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FINDING DIFFERENTIAL EQUATIONS FOR SYMMETRIC GENERALIZED ULTRASPHERICAL POLYNOMIALS BY USING INVERSION METHODS

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Abstract

We find all differential equations of the form

$$M\sum_{i=0}^{\infty}a_i(x)y^{(i)}(x)+(1-x^2)y''(x)-2(\alpha+1)xy'(x)+n(n+2\alpha+1)y(x)=0,$$

where the coefficients $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n and $a_0(x):=a_0(n,\alpha)$ is independent of x, satisfied by the symmetric generalized ultraspherical polynomials $\left\{P_n^{\alpha,\alpha,M,M}(x)\right\}_{n=0}^{\infty}$ which are orthogonal on the interval $[-1,1]$ with respect to the weight function

$$\frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\left\{\Gamma(\alpha+1)\right\}^2}\left(1-x^2\right)^{\alpha}+M\left[\delta(x+1)+\delta(x-1)\right],$$

where $\alpha>-1$ and $M\geq 0$.

In order to find explicit formulas for the coefficients of these differential equations we have to solve systems of equations of the form

$$\sum_{i=1}^{\infty}A_i(x)D^iP_n^{(\alpha,\alpha)}(x)=F_n(x),\ n=1,2,3,\ldots,$$

where the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are independent of n. This system of equations has a unique solution given by

$$A_i(x)=2^i\sum_{j=1}^{i}\frac{2\alpha+2j+1}{(2\alpha+j+1)_{i+1}}P_{i-j}^{(-\alpha-i-1,-\alpha-i-1)}(x)F_j(x),\ i=1,2,3,\ldots.$$

This is a consequence of the inversion formula

$$\sum_{k=j}^{i}\frac{2\alpha+2k+1}{(2\alpha+k+j+1)_{i-j+1}}\times$$
$$\times P_{i-k}^{(-\alpha-i-1,-\alpha-i-1)}(x)P_{k-j}^{(\alpha+j,\alpha+j)}(x)=\delta_{ij},\ j\leq i,\ i,j=0,1,2,\ldots.$$

1 Introduction

Let $\alpha>-1$. In [4] we found all differential equations of the form

$$M\sum_{i=0}^{\infty}a_i(x)y^{(i)}(x)+(1-x^2)y''(x)-2(\alpha+1)xy'(x)+n(n+2\alpha+1)y(x)=0,\tag{1}$$

where the coefficients $\{a_i(x)\}_{i=0}^{\infty}$ are continuous functions on the real line and $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n, satisfied by the symmetric generalized ultraspherical polynomials $\left\{P_n^{\alpha,\alpha,M,M}(x)\right\}_{n=0}^{\infty}$ defined by

$$P_n^{\alpha,\alpha,M,M}(x) = C_0 P_n^{(\alpha,\alpha)}(x) - C_1 x D P_n^{(\alpha,\alpha)}(x),\ n = 0,1,2,\ldots,$$

where $D = \dfrac{d}{dx}$ denotes the differentiation operator and

$$\begin{cases} C_0 = 1 + \dfrac{2Mn}{\alpha+1}\dbinom{n+2\alpha+1}{n} + 4M^2\dbinom{n+2\alpha+1}{n-1}^2 \\[2ex] C_1 = \dfrac{2M}{2\alpha+1}\dbinom{n+2\alpha}{n} + \dfrac{2M^2}{\alpha+1}\dbinom{n+2\alpha}{n-1}\dbinom{n+2\alpha+1}{n}. \end{cases}$$

The case $2\alpha+1=0$ must be understood by continuity. These polynomials form a special case ($\beta = \alpha$ and $N = M$) of the generalized Jacobi polynomials $\left\{P_n^{\alpha,\beta,M,N}(x)\right\}_{n=0}^{\infty}$ introduced by T.H. Koornwinder in [6].

