©1997 M. Alfaro et al. (Eds.) Proceedings of the
International Workshop on Orthogonal Polynomials
in Mathematical Physics. Leganés, 24-26 June, 1996.

CONTINUOUS SOLUTIONS OF DIFFERENCE EQUATIONS AND THEIR ORTHOGONALITY RELATIONS.

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Abstract

Some methods for solving hypergeometric–type difference equations are reviewed. In particular, a method of constructing solutions in the form of a development in "generalized powers" of the lattice $x(s)$, is discussed in detail. Various forms of the continuous orthogonalities for these solutions are also presented.

1 CLASSICAL SPECIAL FUNCTIONS

Classical special functions of mathematical physics arise in solving many theoretical and applied problems in various branches of natural science. They form a fundamental part of our mathematical language, helping us to express precisely our knowledge about the world in which we live. Their importance is therefore difficult to overestimate.

Well-known examples of special functions are the classical Jacobi, Laguerre and Hermite polynomials [26], satisfying the second-order linear differential equation of Sturm-Liouville type:

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0\,, \tag{1.1}$$

where primes denote differentiation with respect to the independent variable x, $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree, respectively, and λ is a constant. Observe that the generic case of (1.1) with $\sigma(x) = ax^2 + bx + c$ is reduced to the differential equation

$$x(1-x)y''(x) + [\gamma - (\alpha+\beta+1)x]y'(x) - \alpha\beta y(x) = 0 \tag{1.2}$$

for the Gaussian hypergeometric function $F(\alpha, \beta; \gamma; x)$, upon making the linear change of the variable $x \mapsto (x_2 - x_1)x + x_1$, where x_1 and x_2 are distinct zeros of $\sigma(x)$ (*i.e.*, $\sigma(x_1) = \sigma(x_2) = 0$ and $x_1 \neq x_2$). The explicit form of $F(\alpha, \beta; \gamma; x)$ is given by a power series in the variable x, namely

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}\,\frac{x^n}{n!}\,, \tag{1.3}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)\,, n = 0, 1, 2, ...$, is the shifted factorial.

The Jacobi, Laguerre and Hermite polynomials share the following common properties:

- they satisfy three-term recurrence relations,
- they can be represented by a Rodrigues formula (*i.e.*, as an nth derivative of an elementary function),
- they are orthogonal with respect to continuous weight functions (on the intervals $[-1, 1]$, $[0, \infty)$, and $(-\infty, \infty)$, respectively), and
- the derivative of each of them is again a polynomial of the same type.

Recently it has become clear that there exists a more general set of orthogonal polynomials which are basic hypergeometric extensions of the classical orthogonal polynomials [7]. We shall discuss them in section 3.2.

The standard way of solving the equation (1.1) is to look for its solution in the form of power series in the independent variable x. Indeed, if x_1 is a root of the equation $\sigma(x) = 0$, then the equation (1.1) has a particular solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_1)^n \,, \tag{1.4}$$

where the coefficients c_n satisfy the two-term recurrence relation [21]

$$(n+1)[\tau(x_1) + n\sigma'(x_1)]c_{n+1} + \{\lambda + n[\tau' + \tfrac{1}{2}(n-1)\sigma'']\}c_n = 0 \,. \tag{1.5}$$

For proving this, it is more convenient to rewrite the equation (1.1) in the self-adjoint form

$$[\sigma(x)\rho(x)y'(x)]' + \lambda\rho(x)y(x) = 0 \,, \qquad [\sigma(x)\rho(x)]' = \tau(x)\rho(x) \,, \tag{1.6}$$

where the second line is the first-order differential equation for defining a function $\rho(x)$. Now the proof that (1.4) and (1.5) represent a solution to (1.1) follows from the identity

$$\rho^{-1}(x)\frac{d}{dx}\left[\sigma(x)\rho(x)\frac{d}{dx}(x-\xi)^n\right] = n(n-1)\sigma(\xi)(x-\xi)^{n-2}$$

$$+n[\tau(\xi) + (n-1)\sigma'(\xi)](x-\xi)^{n-1} - \lambda_n (x-\xi)^n \,, \tag{1.7}$$

which can be easily verified; here $\lambda_n = -n\tau' - n(n-1)\sigma''/2$ and ξ is an arbitrary value of the independent variable x (not necessarily a zero of $\sigma(x)$!). Futhermore, as is evident from (1.7) , in the case when $\sigma(x)$ has no zeros, the series (1.4) will satisfy the equation (1.1) if

$$(n+1)(n+2)\sigma c_{n+2} + (\lambda + n\tau')c_n = 0 \tag{1.8}$$

and x_1 is a root of the equation $\tau(x) = 0$.

2 DIFFERENCE OPERATORS

Those of you who are versed in non-relativistic quantum mechanics probably remember that the notion of difference operator appears at an early stage. Indeed, consider a quantum-mechanical system of N particles with positions in three-dimensional space, that are characterized by the radius vectors $\vec{r}_n, 1 \le n \le N$ [29]. Then an infinitely small parallel displacement (or translation) in this space over a distance $\delta\vec{r}$ signifies a transformation under which the radius vectors $\vec{r}_n$ of all the particles receive the same increment $\delta\vec{r}$: $\vec{r}_n \Rightarrow \vec{r}_n + \delta\vec{r}$. The operator of a parallel displacement over any finite (not only infinitesimal) distance $\vec{a}$ can be expressed in terms of the momentum operator $\hat{\vec{p}} = -i\hbar\frac{\partial}{\partial\vec{r}} = -i\hbar\vec{\nabla}$. By the definition of this operator $\hat{T}_{\vec{a}}$ we must have

$$\hat{T}_{\vec{a}}\,\psi(\vec{r}) = \psi(\vec{r} + \vec{a}) \,. \tag{2.1}$$

Expanding the function $\psi(\vec{r} + \vec{a})$ in a Taylor series gives

$$\psi(\vec{r} + \vec{a}) = \psi(\vec{r}) + \vec{a}\,\frac{\partial\psi(\vec{r})}{\partial\vec{r}} + \cdots \,, \tag{2.2}$$

or, in terms of the momentum $\hat{\vec{p}}$,

$$\psi(\vec{r} + \vec{a}) = [1 + \frac{i}{\hbar}\vec{a}\,\hat{\vec{p}} + \frac{1}{2}(\frac{i}{\hbar}\vec{a}\,\hat{\vec{p}})^2 + \cdots]\psi(\vec{r}) \,. \tag{2.3}$$

The expression in brackets, $\hat{T}_{\vec{a}} = \exp(\frac{i}{\hbar}\vec{a}\,\hat{\vec{p}})$, is the required operator of the finite displacement. From (2.2) or (2.3) it is also evident, that the momentum $\hat{\vec{p}}$ itself (multiplied by the constant factor $i/\hbar$) represents the operator of infinitesimal displacements, or, in the terminology of group theory, it is the generator of displacements.

Once the action of the simple difference operator $\exp(a\,\partial_x)$, $\partial_x \equiv d/dx$, is clear from (2.1), we can try to solve a difference equation of the form

$$[\alpha_l(x)\, e^{-\partial_x} + \alpha_l(x+1)\, e^{\partial_x}]\, f_n(x;l) = \lambda_n(l)\, f_n(x;l)\,, \tag{2.4}$$

where $\alpha_l(x) = \sqrt{(x+l)(l+1-x)}$ and $-l \le x \le l$. Observe that in view of the relation $\alpha_l(x) = \alpha_l(1-x)$, the equation (2.4) is invariant with respect to the change of the sign of x. As usual, one starts with a "ground state" $f_0(x;l)$, which defines "asymptotic" behaviour of solutions $f_n(x;l)$ at the ends of the interval $[-l,l]$. Since

$$\begin{gathered} e^{\partial_x}\,\Gamma(x+a) = \Gamma(x+a+1) = (x+a)\,\Gamma(x+a)\,, \\ e^{\partial_x}\,\Gamma(a-x) = \Gamma(a-x-1) = (a-x-1)^{-1}\,\Gamma(a-x)\,, \end{gathered} \tag{2.5}$$

and the action of the second operator $\exp(-\partial_x)$ on the same gamma-functions is readily obtained from (2.5) by reversing the sign of the variable x, we can look for a solution

$$f_0(x;l) = \Gamma^{\beta}(x+a)\,\Gamma^{\beta}(a-x)\,, \tag{2.6}$$

where a and β are some constants to be found. Substituting (2.6) into the left-hand side of (2.4) gives

$$\begin{gathered} \alpha_l(x)\, f_0(x-1;l) + \alpha_l(x+1)\, f_0(x+1;l) = \\ [\alpha_l(x)\,(a-x)^{\beta}(a+x-1)^{-\beta} + \alpha_l(x+1)\,(x+a)^{\beta}\,(a-x-1)^{-\beta}]\, f_0(x;l)\,. \end{gathered} \tag{2.7}$$

