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ORTHOGONAL POLYNOMIALS AND QUASY-EXACTLY SOLVABLE POTENTIALS ON THE LINE

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Abstract

In this paper, we survey recent results relating quasi-exactly solvable systems with the theory of orthogonal polynomials. The necessary and sufficient conditions for a quasi-exactly solvable potential on the line to generate a family of weakly orthogonal polynomials are presented. We also introduce a new characterization theorem for weakly orthogonal polynomials.

1 Introduction

In a recent paper, [?], Bender and Dunne found a new connection between *quasi-exactly solvable* (QES) models in quantum mechanics and the theory of orthogonal polynomials. They considered the following well known class of QES Hamiltonians:

$$H = -\partial_x^2 + \frac{(4s-1)(4s-3)}{4x^2} - (4s+4n+2)x^2 + x^6\,, \tag{1}$$

with n a positive integer and $s \in \mathbb{R}$. Inserting the asymptotic expansion

$$\psi_E(x) = e^{-\frac{x^4}{4}} x^{2s-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{P_k(E)}{\Gamma(k+2s)} \left(\frac{x^2}{4}\right)^k$$

in the time-independent Schrödinger equation, the coefficients $P_k(E)$ are easily shown to satisfy the three-term recursion relation

$$P_k(E) = E\,P_{k-1}(E) + 16(k-1)(k-n)(k+2s-2)\,P_{k-2}(E)\,, \tag{2}$$

for $k \geq 1$. Imposing the initial condition $P_0(E) = 1$, each coefficient $P_k(E)$ becomes a monic polynomial of the k-th degree. The form of the recursion relation (**??**) has some remarkable implications. In the first place, the polynomials $P_k(E)$ are orthogonal with respect to some Stieltjes measure $d\omega(E)$, [**?**]:

$$\mathcal{L}(P_k\,P_l) = \int_{\mathbb{R}} P_k(E) P_l(E) d\omega(E) = 0 \qquad \text{if } k \neq l\,.$$

Secondly, the norms of the polynomials vanish for $k \geq n+1$, and therefore $\{P_k\}_{k=0}^{\infty}$ defines what is known as a *weakly orthogonal* polynomial system, [**?**]. Another interesting consequence of (**??**) is the factorization property:

$$P_{k+n+1}(E) = Q_k(E)\,P_{n+1}(E)\,, \qquad \text{for} \quad k \geq 0\,,$$

where Q_k is a (monic) k-th degree polynomial. Finally, the roots E_l of the *critical polynomial* P_{n+1} belong to the point spectrum of the Hamiltonian (**??**), the corresponding eigenfunctions being ψ_{E_l}.

Unlike some previously known results relating classical orthogonal polynomials and QES models, [**?**], [**?**], in Bender and Dunne's approach the polynomials $P_k(E)$ are the coefficients of a certain expansion of a solution to a QES equation, but are *not* solutions themselves.

It was shown in [?] that a family of weakly orthogonal polynomials satisfying properties analogous to those obtained by Bender and Dunne can indeed be constructed for *any* QES Hamiltonian on the line, provided some very mild conditions are satisfied. Some additional properties of these weakly-orthogonal polynomials were also studied in [?]. In particular, it was rigorously proved that their associated Stieltjes measure $d\omega(E)$ is supported on a finite set, as claimed in [?], and is unique. Finally, the moments of $d\omega(E)$ were shown to satisfy a recursion relation with constant coefficients. In this paper, we survey these results and also introduce a theorem characterizing weakly orthogonal polynomials independently of their connection to QES spectral problems.

The paper is organized as follows. In Section **??** we make a concise introduction to QES potentials on the line. In Section **??**, we construct the family of weakly orthogonal polynomials associated to each of the canonical forms of a QES Hamiltonian discussed in the previous section. Finally, in Section **??** we review some properties of weakly orthogonal polynomials including the above mentioned theorem.