In [2] we gave a proof of the Jacobi inversion formula. The special case $\beta = \alpha$ of this inversion formula reads

$$\sum_{k=j}^{i} \frac{2\alpha+2k+1}{(2\alpha+k+j+1)_{i-j+1}} \times \\ \times P_{i-k}^{(-\alpha-i-1,-\alpha-i-1)}(x) P_{k-j}^{(\alpha+j,\alpha+j)}(x) = \delta_{ij},\ j \le i,\ i,j = 0,1,2,\ldots. \tag{2}$$

Again, the case $2\alpha+1=0$ must be understood by continuity. If we apply this inversion formula to the system of equations

$$\sum_{i=1}^{\infty} A_i(x) D^i P_n^{(\alpha,\alpha)}(x) = F_n(x),\ n = 1,2,3,\ldots, \tag{3}$$

where the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are independent of n, then we find

$$A_i(x) = 2^i \sum_{j=1}^{i} \frac{2\alpha+2j+1}{(2\alpha+j+1)_{i+1}} P_{i-j}^{(-\alpha-i-1,-\alpha-i-1)}(x) F_j(x),\ i = 1,2,3,\ldots. \tag{4}$$

This will be used to find all differential equations of the form (1), where the coefficients $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n and $a_0(x) := a_0(n,\alpha)$ is independent of x.

We will also need the formula

$$\sum_{k=j}^{i} \frac{2\alpha+2k+1}{(2\alpha+k+j+1)_{i-j+1}} \times \\ \times P_{i-k}^{(-\alpha-i-1,-\alpha-i-1)}(-x) P_{k-j}^{(\alpha+j,\alpha+j)}(x) = \frac{x^{i-j}}{(i-j)!},\ j \le i,\ i,j = 0,1,2,\ldots, \tag{5}$$

which is also proved in [2]. The case $2\alpha+1=0$ must be understood by continuity again.

In this paper we will give the main results. For more details the reader is referred to the report [2] where complete proofs are given.

2 The classical ultraspherical polynomials

In this section we list the definitions and some properties of the classical ultraspherical polynomials which we will use in this paper. For details the reader is referred to [1], [5], [8] and the report [2].

The classical ultraspherical polynomials $\left\{P_n^{(\alpha,\alpha)}(x)\right\}_{n=0}^{\infty}$ can be defined by

$$\begin{aligned} P_n^{(\alpha,\alpha)}(x) &= \sum_{k=0}^{n} \frac{(n+2\alpha+1)_k}{k!}\frac{(\alpha+k+1)_{n-k}}{(n-k)!}\left(\frac{x-1}{2}\right)^k,\ n=0,1,2,\ldots && (6)\\ &= (-1)^n\sum_{k=0}^{n} \frac{(-n-k-2\alpha)_k}{k!}\frac{(-n-\alpha)_{n-k}}{(n-k)!}\left(\frac{x-1}{2}\right)^k,\ n=0,1,2,\ldots && (7)\\ &= 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\alpha}{k}(x-1)^k(x+1)^{n-k},\ n=0,1,2,\ldots && (8)\end{aligned}$$

for all α. For all $n\in\{0,1,2,\ldots\}$ we have

$$D^iP_n^{(\alpha,\alpha)}(x)=\frac{(n+2\alpha+1)_i}{2^i}P_{n-i}^{(\alpha+i,\alpha+i)}(x),\ i=0,1,2,\ldots,n. \tag{9}$$

The ultraspherical polynomials satisfy the symmetry formula

$$P_n^{(\alpha,\alpha)}(-x)=(-1)^nP_n^{(\alpha,\alpha)}(x),\ n=0,1,2,\ldots \tag{10}$$

and the linear second order differential equation

$$(1-x^2)y''(x)-2(\alpha+1)xy'(x)+n(n+2\alpha+1)y(x)=0. \tag{11}$$

Further we list some formulas involving ultraspherical polynomials which we will need in this paper. For details the reader is referred to the report [2]. First of all we have

$$2xDP_n^{(\alpha,\alpha)}(x)=2nP_n^{(\alpha,\alpha)}(x)+(n+\alpha)P_{n-2}^{(\alpha+1,\alpha+1)}(x),\ n=2,3,4,\ldots. \tag{12}$$