The equation (2.4) for $f_0(x;l)$ will be satisfied if the expression in the square brackets in (2.7) turns out to be a constant. So if we choose $a = l+1$ and $\beta = -1/2$, this expression becomes $x+l+l-x = 2l$. Thus,

$$f_0(x;l) = [\Gamma(x+l+1)\,\Gamma(l+1-x)]^{-1/2} \tag{2.8}$$

and the corresponding eigenvalue $\lambda_0(l)$ is equal to $2l$. Having defined $f_0(x;l)$, one can successively find all $2l+1$ linearly independent "excited states"

$$f_n(x;l) = d_n^{-1}\, p_n(x;l)\, f_0(x;l)\,, \qquad 0 \le n \le 2l\,, \tag{2.9}$$

which correspond to the eigenvalues $\lambda_n(l) = 2(l-n)$. The polynomials $p_n(x;l)$ in (2.9) satisfy the three-term recurrence relation

$$2xp_n(x;l) = (n+1)\,p_{n+1}(x;l) + (2l+1-n)\,p_{n-1}(x;l) \tag{2.10}$$

and they are particular case of the well-known Kravchuk polynomials

$$k_n^{(p)}(x,N) = (-1)^n C_N^n\, p^n\, F(-n,-x;-N;p^{-1})$$

with $p = 1/2$ and $N = 2l$ [30] . The normalization constant $d_n = [(2l)!/n!(2l-n)!]^{1/2}$ in (2.9) is chosen in such a way, that the functions $f_n(x;l)$ satisfy the discrete orthogonality relation

$$\sum_{j=0}^{2l} f_m(x_j;l)\, f_n(x_j;l) = \delta_{m,n} \tag{2.11}$$

at the points $x_j = j - l$.

The reason for picking up this particular example of difference equation (2.4) is that it actually represents a discrete model of the harmonic oscillator [22].

We recall in this connection that the quantum-mechanical analogue of Newton's equation is $m\dot{v} = -\partial U/\partial x$, where m is the mass, $v = \dot{x} = i\hbar^{-1}[H, x]$ is the velocity operator, and $H(x) = p^2/2m + U(x)$ is the Hamiltonian [29]. For the harmonic oscillator, *i.e.*, when we have $U(x) = m\,\omega^2 x^2/2$, this equation takes the form

$$[H(x), [H(x), x]] = (\hbar\omega)^2 x\,. \tag{2.12}$$

Now it is easy to check that the difference operator

$$D_l(x) = \frac{1}{2}[\alpha_l(x)\, e^{-\partial_x} + \alpha_l(x+1)\, e^{\partial_x}] \tag{2.13}$$

satisfies the same commutation relation

$$[D_l(x), [D_l(x), x]] = x \tag{2.14}$$

with the independent variable x as in (2.12) .

3 HYPERGEOMETRIC-TYPE DIFFERENCE EQUATIONS

Having discussed some particular examples of difference operators and difference equations in section 2, we consider now a general approach to the problem of constructing a hypergeometric-type difference equation by approximating the differential equation (1.1) on certain lattices [30]. The standard way of approximating any differential equation is based on the use of the definition of the derivative

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{(x+h) - x} \tag{3.1}$$

and involves the substitution

$$y'(x) \Rightarrow \frac{1}{h}[y(x+h) - y(x)] = \frac{1}{h}(e^{h\partial_x} - 1)\, y(x) \tag{3.2}$$

for each derivative of the unknown function $y(x)$. In the case of the differential equation (1.1) the corresponding difference equation of hypergeometric type

$$\begin{aligned} &\frac{\sigma(x)}{h}\left[\frac{y(x+h) - y(x)}{h} - \frac{y(x) - y(x-h)}{h}\right] + \\ &+\frac{\tau(x)}{2}\left[\frac{y(x+h) - y(x)}{h} + \frac{y(x) - y(x-h)}{h}\right] + \lambda y(x) = 0 \end{aligned} \tag{3.3}$$

approximates (1.1) up to the second-order of accuracy with respect to the constant h. We remind the reader that the difference operator L_h approximates the differential operator L at point x to the order of accuracy n with respect to h if $(L_h - L)\, y(x) = O(h^n)$ when $h \to 0$ [30].

This scheme is often employed for obtaining (up to some fixed order of accuracy in the constant h) approximate solutions to differential equations, which are difficult to solve exactly. But a more general case arises when one introduces a new indepent variable s and approximates a differential equation on an arbitrary lattice $x(s)$, *i.e.*, by replacing

$$y'(x) \Rightarrow \frac{y[x(s+h)] - y[x(s)]}{x(s+h) - x(s)}\,. \tag{3.4}$$

The difference equation of hypergeometric type corresponding to (1.1) and (3.3) on a lattice $x(s)$ has the form

$$\frac{\sigma[x(s)]}{x(s+\frac{h}{2})-x(s-\frac{h}{2})}\left[\frac{y(s+h)-y(s)}{x(s+h)-x(s)}-\frac{y(s)-y(s-h)}{x(s)-x(s-h)}\right]+$$
$$+\frac{\tau[x(s)]}{2}\left[\frac{y(s+h)-y(s)}{x(s+h)-x(s)}+\frac{y(s)-y(s-h)}{x(s)-x(s-h)}\right]+\lambda y(s)=0\,. \tag{3.5}$$

Naturally, in the particular case of the linear lattice $x(s)=s$, (3.5) reduces to the difference equation (3.3). Observe that the linear change of the independent variable $s\to hs$ and the replacement of the lattice $x(hs)\to x(s)$ lead to the more compact form of equation (3.5) :

$$\sigma[x(s)]\frac{\triangle}{\nabla x(s+\frac{1}{2})}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\frac{\tau[x(s)]}{2}\left[\frac{\triangle y(s)}{\triangle x(s)}+\frac{\nabla y(s)}{\nabla x(s)}\right]+\lambda y(s)=0\,, \tag{3.6}$$

where the difference operators $\triangle$ and ∇ are defined as

$$\triangle y(s)=y(s+1)-y(s)=(e^{\partial_s}-1)\,y(s)\,, \tag{3.7a}$$

$$\nabla y(s)=y(s)-y(s-1)=(1-e^{-\partial_s})\,y(s)\,. \tag{3.7b}$$

Assume that the difference equation (3.6) has some polynomial solution $y[x(s)]$ and we require (by analogy with the classical case of the Jacobi, Laguerre and Hermite polynomials) that the difference-derivative $\triangle y[x(s)]/\triangle x(s)$ of this solution satisfies an equation of the same kind. This requirement holds if and only if the lattice $x(s)$ is a quadratic or q-quadratic lattice of the form [30, 19]

$$x(s)=\left\{\begin{array}{c}C_1s^2+C_2s\,,\\ C_1q^{-s}+C_2q^s\,.\end{array}\right. \tag{3.8}$$

Difference equations, which approximate differential ones on certain lattices $x(s)$, can be solved exactly, *i.e.*, without expanding solutions in powers of the lattice mesh h. Evidently, this important fact means that there exists a more general set of solutions for each such difference equation, which coincides with the set of solutions for the corresponding differential equation only in the limit of vanishing mesh h.