2 QES Potentials on the Line

In this section we introduce the elements of the theory of quasi-exactly solvable potentials on the line that we need in the sequel, referring the reader to [?] and [?] for an in-depth review of the field. We start with the following

Definition 1 *A finite-dimensional Lie algebra $\mathfrak{g}$ of first order differential operators on the line*

$$\mathfrak{g} = \mathrm{Span}\{\, T_a\,; \quad 1 \le a \le r\,\}\,, \qquad T_a = \xi_a(x)\partial_x + \eta_a(x)\,,$$

is called quasi-exactly solvable (QES) if it preserves a finite-dimensional subspace $\mathcal{N}$ of $C^\infty(\mathbb{R})$ functions. A Hamiltonian (or Schrödinger operator)

$$H = -\partial_x^2 + V(x) \tag{3}$$

is QES if it lies in the universal enveloping algebra of a QES Lie algebra $\mathfrak{g}$.

In other words, a QES Hamiltonian is a polynomial (of the second degree, as it turns out) in the generators of a QES Lie algebra. It follows from this definition that the subspace $\mathcal{N}$ is also preserved by H, and therefore a number of its eigenvalues and eigenfunctions may be computed by diagonalizing the finite-dimensional matrix which represents it in $\mathrm{End}(\mathcal{N})$, the set of endomorphisms of $\mathcal{N}$. We shall use the term *algebraic* to refer to these eigenfunctions and eigenvalues. If we impose appropriate boundary conditions on the functions spanning $\mathcal{N}$, —like square integrability if we are looking for bound states of H, periodicity, etc.— which amounts to requiring H to be a self-adjoint operator, then the matrix representing H in $\mathrm{End}(\mathcal{N})$ will be Hermitian. In this case, there are exactly $\dim \mathcal{N}$ algebraic eigenfunctions of H whose corresponding *real* eigenvalues may be computed algebraically, and H is said to be *fully algebraic.*

The classification of QES Hamiltonians involves (roughly speaking) two steps. In the first place, one needs to classify QES Lie algebras under a suitable notion of equivalence. In the second step, these Lie algebras are used to construct QES Schrödinger operators, and appropriate boundary conditions are imposed. Both of these problems have been solved in the case of QES Hamiltonians acting on the line. It is worth mentioning that only the first step has been carried out for QES Hamiltonians in two variables, [?], and not even this for QES Hamiltonians in more than two variables. The classification of QES Lie algebras on the line is provided by the following well known result, [?], [?]:

Theorem 2 *Every QES Lie algebra $\mathfrak{g}$ is equivalent under a local diffeomorphism and a gauge transformation by a non-vanishing function to (a subalgebra of) one of the Lie algebras $\mathfrak{g}_n = \mathfrak{h}_n \oplus \mathbb{R}$, where $\mathfrak{h}_n = \mathrm{Span}\{J_n^+, J_n^0, J^-\}$, and*

$$J_n^+ = z^2\partial_z - nz\,, \qquad J_n^0 = z\partial_z - \frac{n}{2}\,, \qquad J^- = \partial_z\,, \tag{4}$$

with $n \in \mathbb{Z}^+$. The invariant $\mathfrak{g}_n$-module is $\mathcal{P}_n = \mathrm{Span}\{1, z, \ldots, z^n\}$.

It follows from this theorem that every QES Hamiltonian H on the line may always be written in "gauge" form as a second degree polynomial in the generators of $\mathfrak{h}_n$:

$$\begin{aligned} -H_{\mathrm{gauge}}(z) &= -\frac{1}{\mu(z)} \cdot H(x) \cdot \mu(z)\Big|_{x=\zeta^{-1}(z)} \\ &= \sum_{a,b} c_{ab}\, J_n^a\, J_n^b + \sum_a c_a\, J_n^a + c_*\,, \end{aligned}$$

where the indices a and b run through $+, 0, -$. Using the explicit expressions (**??**), we obtain:

$$-H_{\mathrm{gauge}}(z) = P\,\partial_z^2 + \left\{Q - \frac{n-1}{2}P'\right\}\partial_z + \left\{R - \frac{n}{2}Q' + \frac{n(n-1)}{12}P''\right\}, \tag{5}$$

where P, Q, and R are polynomials in z of degrees 4, 2, and 0, respectively, given by

$$\begin{aligned} P &= c_{++}\, z^4 + 2c_{+0}\, z^3 + c_{00}\, z^2 + 2c_{0-}\, z + c_{--}\,, \\ Q &= c_+\, z^2 + c_0\, z + c_-\,, \\ R &= \frac{n(n+2)}{12}c_{00} + c_*\,. \end{aligned} \tag{6}$$