Further we have (see for instance [8])

$$\begin{aligned}&(n+2\alpha+1)(n+2\alpha+2)P_n^{(\alpha+1,\alpha+1)}(x)-(n+\alpha)(n+\alpha+1)P_{n-2}^{(\alpha+1,\alpha+1)}(x)\\ &\quad=2(n+\alpha+1)(2n+2\alpha+1)P_n^{(\alpha,\alpha)}(x),\ n=2,3,4,\ldots.\end{aligned} \tag{13}$$

Finally we will need the formula

$$(\alpha+1)P_n^{(\alpha+1,\alpha+1)}(x)-(n+\alpha+1)P_n^{(\alpha,\alpha)}(x)=\frac{1}{4}(n+\alpha+1)(1-x^2)P_{n-2}^{(\alpha+2,\alpha+2)}(x), \tag{14}$$

which also holds for $n=2,3,4,\ldots$.

3 The computation of the coefficients

Let $\alpha>-1$. In [4] we found the coefficients $\{a_i(x)\}_{i=0}^{\infty}$ of the differential equation (1) for the symmetric generalized ultraspherical polynomials $\left\{P_n^{\alpha,\alpha,M,M}(x)\right\}_{n=0}^{\infty}$. In order to do this we had to solve the following two systems of equations for the coefficients $\{a_i(x)\}_{i=0}^{\infty}$:

$$\sum_{i=0}^{\infty}a_i(x)D^iP_n^{(\alpha,\alpha)}(x)=\frac{4}{2\alpha+1}\binom{n+2\alpha}{n}D^2P_n^{(\alpha,\alpha)}(x) \tag{15}$$

and

$$\sum_{i=0}^{\infty} ia_i(x)D^iP_n^{(\alpha,\alpha)}(x) + x\sum_{i=0}^{\infty} a_i(x)D^{i+1}P_n^{(\alpha,\alpha)}(x) = 4\binom{n+2\alpha+1}{n-1}D^2P_n^{(\alpha,\alpha)}(x) \tag{16}$$

for $n = 0, 1, 2, \ldots$, where the coefficients $\{a_i(x)\}_{i=0}^{\infty}$ are continuous functions on the real line and $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n. Now we suppose that $a_0(x) := a_0(n, \alpha)$ is independent of x as we did in [3]. Then it is clear (see for instance lemma 1 in [3]) that $a_i(x)$ must be a polynomial in x of degree at most i for each $i = 1, 2, 3, \ldots$. In [4] we showed that the solution for $\{a_i(x)\}_{i=0}^{\infty}$ is not unique. In fact it was shown that

$$a_0(x) := a_0(n, \alpha) = a_0(1, \alpha)b_0(n, \alpha) + c_0(n, \alpha),\ n = 0, 1, 2, \ldots \tag{17}$$

and that

$$a_i(x) = a_0(1, \alpha)b_i(x) + c_i(x),\ i = 1, 2, 3, \ldots, \tag{18}$$

where $a_0(1, \alpha)$ is arbitrary and

$$b_0(n, \alpha) = \frac{1}{2}\left[1 - (-1)^n\right],\ n = 0, 1, 2, \ldots, \tag{19}$$

$$c_0(n, \alpha) = 4(2\alpha + 3)\binom{n+2\alpha+2}{n-2},\ n = 0, 1, 2, \ldots, \tag{20}$$

$$b_i(x) = \frac{2^{i-1}}{i!}(-x)^i,\ i = 1, 2, 3, \ldots, \tag{21}$$

$$c_1(x) = 0 \text{ and } c_i(x) = (2\alpha + 3)(1 - x^2)\frac{2^i}{i!}P_{i-2}^{(\alpha-i+3,\alpha-i+3)}(x),\ i = 2, 3, 4, \ldots. \tag{22}$$

In this paper we will give an alternative proof of this by using the inversion formula (2).