3.1 "Power" series developments

It is possible to construct solutions of the difference equation (3.6) in the form of a development in "generalized powers" of the lattice $x(s)$, defined as [20]

$$[x(s)-x(s_0)]^{(n)}=\prod_{k=0}^{n-1}[x(s)-x(s_0-k)]\,. \tag{3.9}$$

But, as in the case of the differential equation (1.1), it is more convenient first to represent (3.6) in the self-adjoint form [30]

$$\frac{\triangle}{\nabla x(s+\frac{1}{2})}\left[\tilde{\sigma}(s)\,\rho(s)\,\frac{\nabla y(s)}{\nabla x(s)}\right]+\lambda\rho(s)y(s)=0\,,$$

$$\frac{\triangle}{\nabla x(s+\frac{1}{2})}\left[\tilde{\sigma}(s)\,\rho(s)\right]=\tau(s)\,\rho(s)\,, \tag{3.10}$$

where $\widetilde{\sigma}(s)=\sigma[x(s)]-\frac{1}{2}\tau[x(s)]\bigtriangledown x(s+\frac{1}{2})$.

The series of the form

$$y(s)=y[x(s)]=\sum_{n=0}^{\infty}c_n[x(s)-x(r+n-1)]^{(n)}\,, \tag{3.11}$$

where r is a finite root of the equation

$$\widetilde{\sigma}(s)=\sigma[x(s)]-\tfrac{1}{2}\tau[x(s)]\bigtriangledown x(s+\tfrac{1}{2})=0 \tag{3.12}$$

and

$$\frac{c_{n+1}}{c_n}=-\frac{\{\lambda+\gamma(n)[\alpha(n-1)\tau'+\gamma(n-1)\sigma''/2\}\bigtriangledown x[r+(n-1)/2]}{\gamma(n+1)[\widetilde{\sigma}(r+n)+\tau(r+n)\bigtriangledown x(r+n+1/2)]}\,, \tag{3.13}$$

satisfies a non-homogeneous equation of the type (3.6) on the non-uniform lattices (3.8) with the right-hand side

$$G(s)=\lim_{N\to\infty}(\lambda-\lambda_N)[x(s)-x(r+N-1)]^{(N)}\,. \tag{3.14}$$

This can be verified by using the equality

$$\begin{aligned}&\rho^{-1}(s)\frac{\triangle}{\bigtriangledown x_1(s)}\left\{\widetilde{\sigma}(s)\rho(s)\frac{\bigtriangledown}{\bigtriangledown x(s)}[x(s)-x(s_0)]^{(n)}\right\}=\\ &=\gamma(n)\gamma(n-1)\widetilde{\sigma}(s_0-n+1)\,[x(s)-x(s_0-1)]^{(n-2)}+\\ &+\gamma(n)\tau_{n-1}(s_0-n+1)[x(s)-x(s_0-1)]^{(n-1)}-\lambda_n[x(s)-x(s_0)]^{(n)}\,,\end{aligned} \tag{3.15}$$

where the definitions

$$\begin{aligned}\tau_\nu(s)\bigtriangledown x_{\nu+1}(s)&=\widetilde{\sigma}(s+\nu)+\tau(s+\nu)\bigtriangledown x_1(s+\nu)-\widetilde{\sigma}(s)\,,\\ \lambda_\mu&=-\gamma(\mu)\,[\alpha(\mu-1)\tau'+\gamma(\mu-1)\sigma''/2]\,,\\ x_\nu(s)&=x(s+\nu/2)\end{aligned} \tag{3.16}$$

and the notations

$$\alpha(\mu)=\left\{\begin{array}{l}1,\\ \frac{1}{2}(q^{\mu/2}+q^{-\mu/2})\,,\end{array}\right.\qquad \gamma(\mu)=\left\{\begin{array}{r}\mu\,,\\ \dfrac{q^{\mu/2}-q^{-\mu/2}}{q^{1/2}-q^{-1/2}}\end{array}\right. \tag{3.17}$$

have been used for the quadratic and q-quadratic lattices, respectively.

Indeed, let us write the difference equation of hypergeometric type (3.6) in the self-adjoint form (3.10) and look for its solutions in the form of the following development in generalized powers (3.9)

$$y(s)=\lim_{N\to\infty}\sum_{n=0}^{N}c_n[x(s)-x(s_0)]^{(n)}\,, \tag{3.18}$$

where s_0 is a constant. Substituting (3.18) into (3.10), with the aid of identity (3.15) with $s_0=r+n-1$ and $\sigma(r)=0$, we obtain

$$\begin{aligned}&\left[\rho^{-1}(s)\frac{\triangle}{\bigtriangledown x_1(s)}\left(\sigma(s)\rho(s)\frac{\bigtriangledown}{\bigtriangledown x(s)}\right)+\lambda\right]y(s)=\\ &=\lim_{N\to\infty}\sum_{n=0}^{N}c_n\left[\rho^{-1}(s)\frac{\triangle}{\bigtriangledown x_1(s)}\left(\sigma(s)\rho(s)\frac{\bigtriangledown}{\bigtriangledown x(s)}\right)+\lambda\right][x(s)-x(s_0)]^{(n)}=\end{aligned}$$

$$= \lim_{N\to\infty}\left\{\sum_{n=0}^{N} c_n\gamma(n)\tau_{n-1}(r)[x(s)-x(r+n-2)]^{(n-1)}+\right.$$

$$\left.+\sum_{n=0}^{N} c_n(\lambda-\lambda_n)[x(s)-x(r+n-1)]^{(n)}\right\} = G(z)\,, \tag{3.19}$$

provided that the coefficients c_n satisfy the two-term recurrence relation

$$\gamma(n+1)\tau_n(r)c_{n+1} = (\lambda_n-\lambda)\,c_n\,. \tag{3.20}$$

From (3.20), in view of (3.16), (3.13) follows.

Observe that when $\lambda=\lambda_\nu$ and $\nu=n=0,1,2,\ldots$, solution (3.11) is a polynomial of degree n and $G(s)\equiv 0$.

3.2 The Askey-Wilson polynomials

Particular choices of the lattice $x(s)$, generically defined by (3.8), lead to various families of polynomial solutions of the hypergeometric-type difference equation (3.6). The most general one is the five-parameter set of the Askey–Wilson polynomials, which contains as special or limiting cases all the classical polynomials, satisfying orthogonality relations with continuous weight functions. They are defined as [7]

$$p_n(x;a,b,c,d|q) = (ab,ac,ad;q)_n\,a^{-n}\,{}_4\phi_3\left[\begin{matrix} q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta} \\ ab,ac,ad\end{matrix};q,q\right]\,, \tag{3.21}$$

where $x=\cos\theta$, $(a;q)_0=1$ and $(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$, $n=1,2,3,\ldots$, define the q-shifted factorial, $(a_1,\ldots,a_k;q)_n=\prod_{j=1}^{k}(a_j;q)_n$ is the contracted notation for the multiple q-shifted factorials and the basic hypergeometric series ${}_4\phi_3$ is

$${}_4\phi_3\left[\begin{matrix} a_1,a_2,a_3,a_4 \\ b_1,b_2,b_3\end{matrix};q,z\right] = \sum_{k=0}^{\infty}\frac{(a_1,a_2,a_3,a_4;q)_k}{(b_1,b_2,b_3,q;q)_k}\,z^k\,. \tag{3.22}$$

To verify that the formula (3.21) constitutes a polynomial of nth degree in the variable $x=\cos\theta$ we have to take into account the following two facts. First, since the q-shifted factorial $(q^{-n};q)_k=\prod_{j=0}^{k-1}(1-q^{j-n})$ vanishes for any positive integer $k\geq n+1$ by its definition, from (3.22) it follows that the basic hypergeometric series ${}_4\phi_3$ in (3.21) represents the finite sum

$${}_4\phi_3\left[\begin{matrix} q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta} \\ ab,ac,ad\end{matrix};q,q\right] = \sum_{k=0}^{n}\frac{(q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta};q)_k}{(ab,ac,ad,q;q)_k}\,q^k\,. \tag{3.23}$$

Second, since the product of two θ-dependent factors in (3.23) can be written as

$$(ae^{i\theta},ae^{-i\theta};q)_k = \prod_{j=0}^{k-1}(1-aq^je^{i\theta})\,(1-aq^je^{-i\theta}) = \prod_{j=0}^{k-1}[1-2aq^j\cos\theta+a^2q^{2j}]\,, \tag{3.24}$$

it certainly forms a polynomial of kth degree in $x=\cos\theta$.

