $b$.

### 3.4 A general convolution

From the Clebsch-Gordan decomposition we see that $e^k_n$ is also an orthonormal basis of the representation space. So $\Upsilon\, e^k_n$ gives another orthonormal basis in $L^2(\mathbb{R}^2, dm(x_1,x_2))$.

**Proposition 3.3** *Let* $k=k_1+k_2+j$ *for* $j\in\mathbb{Z}_+$, *and* $x_1=\mu(w_1)$ *then*

$$\begin{aligned}
(\Upsilon\, e^k_n)(x_1,x_2) &= S_n(x_1;q^{2k}s,q^{2k}/s|q^2)\,(\Upsilon\, e^k_0)(x_1,x_2),\\
(\Upsilon\, e^k_0)(x_1,x_2) &= C\,p_j(x_2;q^{2k_1}w_1,q^{2k_1}/w_1,q^{2k_2}s,q^{2k_2}/s|q^2),\\
C^{-1} &= \sqrt{(q^2,q^{4k_1},q^{4k_2},q^{4k_1+4k_2+2j-2};q^2)_j}.
\end{aligned}$$

**Sketch of proof.** For the first part of Proposition 3.3 we use that $\Upsilon$ intertwines $\pi_{k_1}\otimes\pi_{k_2}(\Delta(Y_sA))$ with a certain multiplication operator, leading to a three-term recurrence relation for $\Upsilon\, e^k_n$ with respect to $n$. The second part is then proved by identifying orthogonality measures as follows. Since $\Upsilon$ is unitary we have the orthogonality relations $\delta_{nm}\delta_{kl}=\langle\Upsilon e^k_n,\Upsilon e^l_m\rangle=$

$$\int S_n(x_1;q^{2k}s,q^{2k}/s|q^2)S_m(x_1;q^{2l}s,q^{2l}/s|q^2)\int \Upsilon e^k_0(x_1,x_2)\Upsilon e^l_0(x_1,x_2)\,dm(x_1,x_2),$$

by the first part of the proposition. Since the moment problem for the Al-Salam and Chihara polynomials is determined we see that for $k=l$ we get

$$\int \Upsilon e^k_0(x_1,x_2)\Upsilon e^l_0(x_1,x_2)\,dm(x_1,x_2)=\delta_{kl}\,dm(x_1;q^{2k}s,q^{2k}/s|q^2)$$

as measures with respect to functions in the variable $x_1$. It also holds for $k\neq l$, since then $\int\Upsilon e^k_0(x_1,x_2)\Upsilon e^l_0(x_1,x_2)\,dm(x_1,x_2)=0$ because integrated against any polynomial it gives zero. Knowing that $\Upsilon e^k_0(x_1,x_2)$ is a polynomial of degree $j$, $k=k_1+k_2+j$, and the explicit expression for the orthogonality measures involved we find the second part of the proposition. □

We can now apply $\Upsilon$ to both sides of (3). Apart from the explicit value of the Clebsch-Gordan coefficient we have a complete description of the resulting identity in $L^2(\mathbb{R}^2, dm(x_1,x_2))$ due to Propositions 3.2 and 3.3. The explicit value of the Clebsch-Gordan coefficients can be found in e.g. [12], but it can also be obtained from the identity arising from the application of $\Upsilon$ to both sides of (3). Specialising $x_1$ and $x_2$ suitably and observing that the Clebsch-Gordan coefficients are independent of $s$, we obtain a generating function from which the value for the Clebsch-Gordan coefficients can be determined;

$$C^{k_1,k_2,k_1+k_2+j}_{n_1,n_2,n} = C\ Q_j(q^{-2n_1}; q^{4k_1-2}, q^{4k_2-2}, n+j; q^2)$$

with $n_1 + n_2 = n + j$. The constant is explicitly known and $Q_j$ is the $q$-Hahn polynomial;

$$Q_n(q^{-x}; a, b, N; q) = {}_3\varphi_2 \left( \begin{matrix} q^{-n}, q^{-x}, abq^{n+1} \\ aq,\ q^{-N} \end{matrix} ; q, q \right).$$

Plugging this information back into the formula we obtain the following theorem.