By considering (15) and (16) for $n = 0$ and $n = 1$ we conclude that $a_0(0, \alpha) = 0$, $a_0(1, \alpha)$ is arbitrary and $a_1(x) = -a_0(1, \alpha)x$. For $n = 2, 3, 4, \ldots$ it turns out to be more convenient to use another system of equations instead of (16). By using (12) we find for $i = 0, 1, 2, \ldots$

$$\begin{aligned} & iD^iP_n^{(\alpha,\alpha)}(x) + xD^{i+1}P_n^{(\alpha,\alpha)}(x) = D^i\left[xDP_n^{(\alpha,\alpha)}(x)\right] \\ = \ & nD^iP_n^{(\alpha,\alpha)}(x) + \frac{1}{2}(n + \alpha)D^iP_{n-2}^{(\alpha+1,\alpha+1)}(x),\ n = 2, 3, 4, \ldots. \end{aligned}$$

Combining (15) and (16) we now obtain

$$\sum_{i=0}^{\infty} a_i(x)D^iP_{n-2}^{(\alpha+1,\alpha+1)}(x) = \frac{8}{n+\alpha}\binom{n+2\alpha}{n-2}D^2P_n^{(\alpha,\alpha)}(x),\ n = 2, 3, 4, \ldots. \tag{23}$$

So we conclude that (16) for $n = 2, 3, 4, \ldots$ may be replaced by (23). Note that for $n = 2$ this implies that $a_0(2, \alpha) = 4(2\alpha + 3)$.

Since $a_i(x)$ must be a polynomial in x of degree at most i for each $i = 1, 2, 3, \ldots$ we may write

$$a_i(x) = k_ix^i + \text{ lower order terms },\ i = 1, 2, 3, \ldots.$$

By comparing the coefficients of highest degree in (15) and (23) we find by using (6) :

$$\frac{a_0(n, \alpha)}{n!} + \sum_{i=1}^{n} \frac{k_i}{(n-i)!} = 0,\ n = 1, 2, 3, \ldots$$

and

$$\frac{a_0(n, \alpha)}{(n-2)!} + \sum_{i=1}^{n-2} \frac{k_i}{(n-i-2)!} = 4(2n + 2\alpha - 1)\binom{n+2\alpha}{n-2}\frac{1}{(n-2)!},\ n = 3, 4, 5, \ldots.$$

Since k_i is independent of n for $i = 1, 2, 3, \dots$ and $a_0(2, \alpha) = 4(2\alpha + 3)$ we conclude that

$$a_0(n+2,\alpha) - a_0(n,\alpha) = 4(2n+2\alpha+3)\binom{n+2\alpha+2}{n}, \; n = 0, 1, 2, \dots, \tag{24}$$

where $a_0(0, \alpha) = 0$ and $a_0(1, \alpha)$ is arbitrary. Hence

$$a_0(2n,\alpha) - a_0(0,\alpha) = 4\sum_{k=0}^{n-1}\binom{2k+2\alpha+2}{2k}(4k+2\alpha+3), \; n = 1, 2, 3, \dots$$

and

$$a_0(2n+1,\alpha) - a_0(1,\alpha) = 4\sum_{k=0}^{n-1}\binom{2k+2\alpha+3}{2k+1}(4k+2\alpha+5), \; n = 1, 2, 3, \dots.$$

Note that we have

$$(2n+2\alpha+3)\binom{n+2\alpha+2}{n} = (2\alpha+3)\left[\binom{n+2\alpha+4}{n} - \binom{n+2\alpha+2}{n-2}\right], \; n = 0, 1, 2, \dots.$$

Hence, by using the telescoping property of the sums we find that

$$\sum_{k=0}^{n-1}\binom{2k+2\alpha+2}{2k}(4k+2\alpha+3) = (2\alpha+3)\binom{2n+2\alpha+2}{2n-2}, \; n = 1, 2, 3, \dots$$

and

$$\sum_{k=0}^{n-1}\binom{2k+2\alpha+3}{2k+1}(4k+2\alpha+5) = (2\alpha+3)\binom{2n+2\alpha+3}{2n-1}, \; n = 1, 2, 3, \dots.$$

So we conclude that (17), (19) and (20) hold.