The Askey–Wilson polynomials (3.21) satisfy the orthogonality relation [7]

$$\int_{-1}^{1} p_m(x;a,b,c,d|q)\,p_n(x;a,b,c,d|q)\,w(x;a,b,c,d|q)\,dx = \delta_{mn}\,I_n(a,b,c,d|q) \tag{3.25}$$

on the finite interval $-1 \le x = \cos\theta \le 1$ with respect to the continuous weight function

$$w(x; a, b, c, d|q) = 4\sin\theta \, \frac{\prod_{v=a,b,c,d} e_q(ve^{i\theta})\, e_q(ve^{-i\theta})}{e_q(qe^{2i\theta})\, e_q(qe^{-2i\theta})}\,, \tag{3.26}$$

where $e_q(z)$ is the q-exponential function, defined as [27]

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}\,, \tag{3.27}$$

and $I_n(a, b, c, d|q)$ is equal to

$$I_n(a, b, c, d|q) = 2\pi e_q(q^{n+1}) \frac{e_q(abq^n)\, e_q(acq^n)\, e_q(adq^n)\, e_q(bcq^n)\, e_q(bdq^n)\, e_q(cdq^n)}{(1 - abcdq^{2n-1})\, e_q(abcdq^{n-1})}\,. \tag{3.28}$$

3.3 The transformation $q \to q^{-1}$

Jacobi's triple product identity

$$\theta_1(z, q) = 2q^{\frac{1}{4}} \sin z \, (q^2, q^2 e^{2iz}, q^2 e^{-2iz}; q^2)_\infty \tag{3.29}$$

and Euler's formula

$$e_q(z) = (z; q)_\infty^{-1} \tag{3.30}$$

enable the weight function (3.26) for the Askey–Wilson polynomials to be written as

$$w(x; a, b, c, d|q) = 2q^{-\frac{1}{8}}\, e_q(q)\, \vartheta_1(\theta, q^{1/2}) \prod_{v=a,b,c,d,} e_q(ve^{i\theta})\, e_q(ve^{-i\theta})\,. \tag{3.31}$$

Since the theta-function in (3.31) is defined as

$$\vartheta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)\, x\,, \tag{3.32}$$

it is evident that this series is convergent only if the parameter q satisfies $|q| < 1$, therefore the common notation in the theory of special functions is $q \equiv e^{\pi i \tau}$, $\Im \tau > 0$. This means that the representation (3.25) does not admit the transformation $q \to q^{-1}$. But using the modular and periodicity properties of the theta-function $\vartheta_1(\theta, q^{1/2})$ it turns out to be possible to replace them by quadratically decreasing exponential functions. The Ramanujan-type measures thus obtained (with support on the infinite interval) admit the transformation $q \to q^{-1}$.

Indeed, let us consider first the Askey–Wilson q-beta integral

$$I_0(a, b, c, d|q) = \int_{-1}^{1} w(x; a, b, c, d|q)\, dx\,, \tag{3.33}$$

which corresponds to the orthogonality relation (3.25) with $m = n = 0$. From the point of view of symmetry it is more convenient to make the change of variable $\theta = \frac{\pi}{2} - \varphi$, $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$ in (3.33). Consequently, the integral in (3.33) can be written as

$$I_0(a, b, c, d|q) = 2q^{-\frac{1}{8}} e_q(q) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vartheta_2(\varphi, q^{1/2})\, \rho(\sin\varphi; a, b, c, d|q) cos\varphi d\varphi\,, \tag{3.34}$$

where, according to (3.31),

$$\rho(\sin\varphi; a,b,c,d|q) = \prod_{v=a,b,c,d} e_q(ive^{-i\varphi})e_q(-ive^{i\varphi})\,. \tag{3.35}$$

With the aid of the modular transformation [37]

$$\vartheta_2(z|\tau) = \frac{\exp(-iz^2/\pi\tau)}{(-i\tau)^{1/2}}\,\vartheta_4(z\tau^{-1}|-\tau^{-1})\,, \qquad \tau = \frac{i\kappa^2}{\pi}\,, \tag{3.36}$$

and the change of variable $\varphi = \kappa x$, the integral (3.34) is represented as

$$I_0(a,b,c,d|q) = 2\sqrt{\pi}q^{-\frac{1}{8}}e_q(q)\int\limits_{-\pi/2\kappa}^{\pi/2\kappa}\vartheta_4(\pi ix/\kappa, e^{-\pi^2/\kappa^2})\times$$

$$\rho(\sin\kappa x; a,b,c,d|q)\,e^{-x^2}\,\cos\kappa x dx\,. \tag{3.37}$$

Using the expansion

$$\vartheta_4(z,q) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}e^{2ikz} \tag{3.38}$$

and taking into account the uniform convergence of the series (3.38) in any bounded domain of values of z [37], we substitute (3.38) into (3.37) and integrate this series termwise. This gives

$$I_0(a,b,c,d|q) = 2\sqrt{\pi}q^{-\frac{1}{8}}e_q(q)\sum_{n=-\infty}^{\infty}(-1)^n\int\limits_{-\pi/2\kappa}^{\pi/2\kappa}\rho(\sin\kappa x; a,b,c,d|q)e^{-(x+\pi/n\kappa)^2}\cos\kappa x dx\,. \tag{3.39}$$

The change of variable $x_n = x + \frac{\pi}{\kappa}n$, $x_n^{\min} = \frac{\pi}{\kappa}(n-\frac{1}{2}) \le x_n \le \frac{\pi}{\kappa}(n+\frac{1}{2}) = x_n^{\max}$ allows to sum the right-hand side of (3.39) with respect to n. This yields the following representation for the Askey–Wilson q-beta integral [10]

$$I_0(a,b,c,d|q) = 2\sqrt{\pi}q^{-\frac{1}{8}}e_q(q)\int\limits_{-\infty}^{\infty}\rho(\sin\kappa x; a,b,c,d|q)e^{-x^2}\cos\kappa x dx\,. \tag{3.40}$$

The weight function (3.26) with the parameters $a = -b = \beta^{1/2}$, and $c = -d = aq^{1/2} = (\beta q)^{1/2}$, corresponds to the continuous q-ultraspherical polynomials $C_n(x;\beta|q)$ [6]. The relations [27]

$$(a;q)_\infty = (a,aq;q^2)_\infty\,, \qquad (a,-a;q)_\infty = (a^2;q^2)_\infty$$

enable the representation (3.40) for this particular case to be simplified to

$$\int\limits_{-\infty}^{\infty}e_q(-\beta e^{2i\kappa x})e_q(-\beta e^{-2i\kappa x})e^{-x^2+i\kappa x}dx = \sqrt{\pi}q^{1/8}e_q(\beta^2)E_q(-\beta)E_q(-q\beta)\,. \tag{3.41}$$

where $E_q(z)$ is the q-exponential function, *i.e.*,

$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n}z^n = (-z;q)_\infty\,. \tag{3.42}$$

If one compares (3.41) with the Ramanujan integral [33, 4]

$$\int\limits_{-\infty}^{\infty}e^{-x^2+2mx}e_q(aq^{1/2}e^{2i\kappa x})e_q(bq^{1/2}e^{-2i\kappa x})dx = \sqrt{\pi}e^{m^2}e_q(abq)E_q(aqe^{2im\kappa})E_q(bqe^{-2im\kappa}), \tag{3.43}$$

it is easy to verify that (3.41) agrees with (3.43) if one sets $2m = i\kappa$ and $a = b = -\beta q^{1/2}$.

4 RAMANUJAN-TYPE MEASURES FOR THE ASKEY–WILSON POLYNOMIALS

In this section it will be shown that Ramanujan-type measures for a hierarchy of Askey–Wilson polynomials can be systematically built from simple cases of the continuous q-Hermite and q^{-1}-Hermite polynomials by using the Berg-Ismail procedure of attaching generating functions to measures [23]. Starting with the orthogonality relation at the lower lever of the hierarchical ladder, namely, for the q-Hermite polynomials, one can move up with the aid of the corresponding generating functions and eventually obtain the orthogonality for the most general case, i.e., Askey-Wilson polynomials.