**Theorem 3.4** *Let $p_n(x; a, b, c, d|q)$ be Askey-Wilson polynomials as in* [5] *and let the Al-Salam and Chihara polynomials $s_n(x; a, b|q) = p_n(x; a, b, 0, 0|q)$. With $x_i = (w_i + w_i^{-1})/2$ $(i = 1, 2)$, $n, j \in \mathbb{Z}_+$, $k_1, k_2 > 0$ we have*

$$\begin{gathered}(q^{4k_1}; q^2)_j \sum_{l=0}^{n+j} q^{2k_1(n-l)} \begin{bmatrix} n+j \\ l \end{bmatrix}_{q^2} Q_j(q^{-2l}; q^{4k_1-2}, q^{4k_2-2}, n+j; q^2) \\ \times s_l(x_1; q^{2k_1} w_2, q^{2k_1}/w_2|q^2)\, s_{n+j-l}(x_2; q^{2k_2} s, q^{2k_2}/s|q^2) = \\ s_n(x_1; q^{2k_1+2k_2+2j} s, q^{2k_1+2k_2+2j}/s|q^2)\, p_j(x_2; q^{2k_1} w_1, q^{2k_1}/w_1, q^{2k_2} s, q^{2k_2}/s|q^2).\end{gathered}$$

**Remark 3.5** (i) Theorem 3.4 is a connection coefficient formula for orthogonal polynomials in two variables, orthogonal for the same measure $dm(x_1, x_2)$ of Proposition 3.2, where the connection coefficients are given by the $q$-Hahn polynomials. The dual connection coefficient problem follows from the orthogonality for the Clebsch-Gordan coefficients or, equivalently, from the orthogonality relations for the dual $q$-Hahn polynomials.

(ii) The case $j = 0$ gives a simple convolution property for the Al-Salam and Chihara polynomials, since the $q$-Hahn and the Askey-Wilson polynomial reduce to 1. This was the motivation for Al-Salam and Chihara [2] to introduce the Al-Salam and Chihara polynomials as the most general set of orthogonal polynomials still satisfying a convolution property. The case $n = 0$ is also of interest, since then the $q$-Hahn polynomial can be evaluated and the Al-Salam and Chihara polynomial on the right hand side reduces to 1. In both cases we have a free parameter in the sum.

(iii) Using Theorem 3.4 we can also work in the three-fold tensor product representation of positive discrete series representations. Instead of Clebsch-Gordan coefficients we have to work with Racah coefficients. We obtain a similar formula with the $q$-Hahn and the Al-Salam and Chihara polynomials replaced by the $q$-Racah and Askey-Wilson polynomials, see [19, §4]. The resulting formula has the same structure as Theorem 3.4 due to the Wigner-Eckhart theorem, and it contains Theorem 3.4 as a special case. A special case of this general convolution property has been obtained by Dunkl [8] by working on the general linear group over a finite field with $q$ elements. (Here $q$ is a prime power.) This is the "$q$-analogue" (in Stanton's [27] terminology) of Dunkl's [9] proof of a special case of [19, Thm. 3.13] using the representation theory of the symmetric group, which is the Weyl group of the general linear group over a finite field with $q$ elements.

(iv) Using the the result of the previous remark and the interpretation of a two-parameter family of Askey-Wilson polynomials on the quantum $SU(2)$ group, see §2, we can find the explicit linearisation coefficients for these polynomials as the product of two terminating balanced ${}_4\varphi_3$-series, see [19, §5].

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## References

[1] W.A. Al-Salam: *Characterization theorems for orthogonal polynomials* in ORTHOGONAL POLYNOMIALS: THEORY AND PRACTICE (P. Nevai ed.), NATO ASI Series C **294** (Kluwer Acad. Publ., Dordrecht, 1990), 1–24.

[2] W.A. Al-Salam and T.S. Chihara: *Convolutions of orthonormal polynomials*, SIAM J. MATH. ANAL. **7** (1976), 16–28.

[3] R. Askey: ORTHOGONAL POLYNOMIALS AND SPECIAL FUNCTIONS, Reg. Conf. Ser. Appl. Math. **21** (SIAM, 1975).

[4] R.A. Askey, M. Rahman and S.K. Suslov: *On a general q-Fourier transformation with nonsymmetric kernels*, J. COMP. APPL. MATH. **68** (1996), 25–55.

[5] R. Askey and J. Wilson: *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, MEM. AMER. MATH. SOC. **54**, no. 319 (1985).