The systems of equations (15) and (23) lead to

$$\sum_{i=1}^{\infty} a_i(x) D^i P_n^{(\alpha,\alpha)}(x) = \frac{4}{2\alpha+1}\binom{n+2\alpha}{n} D^2 P_n^{(\alpha,\alpha)}(x) - a_0(n,\alpha) P_n^{(\alpha,\alpha)}(x) \tag{25}$$

for $n = 0, 1, 2, \dots$ and

$$\sum_{i=1}^{\infty} a_i(x) D^i P_{n-2}^{(\alpha+1,\alpha+1)}(x) = \frac{8}{n+\alpha}\binom{n+2\alpha}{n-2} D^2 P_n^{(\alpha,\alpha)}(x) - a_0(n,\alpha) P_{n-2}^{(\alpha+1,\alpha+1)}(x) \tag{26}$$

for $n = 2, 3, 4, \dots$.

First we remark that (25) is true for $n = 0$ and $n = 1$ since $a_0(0, \alpha) = 0$ and $a_1(x) = -a_0(1, \alpha)x$. Then we will show that every solution of (26) also satisfies (25). Suppose that $\{a_i(x)\}_{i=1}^{\infty}$ is a solution of (26). Now we use (9), (13), (24) and the fact that $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n to obtain for $n = 2, 3, 4, \dots$ (see [2] for more details)

$$\begin{aligned}
& 2(n+\alpha+1)(2n+2\alpha+1)\sum_{i=1}^{\infty} a_i(x) D^i P_n^{(\alpha,\alpha)}(x) \\
= \; & (n+2\alpha+1)(n+2\alpha+2)\sum_{i=1}^{\infty} a_i(x) D^i P_n^{(\alpha+1,\alpha+1)}(x) + \\
& \quad - (n+\alpha)(n+\alpha+1)\sum_{i=1}^{\infty} a_i(x) D^i P_{n-2}^{(\alpha+1,\alpha+1)}(x) \\
= \; & (n+2\alpha+1)(n+2\alpha+2) \times
\end{aligned}$$

$$
\begin{aligned}
& \times \left[\frac{8}{n+\alpha+2}\binom{n+2\alpha+2}{n} D^2 P_{n+2}^{(\alpha,\alpha)}(x) - a_0(n+2,\alpha) P_n^{(\alpha+1,\alpha+1)}(x)\right] + \\
& - (n+\alpha)(n+\alpha+1) \times \\
& \qquad \times \left[\frac{8}{n+\alpha}\binom{n+2\alpha}{n-2} D^2 P_n^{(\alpha,\alpha)}(x) - a_0(n,\alpha) P_{n-2}^{(\alpha+1,\alpha+1)}(x)\right] \\
= \; & 2(n+\alpha+1)(n+2\alpha+1)(n+2\alpha+2)\binom{n+2\alpha+2}{n} P_{n-2}^{(\alpha+2,\alpha+2)}(x) + \\
& - 8(n+\alpha+1)\binom{n+2\alpha}{n-2} D^2 P_n^{(\alpha,\alpha)}(x) + \\
& - 2(n+\alpha+1)(2n+2\alpha+1) a_0(n,\alpha) P_n^{(\alpha,\alpha)}(x) \\
= \; & 2(n+\alpha+1)(2n+2\alpha+1)\left[\frac{4}{2\alpha+1}\binom{n+2\alpha}{n} D^2 P_n^{(\alpha,\alpha)}(x) - a_0(n,\alpha) P_n^{(\alpha,\alpha)}(x)\right].
\end{aligned}
$$

Since $\alpha > -1$ this proves that every solution of (26) also satisfies (25).