Let us remark that gauge transformations and changes of variables respect the Lie bracket between differential operators. They also preserve the Schrödinger eigenvalue equation. In fact, if $\chi(z)$ is an eigenfunction of H_{gauge} with eigenvalue E, $\psi(x) = \mu(z)\chi(z)|_{z=\zeta(x)}$ is an eigenfunction of H with the same eigenvalue. Note that we do not require $\mu(z)$ to be unimodular, so square-integrability is *not* preserved. Regarding the second step in the construction of QES Schrödinger operators, there are explicit formulas for the change of variables and the gauge factor needed to transform the differential operator (**??**) into Schrödinger form (**??**), [**?**]:

$$x = \zeta^{-1}(z) = \int^z \frac{dy}{\sqrt{P(y)}}, \qquad \mu(z) = P(z)^{-n/4} \exp\left\{\int^z \frac{Q(y)}{2P(y)}dy\right\}\,,$$

where z belongs to an interval I in which $P(z) > 0$. The potential is given by:

$$V(x) = -R - \frac{1}{12P}\left[n(n+2)(PP'' - \frac{3}{4}P'^2) + 3(n+1)(QP' - 2PQ') - 3Q^2\right]_{z=\zeta(x)}\,,$$

where the prime denotes derivative with respect to z. Therefore, any second degree polynomial in the generators (**??**) may be written in Schrödinger form up to sign (this property is no longer true in more than one variable, [**?**]). If $V(x)$ turns out to be independent of the "spin" parameter n, we can compute an arbitrary large number of its eigenvalues and eigenfunctions algebraically (leaving boundary conditions aside). In this case we shall say that V is *exactly-solvable*.

The "gauge" form (**??**) of the QES Hamiltonian H is *not* unique (the physical form (**??**) is unique up to a traslation in x). There is a residual group of transformations which preserve the

Lie algebras $\mathfrak{h}_n$. These transformations are the combination of a change of variables given by the projective action of GL$(2,\mathbb{R})$ on $\mathbb{R}$P:

$$z = \zeta(w) = \frac{\alpha w + \beta}{\gamma w + \delta}, \qquad \Delta = \alpha\delta - \beta\gamma \neq 0,$$

and a gauge transformation with gauge factor

$$\mu(z) = (\gamma w + \delta)^n .$$

The space $\mathcal{P}_n$ is preserved under these transformations, whose action

$$p(z) \in \mathcal{P}_n \mapsto \hat{p}(w) = (\gamma w + \delta)^n p\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right),$$

defines the (irreducible) multiplier representation ρ_n of GL$(2,\mathbb{R})$ on $\mathcal{P}_n$. We also introduce the tensor product representations $\rho_{n,i} = \rho_n \otimes \det^i$, where det is the usual determinantal representation of GL$(2,\mathbb{R})$. The generators (**??**) then transform according to the representation $\rho_{2,-1}$, and the polynomials P, Q, and R defining H_{gauge} do so under $\rho_{4,-2}$, $\rho_{2,-1}$, and ρ_0, respectively, [**?**]. Note that the eigenfunctions of H_{gauge} transform (by definition) under ρ_n. We may use the GL$(2,\mathbb{R})$ action to reduce the polynomial P to one of the following canonical forms, [**?**]:

$$\begin{array}{llll}
1) & \nu(z^2+1), & 6) & \nu(1-z^2)(1-\kappa^2 z^2), \\
2) & \nu(z^2-1), & 7) & \nu(1-z^2)(1-\kappa^2(1-z^2)), \\
3) & \nu z^2, & 8) & \nu(1+z^2)(1+(1-\kappa^2)z^2), \\
4) & z, & 9) & \nu(1+z^2)^2, \\
5) & 1, & 10) & \nu(1-z^2),
\end{array} \tag{7}$$