[6] I.M. Burban and A.U. Klimyk: *Representations of the quantum algebra $U_q(su_{1,1})$*, J. PHYS. A: MATH. GEN. **26** (1993), 2139–2151.

[7] V. Chari and A. Pressley: A GUIDE TO QUANTUM GROUPS (Cambridge Univ. Press, 1994).

[8] C.F. Dunkl: *Orthogonal polynomials in two variables of q-Hahn and q-Jacobi type*, SIAM J. ALG. DISC. METH. **1** (1980), 137–151.

[9] C.F. Dunkl: *A difference equation and Hahn polynomials in two variables*, PACIFIC J. MATH. **92** (1981), 57–71.

[10] P.G.A. Floris: *A noncommutative discrete hypergroup associated with q-disk polynomials*, J. COMP. APPL. MATH. **68** (1996), 69–78.

[11] G. Gasper and M. Rahman: BASIC HYPERGEOMETRIC SERIES, Encycl. Math. Appl. **35** (Cambridge Univ. Press, 1990).

[12] E.G. Kalnins, H.L. Manocha and W. Miller: *Models of q-algebra representations: tensor products of special unitary and oscillator algebras*, J. MATH. PHYS. **33** (1992), 2365–2383.

[13] C. Kassel: QUANTUM GROUPS, GTM **155** (Springer, 1995).

[14] H.T. Koelink: *The addition formula for continuous q-Legendre polynomials and associated spherical elements on the $SU(2)$ quantum group related to Askey-Wilson polynomials*, SIAM J. MATH. ANAL. **25** (1994), 197–217.

[15] H.T. Koelink: *Askey-Wilson polynomials and the quantum $SU(2)$ group: survey and applications*, ACTA APPLIC. MATH. **44** (1996), 295–352.

[16] H.T. Koelink: *Addition formulas for q-special functions*, FIELDS INSTITUTE COMMUN. (to appear). [1]

[1] Available from `ftp.fwi.uva.nl` in directory `pub/mathematics/reports/Analysis`

[17] H.T. Koelink: *q-Krawtchouk polynomials as spherical functions on the Hecke algebra of type B*, preprint 96-07, Universiteit van Amsterdam (1996). q-alg/9605040

[18] E. Koelink: *8 lectures on quantum groups and q-special functions*, REVISTA COLOMBIANA MAT. (to appear). q-alg/9608018

[19] H.T. Koelink and J. Van der Jeugt: *Convolutions for orthogonal polynomials from Lie and quantum algebra representations*, preprint 96-11, Universiteit van Amsterdam (1996). q-alg/9607010

[20] H.T. Koelink and J. Verding: *Spectral analysis and the Haar functional on the quantum $SU(2)$ group*, COMMUN. MATH. PHYS. **177** (1996), 399–415.

[21] T. Koornwinder: *Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying and addition formula*, J. LONDON MATH. SOC.(2) **18** (1978), 101–114.

[22] T.H. Koornwinder: *The addition formula for little q-Legendre polynomials and the $SU(2)$ quantum group*, SIAM J. MATH. ANAL. **22** (1991), 295–301.

[23] T.H. Koornwinder: *Discrete hypergroups associated with quantum groups*, CONTEMP. MATH. **183** (1995), 213–235.

[24] T.H. Koornwinder: *Special functions and q-commuting variables*, FIELDS INSTITUTE COMMUN. (to appear). q-alg/9608008

[25] M. Rahman and A. Verma: *Product and addition formulas for the continuous q-ultraspherical polynomials*, SIAM J. MATH. ANAL. **17** (1986), 1461–1474.

[26] D. Stanton: *Three addition theorems for some q-Krawtchouk polynomials*, GEOM. DEDICATA **10** (1981), 403–425.

[27] D. Stanton: *Orthogonal polynomials and Chevalley groups* in SPECIAL FUNCTIONS: GROUP THEORETICAL ASPECTS AND APPLICATIONS (R.A. Askey, T.H. Koornwinder and W. Schempp, eds.), (Reidel, Dordrecht, 1984), 87–128.

[28] N.J. Vilenkin and A.U. Klimyk: REPRESENTATION OF LIE GROUPS AND SPECIAL FUNCTIONS, 3 volumes (Kluwer, Dordrecht, 1991-3).

[29] G.N. Watson: THEORY OF BESSEL FUNCTIONS, 2nd ed. (Cambridge Univ. Press, 1944).

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