Now we will solve (26). Shifting n by two we may write, since the coefficients $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n

$$\sum_{i=1}^{\infty} a_i(x) D^i P_n^{(\alpha+1,\alpha+1)}(x) = F_n(x),\ n = 0,1,2,\ldots, \tag{27}$$

where

$$F_n(x) = \frac{8}{n+\alpha+2}\binom{n+2\alpha+2}{n} D^2 P_{n+2}^{(\alpha,\alpha)}(x) - a_0(n+2,\alpha) P_n^{(\alpha+1,\alpha+1)}(x).$$

Since $a_0(2,\alpha) = 4(2\alpha+3)$ we easily find that $F_0(x) = 0$. This implies that the system of equations (27) is of the form (3). So if we apply the inversion formula (2) to the system of equations (27) we obtain by using (4)

$$a_i(x) = 2^i \sum_{j=1}^{i} \frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}} P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x) F_j(x),\ i = 1,2,3,\ldots.$$

Hence, by using (17) we conclude that the coefficients $\{a_i(x)\}_{i=1}^{\infty}$ can be written in the form (18). Moreover, we find by using (19), (10) and (5)

$$
\begin{aligned}
b_i(x) \; = \; & 2^{i-1} \sum_{j=1}^{i} \frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}} P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x) P_j^{(\alpha+1,\alpha+1)}(x) \left[(-1)^j - 1\right] \\
= \; & 2^{i-1} \sum_{j=0}^{i} \frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}} P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x) P_j^{(\alpha+1,\alpha+1)}(-x) + \\
& - 2^{i-1} \sum_{j=0}^{i} \frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}} P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x) P_j^{(\alpha+1,\alpha+1)}(x) \\
= \; & 2^{i-1} \frac{(-x)^i}{i!},\ i = 1,2,3,\ldots,
\end{aligned}
$$

which proves (21). And by using (20) and (9) we obtain

$$c_i(x) = 2^i \sum_{j=1}^{i} \frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}} P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x) G_j(x),\ i = 1,2,3,\ldots,$$

where

$$G_j(x) = \frac{4(2\alpha+3)}{j+\alpha+2}\binom{j+2\alpha+4}{j}\left[(\alpha+2)P_j^{(\alpha+2,\alpha+2)}(x) - (j+\alpha+2)P_j^{(\alpha+1,\alpha+1)}(x)\right].$$

It is clear that $G_1(x) = 0$, which implies that $c_1(x) = 0$. Note that since $b_1(x) = -x$ this also implies that $a_1(x) = -a_0(1,\alpha)x$, which agrees with what we have found before. Now we use (14) to find

$$G_j(x) = (2\alpha+3)(1-x^2)\binom{j+2\alpha+4}{j}P_{j-2}^{(\alpha+3,\alpha+3)}(x),\ j = 2,3,4,\ldots.$$

Hence, for $i = 2,3,4\ldots$ we have

$$c_i(x) = (2\alpha+3)(1-x^2)2^i \times$$
$$\times \sum_{j=2}^{i}\frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}}\binom{j+2\alpha+4}{j}P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x)P_{j-2}^{(\alpha+3,\alpha+3)}(x).$$

Now it remains to show that

$$\sum_{j=2}^{i}\frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}}\binom{j+2\alpha+4}{j}P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x)P_{j-2}^{(\alpha+3,\alpha+3)}(x)$$
$$= \frac{1}{i!}P_{i-2}^{(\alpha-i+3,\alpha-i+3)}(x),\ i = 2,3,4,\ldots. \tag{28}$$