where $\nu > 0$ and $0 < \kappa < 1$. The first five canonical forms lead to *normalizable* QES Hamiltonians (that is, the algebraic eigenfunctions are square-integrable after imposing suitable constraints in the remaining parameters), whereas the remaining five lead to periodic potentials. It is readily seen that the Hamiltonian (**??**) corresponds to the fourth canonical form. Finally, let us mention that this invariance of H_{gauge} was used in [**?**] to classify all normalizable and fully algebraic QES Hamiltonians on the line.

3 The Recursion Relation

In this section, we obtain the generating function and the recursion relation satisfied by the system of weakly orthogonal polynomials associated to each class of QES Hamiltonians determined by the canonical forms (**??**). Let H be a QES Hamiltonian, and let H_{gauge} be one of its gauge forms. We consider the "gauged" Schrödinger equation:

$$(H_{\text{gauge}} - E)\,\chi_E = 0, \tag{8}$$

and we try a formal (no boundary conditions) solution of the form

$$\chi_E(z) = \sum_{k=0}^{\infty} P_k(E)\frac{z^k}{k!}.$$

A straightforward computation leads to the following five-term recursion relation for the coefficients $P_k(E)$:

$$\begin{aligned} -c_{--}P_{k+2} &= \left[(2k-n+1)c_{0-}+c_{-}\right]P_{k+1} \\ &+ \left[E+c_*+c_0\Big(k-\frac{n}{2}\Big)+c_{00}\Big(k-\frac{n}{2}\Big)^2\right]P_k \\ &+ k(k-1-n)\left[(2k-n-1)c_{+0}+c_{+}\right]P_{k-1} \\ &+ k(k-1)(k-1-n)(k-2-n)\,c_{++}P_{k-2}, \qquad k\geq 0. \end{aligned} \tag{9}$$

If c_{--} does not vanish, then $z=0$ is a regular point of the equation (**??**), and $\chi_E(z)$ depends on two arbitrary functions, $P_0(E)$ and $P_1(E)$. If we choose them as polynomials in E, each $P_k(E)$ will also be a polynomial, but not a k-th degree polynomial unless $c_{--}=0$. Moreover, we need $c_{++}=0$ if we want (**??**) to be three-term (so that the polynomials $P_k(E)$ can be orthogonal with respect to some nonzero Stieltjes measure). Note that $c_{--}=c_{++}=0$ simply means that $P(z)$ in (**??**) vanishes at $z=0\,,\infty$ when z takes values on the complex projective line. The idea is trying to use (complex) projective transformations analogous to those discussed in the previous section to map $P(z)$ into another polynomial $\hat{P}(w)$ with roots at $0,\infty$. We assume that $P(z)$ is one of the canonical forms (**??**). We have the following possibilities:
Case 1: P has two different roots, $z_1\neq z_2=\infty$ (canonical forms 1–4,10).
The translation $w=z-z_1$ maps $P(z)$ into $\hat{P}(w)$ vanishing at $0,\infty$. The generating function in the canonical coordinate z is:

$$\chi_E(z)=\sum_{k=0}^{\infty}P_k(E)\frac{(z-z_1)^k}{k!}\,, \tag{10}$$

and the polynomials $P_k(E)$ are determined from eq. (**??**). Note that we are expanding around $z=z_1$, which is a singular point of the differential equation. We could also have written the generating function in the "physical" coordinate x as:

$$\psi_E(x)=\mu(\zeta(x))\chi_E(\zeta(x)).$$

Case 2: P has two different *finite* roots, $z_1\neq z_2$ (canonical forms 6–9).
The appropriate transformation is now $w=(z-z_1)/(z-z_2)$, and the generating function is given by:

$$\chi_E(z)=(z-z_2)^n\sum_{k=0}^{\infty}\frac{1}{k!}P_k(E)\left(\frac{z-z_1}{z-z_2}\right)^k\,. \tag{11}$$