4.1 The continuous q-Hermite ladder $(0 < q < 1)$

We begin with the Ramanujan-type orthogonality relation

$$\int_{-\infty}^{\infty} H_m(\sin\kappa x|q) H_n(\sin\kappa x|q)\, e^{-x^2} \cos\kappa x\, dx = \sqrt{\pi}\, q^{1/8}\, (q;q)_n\, \delta_{mn} \tag{4.1.1}$$

for the continuous q-Hermite polynomials, assuming at first that $0 < q < 1$ and, therefore, using Ramanujan's notation $q = \exp(-2\kappa^2)$. The relation (4.1.1) was derived in [17] by the transformation of the standard orthogonality relation on the finite interval $-1 \le x \le 1$ for the continuous q-Hermite polynomials $H_n(x|q)$ [6], based on the use of the modular and periodicity properties of the theta-function, entering in this measure for $H_n(x|q)$. The validity of (4.1.1) can be verified with the aid of recurrence relation and the explicit form of the difference lowering operator for the q-Hermite polynomials $H_n(\sin\kappa x|q)$ [17]. We will give here another derivation of the orthogonality relation (4.1.1). In view of the symmetry of the integrand in (4.1.1) with respect to n and m we may assume for definiteness that $m \le n$. The Rogers linearization formula [34, 24]

$$H_m(x|q)\, H_n(x|q) = \sum_{k=0}^{m} \frac{(q;q)_n}{(q;q)_{n-m+k}} \begin{bmatrix} m \\ k \end{bmatrix}_q H_{n-m+2k}(x|q) \tag{4.1.2}$$

and the Fourier expansion [6]

$$H_n(\sin\kappa x|q) = i^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(2k-n)\kappa x} \tag{4.1.3}$$

for the q-Hermite polynomials, where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\, (q,q)_{n-k}}$$

is the q-binomial coefficient, enable the left side of (4.1.1) to be represented as

$$I_{mn}(q) = i^{n-m} \sum_{j=0}^{m} \frac{(-1)^j (q;q)_n}{(q;q)_{n-m+j}} \begin{bmatrix} m \\ j \end{bmatrix}_q \sum_{k=0}^{N} (-1)^k \begin{bmatrix} N \\ k \end{bmatrix}_q \int_{-\infty}^{\infty} e^{-x^2+i(2k-N)\kappa x} \cos\kappa x\, dx \tag{4.1.4}$$

with $N = n-m+2j$. The integration over x in (6.1.4) is easily performed by the formula (Fourier transformation for the ground state of the linear harmonic oscillator)

$$\int_{-\infty}^{\infty} dx\, e^{-x^2+2ixy} = \sqrt{\pi}\, e^{-y^2} \tag{4.1.5}$$

and therefore (4.1.4) takes the form

$$I_{mn}(q) = i^{n-m}\,\frac{\sqrt{\pi}}{2}\sum_{j=0}^{m}\frac{(-1)^j(q;q)_n}{(q;q)_{n-m+j}}\begin{bmatrix} m \\ j \end{bmatrix}_q \sum_{k=0}^{N}(-1)^k\begin{bmatrix} N \\ k \end{bmatrix}_q \left[q^{\frac{1}{2}(k-\frac{N-1}{2})^2} + q^{\frac{1}{2}(k-\frac{N+1}{2})^2}\right]. \qquad (4.1.6)$$

The sum over k in (4.1.6) is calculated with the aid of the identity (see, for example, [27])

$$(z;q)_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k\, q^{\frac{k(k-1)}{2}}. \qquad (4.1.7)$$

This leads to the expression

$$q^{\frac{1}{8}(N^2+1)}\left[q^{\frac{N}{4}}(q^{-\frac{N}{2}};q)_N + q^{-\frac{N}{4}}(q^{1-\frac{N}{2}};q)_N\right]. \qquad (4.1.8)$$

Since $(1-a)(aq;q)_n = (1-aq^n)(a;q)_n$, the square bracket in (4.1.8) vanishes for any positive number N, whereas for $N = n-m+2j = 0$ it is equal to two. Thus, we arrive at the right side of (4.1.1).

After the change of variable $\theta = \frac{\pi}{2} - \kappa x$, the conventional generating function for the continuous q-Hermite polynomials [34]

$$e_q(te^{i\theta})\, e_q(te^{-i\theta}) = (te^{i\theta}, te^{-i\theta};q)_\infty^{-1} = \sum_{n=0}^{\infty}\frac{H_n(\cos\theta|q)}{(q;q)_n}\, t^n, \qquad |t|<1, \qquad (4.1.9)$$

can be written as

$$e_q(ite^{-i\kappa x})\, e_q(-ite^{i\kappa x}) = (ite^{-i\kappa x},\, -ite^{i\kappa x};q)_\infty^{-1} = \sum_{n=0}^{\infty}\frac{H_n(\sin\kappa x|q)}{(q;q)_n}\, t^n. \qquad (4.1.10)$$

Now let us multiply both sides of the orthogonality relation (4.1.1) for the q-Hermite polynomials by $t^m\tau^n(q;q)_m^{-1}(q;q)_n^{-1}$ and then sum over m and n from zero to infinity, using the generating function (4.1.10) and definition (3.27)

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}. \qquad (4.1.11)$$

The result is

$$\int_{-\infty}^{\infty}\prod_{v=t,\tau} e_q(ive^{-i\kappa x})\, e_q(-ive^{i\kappa x})\, e^{-x^2}\cos\kappa x\, dx = \sqrt{\pi}\, q^{\frac{1}{8}}\, e_q(t\tau) \qquad (4.1.12)$$

and the integrand in the representation (4.1.12) may serve as a weight function for a family of polynomials, depending on the two more parameters t and τ in addition to q. The possibility of finding such a family is simplified by the circumstance [7, 23], that the multiplication of the integrand in (4.1.12) by the factor $(ive^{-i\kappa x}, -ive^{i\kappa x};q)_k$, $v = t,\tau$, just transforms the parameters v to vq^k, since

$$e_q(aq^k) = (a;q)_k\, e_q(a), \qquad k = 0,1,2,\ldots \qquad (4.1.13)$$

Indeed, the same reasoning as in Ref. [23] *mutatis mutandis* shows that the Al-Salam-Chihara polynomials [2]

$$p_n(\sin\kappa x;t,\tau;q) = \frac{(t\tau;q)_n}{t^n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, ite^{-i\kappa x}, -ite^{i\kappa x} \\ t\tau, 0 \end{matrix}; q, q\right] \qquad (4.1.14)$$

are orthogonal with respect to the weight integrand in (4.1.12), namely,

$$\int_{-\infty}^{\infty} p_m(\sin\kappa x; t,\tau; q)\, p_n(\sin\kappa x; t,\tau; q) \times \tag{4.1.15}$$

$$\prod_{v=t,\tau} e_q(ive^{-i\kappa x})\, e_q(-ive^{i\kappa x})\, e^{-x^2} \cos\kappa x\, dx = \sqrt{\pi}\, q^{\frac{1}{8}}\, (q;q)_n\, e_q(t\tau q^n)\, \delta_{mn}\,.$$

The basic hypergeometric series ${}_3\phi_2$ in (4.1.14) is defined as [27]

$${}_3\phi_2 \left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, a_3; q)_k}{(b_1, b_2, q; q)_k}\, z^k \tag{4.1.16}$$

and the Al-Salam-Chihara polynomials are symmetric in the parameters t and τ [23]. Observe that the normalization in (4.1.14) slightly differs from the one, employed in Refs. [2] and [23], and is chosen in a such way, that the polynomials (4.1.14) are a particular case of the more general five-parameter Askey-Wilson polynomials, *i.e.*, they coincide with $p_n(\sin\kappa x; t, \tau, 0, 0|q)$ (see the formula (4.1.22) below).