In order to do this we write for $i = 2,3,4,\ldots$

$$\sum_{j=2}^{i}\frac{2\alpha+2j+3}{(2\alpha+j+3)_{i+1}}\binom{j+2\alpha+4}{j}P_{i-j}^{(-\alpha-i-2,-\alpha-i-2)}(x)P_{j-2}^{(\alpha+3,\alpha+3)}(x)$$
$$= \sum_{k=0}^{i-2}\frac{2\alpha+2k+7}{(2\alpha+k+5)_{i+1}}\frac{(2\alpha+5)_{k+2}}{(k+2)!}P_{i-k-2}^{(-\alpha-i-2,-\alpha-i-2)}(x)P_k^{(\alpha+3,\alpha+3)}(x).$$

Now we apply definition (6) to $P_k^{(\alpha+3,\alpha+3)}(x)$ and definition (7) to $P_{i-k-2}^{(-\alpha-i-2,-\alpha-i-2)}(x)$ and change the order of summation to obtain for $i = 2,3,4,\ldots$ (see [2])

$$\sum_{k=0}^{i-2}\frac{2\alpha+2k+7}{(2\alpha+k+5)_{i+1}}\frac{(2\alpha+5)_{k+2}}{(k+2)!}P_{i-k-2}^{(-\alpha-i-2,-\alpha-i-2)}(x)P_k^{(\alpha+3,\alpha+3)}(x)$$
$$= \sum_{k=0}^{i-2}\sum_{m=0}^{i-k-2}\sum_{n=0}^{k}(-1)^{i-k-2}\times$$
$$\times \frac{(2\alpha+2k+7)(2\alpha+5)_{k+n+2}(\alpha+n+4)_{i-m-n-2}}{(2\alpha+k+5)_{i-m+1}(k+2)!\,m!\,(i-k-m-2)!\,n!\,(k-n)!}\left(\frac{x-1}{2}\right)^{m+n}$$
$$= \sum_{j=0}^{i-2}\sum_{n=0}^{j}(-1)^{i-n}\frac{(2\alpha+5)_n(\alpha+n+4)_{i-j-2}}{(2\alpha+2n+7)_{i-j-1}(n+2)!\,(j-n)!\,(i-j-2)!\,n!}\left(\frac{x-1}{2}\right)^j\times$$
$$\times\sum_{k=0}^{i-j-2}\frac{(-i+j+2)_k(2\alpha+n+5)_k(2\alpha+2n+7)_k}{(2\alpha+2n+i-j+6)_k(n+3)_k k!}(2\alpha+2n+2k+7).$$

In [2] we proved that for $i-j-2 \in \{0,1,2,\ldots\}$ we have

$$\sum_{k=0}^{i-j-2}\frac{(-i+j+2)_k(2\alpha+n+5)_k(2\alpha+2n+7)_k}{(2\alpha+2n+i-j+6)_k(n+3)_k k!}(2\alpha+2n+2k+7)$$
$$= \frac{(2\alpha+2n+7)_{i-j-1}(-\alpha-1)_{i-j-2}}{(n+3)_{i-j-2}(\alpha+n+4)_{i-j-2}}.$$

See [2] for much more details. By using this, the well-known Vandermonde summation formula and definition (7) we finally obtain for $i = 2, 3, 4, \ldots$