Case 3: P has a quadruple root. This corresponds to the fifth canonical form ($P=1$). In this case, the recursion relation (**??**) reduces to a pair of two-term recursion relations for the even and odd coefficients $P_j^0=P_{2j}$ and $P_j^1=P_{2j+1}$,

$$-P_{j+1}^{\epsilon}=\left[E+c_*+c_0\,(2j+\epsilon-\frac{n}{2})\right]P_j^{\epsilon},\qquad j\geq 0;\quad \epsilon=0,1\,,$$

after we impose normalizability of the algebraic wavefunctions. The (normalizable) potentials associated to this canonical form, $V(x)=\frac{1}{4}c_0^2x^2-c_*$, are exactly solvable. It may be shown that all normalizable exactly solvable potentials lead to a two-term recursion relation, [**?**].

In conclusion, for every canonical form but the fifth one, we can construct a family of polynomials $P_k(E)$ satisfying a three-term recursion relation of the form ($k\geq 1$):

$$\begin{aligned} &-\left[(2k-n-1)\hat{c}_{0-}+\hat{c}_{-}\right]P_k \\ &\qquad = \left[E+c_*+\hat{c}_0\,(k-\frac{n}{2}-1)+\hat{c}_{00}\,(k-\frac{n}{2}-1)^2\right]P_{k-1} \\ &\qquad + (k-1)(k-2-n)\left[(2k-n-3)\hat{c}_{+0}+\hat{c}_{+}\right]P_{k-2}\,, \end{aligned} \tag{12}$$

where $\hat{c}_{ab}$ denote the coefficients of $\hat{P}$ and $\hat{Q}$, the transforms of P and Q under the projective transformation $z \mapsto w$ defined above. There is still an important point to be addressed: one needs to make sure that the coefficient in front of P_k in (**??**) does not vanish for any positive k:

$$(2k-n-1)\hat{c}_{0-} + \hat{c}_{-} \neq 0\,, \qquad k \geq 1\,. \tag{13}$$

If H does *not* correspond to the canonical forms 1, 8, and 9, this is the case provided H is *fully algebraic* and the eigenfunctions are all regular. For the canonical forms 1, 8, and 9, the condition (**??**) must be imposed *ad hoc*; see [**?**] for more details. Finally, if (**??**) holds, we introduce new (normalized) polynomials $\hat{P}_k$ by

$$P_k = \begin{cases} \frac{(-1)^k}{(2\hat{c}_{0-})^k} \dfrac{\hat{P}_k}{\Gamma\left(\frac{\hat{c}_-}{2\hat{c}_{0-}} + k - \frac{n}{2} + \frac{1}{2}\right)}, & \text{if} \quad \hat{c}_{0-} \neq 0; \\ \frac{(-1)^k}{\hat{c}_-^k}\, \hat{P}_k, & \text{if} \quad \hat{c}_{0-} = 0\,, \end{cases} \tag{14}$$

and arrive at our main result:

Theorem 3 *Let H be a* fully algebraic *QES Hamiltonian whose eigenfunctions are all regular. Assume also that eq.* (**??**) *holds, if H corresponds to the canonical forms 1, 8 or 9. Then H defines a family $\{\hat{P}_k\}_{k=0}^{\infty}$ of* (*monic*) weakly orthogonal *polynomials via equations* (**??**), (**??**), *and* (**??**) *satisfying*

$$\hat{P}_{k+1} = (E - b_k)\, \hat{P}_k - a_k\, \hat{P}_{k-1}, \qquad k \geq 0, \tag{15}$$

where

$$\begin{aligned} a_k &= k(k-n-1)\,[\hat{c}_{+0}(2k-n-1) + \hat{c}_+]\,[\hat{c}_{0-}(2k-n-1) + \hat{c}_-] \\ -b_k &= c_* + \hat{c}_0\left(k - \frac{n}{2}\right) + \hat{c}_{00}\left(k - \frac{n}{2}\right)^2 . \end{aligned}$$