A useful consequence of the first step is the following integral for the Al-Salam-Chihara polynomials (4.1.14)

$$\int_{-\infty}^{\infty} p_n(\sin\kappa x; t,\tau; q)\, e_q(ite^{-i\kappa x})\, e_q(-ite^{i\kappa x})\, e^{-x^2}\cos\kappa x\, dx = \sqrt{\pi}\, q^{\frac{1}{2}(n-\frac{1}{2})^2} (-\tau)^n\,. \tag{4.1.17}$$

To verify (4.1.17), use the formulas (4.1.13) – (4.1.16) and the Chu-Vandermonde q-sum

$${}_2\phi_1(a, q^{-n}; c; q, q) = \frac{(ca^{-1}; q)_n}{(c;q)_n}\, a^n\,. \tag{4.1.18}$$

To repeat the same procedure with the Al-Salam-Chihara polynomials as the starting point, we can use the generating function [23]

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n}\, p_n(\sin\kappa x; t_1, t_2; q) = \frac{(tt_1, tt_2; q)_\infty}{(ite^{-i\kappa x}, -ite^{i\kappa x}; q)_\infty}\,. \tag{4.1.19}$$

Indeed, multiplying both sides of the orthogonality relation (4.1.15) (with $t = t_1$ and $\tau = t_2$) by the factor $(q;q)_m^{-1}\, (q;q)_n^{-1}\, t_3^m\, t_4^n$ and then summing over m and n from zero to infinity with the aid of (4.1.19) and the q-binomial formula [27]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, z^n = \frac{(az;q)_\infty}{(z;q)_\infty}\,, \tag{4.1.20}$$

we obtain the Ramanujan-type representation for the Askey-Wilson q-beta integral [10, 8]

$$\int_{-\infty}^{\infty} \prod_{j=1}^{4} e_q(it_je^{-i\kappa x})\, e_q(-it_je^{i\kappa x})\, e^{-x^2}\cos\kappa x\, dx =$$

$$\sqrt{\pi}\, q^{\frac{1}{8}}\, E_q(-t_1t_2t_3t_4) \prod_{1\le j<k\le 4} e_q(t_jt_k)\,, \tag{4.1.21}$$

where $E_q(z)$ is the q-exponential function (3.42). The polynomials, orthogonal with respect to the weight function, represented by the integrand in the left side of (4.1.21), belong to the well-known family of the Askey-Wilson polynomials [7] (*cf.* (3.21))

$$p_n(\sin\kappa x; t_1, t_2, t_3, t_4|q) =$$

$$= (t_1t_2, t_1t_3, t_1t_4; q)_n\, t_1^{-n}\, {}_4\phi_3 \left[\begin{matrix} q^{-n},\ t_1t_2t_3t_4q^{n-1},\ it_1e^{-i\kappa x},\ -it_1e^{i\kappa x} \\ t_1t_2\ ,\ t_1t_3\ ,\ t_1t_4 \end{matrix}; q, q\right]. \tag{4.1.22}$$

The use of the formulas (4.1.13), (4.1.21) and the Pfaff-Saalschütz q-sum [27]

$${}_3\phi_2 \left[\begin{matrix} q^{-n},\ a,\ b \\ c,\ abc^{-1}q^{1-n} \end{matrix}; q\, ,\, q\right] = \frac{(c/a,\ c/b\,; q)_n}{(c,\ c/ab; q)_n} \tag{4.1.23}$$

leads to the Ramanujan-type orthogonality relation

$$\int_{-\infty}^{\infty} p_m(\sin\kappa x; t_1, t_2, t_3, t_4|q)\, p_n(\sin\kappa x; t_1, t_2, t_3, t_4|q)\times$$

$$\prod_{j=1}^{4} e_q(it_je^{-i\kappa x})\, e_q(-it_je^{i\kappa x})\, e^{-x^2}\cos\kappa x\, dx = \tag{4.1.24}$$

$$\sqrt{\pi}\, q^{\frac{1}{8}}\, (q;q)_n \frac{E_q(-t_1t_2t_3t_4q^{n-1})}{1 - t_1t_2t_3t_4q^{2n-1}} \prod_{1\le j<k\le 4} e_q(q^n\, t_j\, t_k)\; \delta_{mn}$$

for the Askey-Wilson polynomials (4.1.22), obtained by a different method in the Refs. [10, 9].

4.2 The continuous q^{-1}-Hermite ladder $(q > 1)$

The existence of the orthogonality relation of Ramanujan-type

$$\int_{-\infty}^{\infty} h_m(\sinh\kappa x|q)\, h_n(\sinh\kappa x|q)\, e^{-x^2}\cosh\kappa x\, dx = \sqrt{\pi}\, q^{-\frac{1}{2}(n+\frac{1}{2})^2}\, (q;q)_n\, \delta_{mn} \tag{4.2.1}$$

for the continuous q^{-1}-Hermite polynomials [5]

$$h_n(x|q) = i^{-n} H_n(ix|q^{-1}) \tag{4.2.2}$$

became clear even before that for the continuous q-Hermite polynomials $H_n(x|q)$. It has emerged in the course of constructing a simple difference realization of the Heisenberg q-algebra on the full real line [16]. From the definition (4.2.2) and inversion formula

$$(q^{-1}; q^{-1})_n = (-1)^n\, q^{-\frac{n(n+1)}{2}}\, (q;q)_n\,, \qquad n = 0, 1, 2, \ldots \tag{4.2.3}$$

it follows that the orthogonality relations (4.1.1) and (4.2.1) are in fact related to each other by the substitution $q \to q^{-1}$ $(\kappa \to i\kappa)$. In other words, derivation of the Ramanujan-type orthogonality relation (4.2.1) goes through in exactly the same way as of (4.1.1) in the previous section, namely, by using the linearization formula $(m \le n)$

$$h_m(x|q)\, h_n(x|q) = \sum_{j=0}^{m} (-q^j)^{j-m} \begin{bmatrix} m \\ j \end{bmatrix}_q (q^{-n}; q)_{m-j}\, h_{n-m+2j}(x|q)\,, \tag{4.2.4}$$

the explicit representation for the continuous q^{-1}-Hermite polynomials

$$h_n(\sinh \kappa x|q) = \sum_{k=0}^{n} (-1)^k \, q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{(n-2k)\kappa x} \,, \tag{4.2.5}$$

and the integral (4.1.5) with $y = it$.

In complete analogy with the case $0 < q < 1$, we can combine the orthogonality relation (4.2.1) with the generating function

$$\sum_{n=0}^{\infty} \frac{(-it)^n \, q^{\frac{n(n-1)}{2}}}{(q;q)_n} h_n(\sinh \kappa x|q) = (ite^{\kappa x}\,, -ite^{-\kappa x}\,; q)_\infty = E_q(ite^{-\kappa x})\, E_q(-ite^{\kappa x}) \tag{4.2.6}$$

to obtain the representation (*cf.* (4.1.12))

$$\int_{-\infty}^{\infty} \prod_{v=t,\tau} E_q(ive^{-\kappa x})\, E_q(-ive^{\kappa x})\, e^{-x^2} \cosh \kappa x \, dx = \sqrt{\pi} q^{-\frac{1}{8}} \, E_q(-q^{-1} t\tau)\,. \tag{4.2.7}$$

Polynomials, orthogonal with respect to the integrand in (4.2.7), are again the Al-Salam-Chihara polynomials (*cf.* the definition (4.2.2))

$$\widetilde{p}_n(\sinh \kappa x; t, \tau; q) = i^{-n} p_n(i \sinh \kappa x; t, \tau; q^{-1}) =$$

$$\frac{(t\tau; q^{-1})_n}{(it)^n} \, {}_3\phi_2 \left[\begin{matrix} q^n, ite^{\kappa x}\,, \; -ite^{-\kappa x} \\ t\tau, \; 0 \end{matrix} ; q^{-1}, q^{-1} \right] , \tag{4.2.8}$$

but corresponding to the case when the parameter $q > 1$. To verify that one needs to reverse the order of summation in (4.2.8) , leading to the explicit sum of the form (*cf.* [23])

$$\widetilde{p}_n(\sinh \kappa x; t, \tau; q) = q^{-\frac{n(n-1)}{2}} \, (it)^n \, (it^{-1} e^{\kappa x}, -it^{-1} e^{-\kappa x}; q)_n \times$$

$$\sum_{n=0}^{n} \frac{(q^{-n}, t\tau q^{1-n}; q)_k \, q^k}{(itq^{1-n} e^{\kappa x}, \; -itq^{1-n} e^{-\kappa x}\,, \; q; q)_k} \,. \tag{4.2.9}$$