$$\begin{aligned}
&\sum_{k=0}^{i-2} \frac{2\alpha+2k+7}{(2\alpha+k+5)_{i+1}} \frac{(2\alpha+5)_{k+2}}{(k+2)!} P_{i-k-2}^{(-\alpha-i-2,-\alpha-i-2)}(x) P_k^{(\alpha+3,\alpha+3)}(x)\\
&= \sum_{j=0}^{i-2}\sum_{n=0}^{j} (-1)^{i-n} \frac{(2\alpha+5)_n(-\alpha-1)_{i-j-2}}{(n+i-j)!\,(j-n)!\,(i-j-2)!\,n!} \left(\frac{x-1}{2}\right)^j\\
&= (-1)^i \sum_{j=0}^{i-2} \frac{(-\alpha-1)_{i-j-2}}{(i-j)!\,(i-j-2)!\,j!} \left(\frac{x-1}{2}\right)^j {}_2F_1\left(\left.\begin{matrix}-j, 2\alpha+5\\ i-j+1\end{matrix}\right| 1\right)\\
&= \frac{(-1)^i}{i!} \sum_{j=0}^{i-2} \frac{(-\alpha-1)_{i-j-2}(i-j-2\alpha-4)_j}{(i-j-2)!\,j!} \left(\frac{x-1}{2}\right)^j\\
&= \frac{(-1)^{i-2}}{i!} \sum_{j=0}^{i-2} \frac{(i-j-2\alpha-4)_j}{j!} \frac{(-\alpha-1)_{i-j-2}}{(i-j-2)!} \left(\frac{x-1}{2}\right)^j = \frac{1}{i!} P_{i-2}^{(\alpha-i+3,\alpha-i+3)}(x),
\end{aligned}$$

which proves (28).

4 Some remarks

By using definition (8) we may write

$$P_{i-2}^{(\alpha-i+3,\alpha-i+3)}(x) = 2^{2-i} \sum_{k=0}^{i-2} \binom{\alpha+1}{i-2-k}\binom{\alpha+1}{k} (x-1)^k (x+1)^{i-2-k},\ i = 2, 3, 4, \ldots.$$

By using (22) this implies that for nonnegative integer values of α we have $c_i(x) = 0$ for all $i > 2\alpha + 4$ and

$$c_{2\alpha+4}(x) = (2\alpha+3)(1-x^2)\frac{2^{2\alpha+4}}{(2\alpha+4)!} P_{2\alpha+2}^{(-\alpha-1,-\alpha-1)}(x) = -\frac{4(2\alpha+3)}{(2\alpha+4)!}(x^2-1)^{\alpha+2} \neq 0.$$

Hence, for nonnegative integer values of α the polynomials $\left\{P_n^{\alpha,\alpha,M,M}(x)\right\}_{n=0}^{\infty}$ satisfy a differential equation of the form

$$M \sum_{i=0}^{2\alpha+4} c_i(x) y^{(i)}(x) + (1-x^2)y''(x) - 2(\alpha+1)xy'(x) + n(n+2\alpha+1)y(x) = 0,$$

which is of finite order $2\alpha + 4$ if $M > 0$. Moreover, in [4] it was shown that for $M > 0$ the differential equation given by (1) has finite order if and only if we choose $a_0(1, \alpha) = 0$ and if α is a nonnegative integer.

Finally, we remark that if we apply the inversion formula (2) to the system of equations (25) instead of (26) we find for $i = 1, 2, 3, \ldots$ that

$$b_i(x) = 2^{i-1} \sum_{j=1}^{i} \frac{2\alpha+2j+1}{(2\alpha+j+1)_{i+1}} P_{i-j}^{(-\alpha-i-1,-\alpha-i-1)}(x) P_j^{(\alpha,\alpha)}(x) \left[(-1)^j - 1\right] \tag{29}$$

and

$$\begin{aligned}
c_i(x) = 2^{i+2} \sum_{j=1}^{i} &\frac{2\alpha+2j+1}{(2\alpha+j+1)_{i+1}} P_{i-j}^{(-\alpha-i-1,-\alpha-i-1)}(x) \times\\
&\times \left[\frac{1}{2\alpha+1}\binom{j+2\alpha}{j} D^2 P_j^{(\alpha,\alpha)}(x) - (2\alpha+3)\binom{j+2\alpha+2}{j-2} P_j^{(\alpha,\alpha)}(x)\right].
\end{aligned} \tag{30}$$

¿From (29) we easily obtain (21) in the same way as before by using (10), (2) and (5). Further we easily find from (30) that $c_1(x) = 0$, but we were not able to derive (22) for $i = 2, 3, 4, \ldots$ from (30).

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