4 Weakly Orthogonal Polynomials

In the previous section we have shown that "almost" every QES Hamiltonian generates a family of weakly orthogonal polynomials, that is a set of polynomials $\{\hat{P}_k\}_{k=0}^{\infty}$ verifying a recursion relation of the form (**??**) with $P_0 = 1$ and $a_{n+1} = 0$ (for a certain n). In this final section, we summarize some properties of weakly-orthogonal polynomials without making reference to their relation with QES spectral problems. Let us first recall a pair of standard results of the theory of orthogonal polynomials, [**?**]. According to Favard's Theorem there is a *unique* moment functional $\mathcal{L}$ defined on $\mathbb{C}[E]$ such that the polynomials are orthogonal under it:

$$\mathcal{L}(\hat{P}_k\, \hat{P}_l) = \gamma_k\, \delta_{kl}\,. \tag{16}$$

Moreover (Boas's Theorem), there is a (not necessarily unique) function of bounded variation ω such that

$$\mathcal{L}(p) = \int_{-\infty}^{\infty} p(E)\, dw(E)\,,$$

for an arbitrary polynomial p. We make use of these results to state the following characterization theorem for weakly orthogonal polynomials:

Theorem 4 *Let $\{\hat{P}_k\}_{k=0}^{\infty}$ be a familiy of polynomials satisfying a three-term recursion relation of the form* (**??**), *with $\hat{P}_{-1} = 0$ and $\hat{P}_0 = 1$. Let $\mathcal{L}$ be the associated moment functional. The following statements are then equivalent:*

1. *The coefficient* a_k *is nonzero for* $1 \leq k \leq n$, *and* $a_{n+1} = 0$.

2. *The norms* γ_k *in* (**??**) *are nonzero for* $0 \leq k \leq n$ *and vanish for* $k > n$.

3. $\hat{P}_{n+2}$ (*and hence* $\hat{P}_{n+1+l}$ *for* $l \geq 1$) *is divisible by* $\hat{P}_{n+1}$, *and* $\hat{P}_{k+1}$ *is not divisible by* $\hat{P}_k$ *for all* $k \leq n$.

4. *Let* c_i *be the coefficient of* E^{n+1-i} *in* $\hat{P}_{n+1}$. *Then, the moments of* $\mathcal{L}$ *satisfy the following recursion relation:*
$$\mu_k = \sum_{i=1}^{n+1} c_i \, \mu_{k-i}, \qquad k \geq n+1 \,.$$

5. *If, in addition,* $a_k > 0$ *and* b_k *is real, the moment functional* $\mathcal{L}$ *is given by*
$$\mathcal{L} = \sum_{l=0}^{n} \omega_l \, \delta(E - E_l) \,, \tag{17}$$
where the numbers ω_l *satisfy*
$$\sum_{l=0}^{n} \hat{P}_k(E_l) \, \omega_l = \delta_{k0} \,, \qquad 0 \leq k \leq n \,, \tag{18}$$
and E_l $(0 \leq l \leq n)$ *are the roots of* $\hat{P}_{n+1}$, *which are all simple and real.*

Proof. The proof was given in [**?**], except for the following statements:

i) $1 \Leftrightarrow 3$: Clearly $a_k = 0$ implies that $\hat{P}_{k+1}$ is divisible by $\hat{P}_k$. Conversely, if $\hat{P}_{k+1} = (E - \tilde{b}_k)\hat{P}_k$, the recursion relation (**??**) implies that $\tilde{b}_k = b_k$ and $a_k = 0$.

ii) $1 \Rightarrow 5$: By uniqueness, we only need to show that $\mathcal{L}$ in (**??**) is consistent with (**??**). This was proved in [**?**] using the fact that the matrix of coefficients $M_{kl} = \hat{P}_k(E_l)$ is non-singular if the polynomials are associated to a fully algebraic QES Hamiltonian. However, since
$$\det M = \prod_{i<j} (E_j - E_i) \,,$$
and it can be shown that the roots of P_{n+1} are real and simple if $a_k > 0$ and b_k is real, the same is true in this case. Q.E.D.

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