In fact, let us consider the product

$$\widetilde{p}_n(\sinh \kappa x; t, \tau; q) \, E_q(iqte^{-\kappa x})\, E_q(-iqte^{\kappa x})\,. \tag{4.2.10}$$

First, the factor $(it^{-1} e^{\kappa x}, -it^{-1} e^{-\kappa x}; q)_n$ from the representation (4.2.9) is easily absorbed into the q-exponential functions in the above product, since

$$E_q(z) \, (-\frac{q}{z}; q)_n = q^{\frac{n(n+1)}{2}} \, z^{-n} \, E_q(zq^{-n})\,. \tag{4.2.11}$$

Then, the resulting q-exponential functions $E_q(itq^{1-n}\, e^{-\kappa x})$ and $E_q(-itq^{1-n}\, e^{\kappa x})$ can be combined with the factors $(-itq^{1-n}\, e^{-\kappa x}; q)_k$ and $(itq^{1-n}\, e^{\kappa x}; q)_k$ in the denominator of the summand in (4.2.9), respectively, by using the formula

$$E_q(z) = (-z; q)_k \, E_q(zq^k)\,. \tag{4.2.12}$$

Thus, the product (4.2.10) may be written as

$$\widetilde{p}_n(\sinh \kappa x; t, \tau; q) \, E_q(iqte^{-\kappa x})\, E_q(-iqte^{\kappa x}) =$$

$$\frac{q^{\frac{n(n-1)}{2}}}{(-it)^n} \sum_{k=0}^{n} \frac{(q^{-n}, t\tau q^{1-n}; q)_k \, q^k}{(q;q)_k} E_q(itq^{k+1-n} \, e^{-\kappa x}) \, E_q(-itq^{k+1-n} \, e^{\kappa x}) \,, \tag{4.2.13}$$

where the entire dependence of this product on the variable x is accumulated in the arguments of two q-exponential functions only. In other words, the derivation of an orthogonality relation for the Al-Salam-Chihara polynomials (4.2.8) reduces to computing the known integral (4.2.7), namely,

$$I_{mn}(t,\tau;q) = \int_{-\infty}^{\infty} \tilde{p}_m(\sinh \kappa x; t, \tau; q) \, \tilde{p}_n(\sinh \kappa x; t, \tau; q) \times$$

$$\prod_{v=t,\tau} E_q(iqve^{-\kappa x}) \, E_q(-iqve^{\kappa x}) \, e^{-x^2} \cosh \kappa x \, dx = \sqrt{\pi} \, \frac{q^{\frac{n(n-1)}{2} + \frac{m(m-1)}{2} - \frac{1}{8}}}{(-it)^n \, (-i\tau)^m} \times$$

$$\sum_{k=0}^{n} \frac{(q^{-n}, t\tau q^{1-n}; q)_k \, q^k}{(q;q)_k} \sum_{j=0}^{m} \frac{(q^{-m}, t\tau q^{1-m}; q)_j \, q^j}{(q;q)_j} E_q(-t\tau q^{k+j-n-m+1}) \,. \tag{4.2.14}$$

With the aid of the formulas (4.2.12) and (4.1.18) we can sum over j , which yields

$$I_{mn}(t,\tau;q) = \sqrt{\pi} \, i^{n+m} q^{\frac{n(n-1)}{2} - \frac{m(m-1)}{2} - \frac{1}{8}} \, t^{m-n} \times$$

$$E_q(-t\tau q^{1-n}) \, (q^{-n};q)_m \sum_{k=0}^{n} \frac{(q^{m-n};q)_k}{(q;q)_k} \,. \tag{4.2.15}$$

The factor $(q^{-n};q)_m$ is non-vanishing only if $m \le n$; but the integral (4.2.14) is symmetric in n and m, therefore it is nonzero for $m = n$ only, i.e.,

$$I_{mn}(t,\tau;q) = \sqrt{\pi} \, q^{-\frac{1}{2}(n+\frac{1}{2})^2} \, (q;q)_n \, E_q(-t\tau q^{1-n}) \, \delta_{mn} \,, \tag{4.2.16}$$

where we have used the formula [27]

$$(q^{-n};q)_n = (-1)^n \, q^{-\frac{n(n+1)}{2}} \, (q;q)_n \,.$$

A counterpart of the integral (4.1.17) for the Al-Salam-Chihara polynomials (4.2.8) in the case under consideration has the form

$$\int_{-\infty}^{\infty} \widetilde{p}_n(\sinh \kappa x; t, \tau; q) \, E_q(iqte^{-\kappa x}) \, E_q(-iqte^{\kappa x}) \, e^{-x^2} \cosh \kappa x \, dx = \sqrt{\pi} \, (i\tau)^n \, q^{-\frac{1}{2}(n-\frac{1}{2})^2} \,. \tag{4.2.17}$$

Having obtained at this intermediate stage the Ramanujan-type orthogonality relation (4.2.16) for the Al-Salam-Chihara polynomials (4.2.8), for the next step to be made we need to determine an analogue of the generating function (4.1.19) for the case when $q > 1$. To this end one can reverse the order of summation in (4.2.8), apply the q-analogue of the Pfaff-Kummer transformation [27] (these two steps are discussed in detail in Ref.[23]), and, finally, use the inversion formula (with respect to the transformation $q \to q^{-1}$)

$$(a;q^{-1})_n = (a^{-1};q)_n \, (-a)^n \, q^{-\frac{n(n-1)}{2}} \,, \tag{4.2.18}$$

to express a multiple of the polynomials (4.2.8) in the form of a Cauchy product of two sequences , namely,

$$\widetilde{p}_n(\sinh \kappa x; t, \tau; q) = q^{-\frac{n(n-1)}{2}} \, (q;q)_n \, (it)^n \sum_{k=0}^{n} \frac{(-i\tau^{-1}e^{-\kappa x};q)_k}{(q;q)_k} \, \frac{(it^{-1}e^{\kappa x};q)_{n-k}}{(q;q)_{n-k}} \left(\frac{\tau}{t}\right)^k \,. \tag{4.2.19}$$

From the q-binomial theorem (4.1.20) and the representation (4.2.19) it follows that the required generating function has the form

$$\sum_{n=0}^{\infty} \frac{(-is)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} \widetilde{p}_n(\sinh \kappa x; t, \tau; q) = \frac{(ise^{\kappa x}, -ise^{-\kappa x}; q)_\infty}{(st, s\tau; q)_\infty} \tag{4.2.20}$$

and it contains (4.2.6) as the particular case when $t = \tau = 0$. The orthogonality relation (4.2.16) and the generating function (4.2.20) imply the Ramanujan-type representation [10, 9]

$$\int_{-\infty}^{\infty} \prod_{j=1}^{4} E_q(iqt_j e^{-\kappa x})\, E_q(-iqt_j e^{\kappa x})\, e^{-x^2} \cosh \kappa x \, dx =$$

$$\sqrt{\pi} q^{-1/8}\, e_q(qt_1t_2t_3t_4) \prod_{1 \le j < k \le 4} E_q(-qt_jt_k) \tag{4.2.21}$$

for the Askey-Wilson q-beta integral [7] with $q > 1$. The corresponding orthogonal polynomials

$$\widetilde{p}_n(\sinh \kappa x; t_1, t_2, t_3, t_4|q) = i^{-n} p_n(i \sinh \kappa x; t_1, t_2, t_3, t_4|q^{-1}) \tag{4.2.22}$$

can be interpreted as the continuous q^{-1}-Askey-Wilson polynomials (*cf.* the formulas (4.1.22) and (4.2.2)).Indeed, with the aid of (4.2.11), (4.2.21) and the Pfaff-Saalschütz q-sum (4.1.23) it is not difficult to verify that [10, 9]

$$\int_{-\infty}^{\infty} \widetilde{p}_m(\sinh \kappa x; t_1, t_2, t_3, t_4|q)\, \widetilde{p}_n(\sinh \kappa x; t_1, t_2, t_3, t_4|q) \times$$

$$\prod_{j=1}^{4} E_q(iqt_j e^{-\kappa x})\, E_q(-iqt_j e^{\kappa x}) e^{-x^2} \cosh \kappa x \; dx = \sqrt{\pi}\, q^{-\frac{1}{2}(n+\frac{1}{2})^2} \times$$

$$(q;q)_n \, \frac{e_q(t_1t_2t_3t_4q^{2-n})}{1 - t_1t_2t_3t_4q^{1-2n}} \prod_{1 \le j < k \le 4} E_q(-t_jt_kq^{1-n}) \delta_{mn}\,, \tag{4.2.23}$$

provided that $t_1t_2t_3t_4 \neq q^k$, where k is an arbitrary integer number. When any two of the parameters t_j, $j = 1, 2, 3, 4$, are equal to zero, the Ramanujan-type orthogonality relation (4.2.23) reproduces the orthogonality relation (4.2.16) for the Al-Salam-Chihara polynomials (4.2.8), whereas when all of them vanish, it reduces to that for the continuous q^{-1}-Hermite polynomials (4.2.1).

5 BIORTHOGONAL RATIONAL FUNCTIONS

The purpose of this section is to apply the same technique of constructing Ramanujan-type measures, as in the previous one, to a family of ${}_4\phi_3$ biorthogonal rational functions [3], which are unit circle analogues of the Askey-Wilson polynomials. It will be shown that Ramanujan-type continuous measures for a hierarchy of ${}_4\phi_3$ biorthogonal rational functions can be systematically built from simple cases of the Rogers-Szegő (for $0 < q < 1$) and Stieltjes–Wigert (for $q > 1$) polynomials by using the Berg–Ismail procedure of attaching generating functions to measures.

5.1 The Rogers-Szegő ladder $(0 < q < 1)$.

The q-analogue of Hermite polynomials on the unit circle are the Rogers-Szegő polynomials [36, 1, 25]

$$H_n(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k = {}_2\phi_0(q^{-n}, 0; q, zq^n)\,, \tag{5.1.1}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient. The basic hypergeometric function ${}_2\phi_0$ in (5.1.1) represents a particular case ($r = 2$, $s = 0$, $a_1 = q^{-n}$, $a_2 = 0$) of the general definition

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n\,. \tag{5.1.2}$$

Szegő [36] has proved that the polynomials (5.1.1) satisfy the orthogonality relation

$$\frac{1}{2\pi i} \oint_{|z|=1} H_m(-q^{-\frac{1}{2}} z^*; q)\, H_n(-q^{-\frac{1}{2}} z; q)\, \vartheta_3(\frac{\log z}{2i}, q^{1/2})\, \frac{dz}{z} = \frac{(q;q)_m}{q^m}\, \delta_{mn}\,, \tag{5.1.3}$$

where $\vartheta_3(z, q)$ is the theta-function, *i.e.*,

$$\vartheta_3(z, q) \equiv \vartheta_3(z|\tau) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2ikz} \tag{5.1.4}$$

and $q = \exp(\pi i \tau)$ [37]. The orthogonality relation (5.1.3) on the unit circle can be transformed into the Ramanujan-type orthogonality of the form [18]

$$\int_{-\infty}^{\infty} H_m(-q^{-\frac{1}{2}} e^{2i\kappa x}; q) H_n(-q^{-\frac{1}{2}} e^{-2i\kappa x}; q)\, e^{-x^2}\, dx = \sqrt{\pi}\, \frac{(q;q)_n}{q^n}\, \delta_{mn}\,. \tag{5.1.5}$$

Once the relation (5.1.5) is established, it is readily verified by a direct evaluation with the aid of the explicit representation (5.1.1), the Fourier transformation

$$\int_{-\infty}^{\infty} e^{-x^2+2ixy}\, dx = \sqrt{\pi}\, e^{-y^2}\,, \tag{5.1.6}$$

and Gauss' identity (4.1.7).

It follows from (5.1.1) and the q-binomial theorem (4.1.20) that the generating function for the Rogers-Szegő polynomials has the form [25]

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(x;q) = e_q(t)\, e_q(xt)\,. \tag{5.1.7}$$

Hence, combining (5.1.7) with the orthogonality relation (5.1.5) leads to the particular case

$$\int_{-\infty}^{\infty} e_q(-te^{2i\kappa x})\, e_q(-\tau e^{-2i\kappa x})\, e^{-x^2}\, dx = \sqrt{\pi}\, e_q(t\tau)\, E_q(-q^{\frac{1}{2}} t)\, E_q(-q^{\frac{1}{2}} \tau) \tag{5.1.8}$$

of a Ramanujan's q-extension of Cauchy's form of the beta integral [33, 4, 32], namely

$$\int_{-\infty}^{\infty} e_q(\alpha e^{2i\kappa x})\, e_q(\beta e^{-2i\kappa x})\, e^{-x^2+2mx}\, dx =$$

$$= \sqrt{\pi}\, e^{m^2}\, e_q(\alpha\beta)\, E_q(\alpha q^{1/2} e^{2im\kappa})\, E_q(\beta q^{1/2} e^{-2im\kappa})\,. \tag{5.1.9}$$

Two points demand comments, however. First, it is common to express (see, for example, Refs.[27, 33, 4, 32]) two integrals of Ramanujan, *i.e.*, (5.1.9) and a q-extension of Euler's beta integral

$$\int_{-\infty}^{\infty} E_q(\alpha q^{1/2}e^{2\kappa x})\, E_q(\beta q^{1/2}e^{-2\kappa x})\, e^{-x^2+2mx}\, dx =$$

$$= \sqrt{\pi}\, e^{m^2}\, E_q(-\alpha\beta) e_q(\alpha e^{2m\kappa})\, e_q(\beta e^{-2m\kappa})\,, \tag{5.1.10}$$

in terms of the infinite product $(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n)$, rather than the q-exponential functions (3.27) and (3.42). The advantage of these forms (5.1.9) and (5.1.10) is that they exhibit the fact that they are related to each other by the transformation $q \to q^{-1}$ (*i.e.*, $\kappa \to i\kappa$) (see Refs.[12, 13, 11] for a more detailed discussion of this point). Second, since the term $2mx$ in the argument of exponential function $\exp(-x^2+2mx)$ can be removed by the translation $x \to x+m$ and redefinition of the parameters α and β, to simplify a proof of (5.1.9) and (5.1.10) it is usually assumed that $m = 0$. Thus the parameter m in (5.1.9) and (5.1.10) is supposed to be real. In fact, these formulas are also valid for arbitrary complex m [15]. This can be proved by a direct evaluation of integrals (5.1.9) and (5.1.10) with the aid of the formulas (5.1.6), (4.1.20) and

$$(-\frac{q}{z};q)_n\, E_q(z) = q^{\frac{n(n+1)}{2}}\, z^{-n}\, E_q(zq^{-n})\,. \tag{5.1.11}$$

As the real part of a complex number m can be recovered by the shift $x \to x - \Re m$, without loss of generality it may be assumed that m is imaginary, *i.e.*, $m = iy$. Consequently, the representations (5.1.9) and (5.1.10) are also instances of Fourier transformations and this circumstance will be used in what follows.

Now we return to the integrand in (5.1.8), which represents a weight function for the biorthogonal polynomials

$$p_n(z;t,\tau;q) := (q^{-\frac{1}{2}}\tau;q)_n\ {}_2\phi_1(q^{-n}, q^{\frac{1}{2}}t;\ q^{\frac{3}{2}-n}/\tau; q, qz/\tau) =$$

$$\frac{(t\tau;q)_n}{(q^{\frac{1}{2}}t)^n}\ {}_3\phi_2\left[\begin{matrix} q^{-n},\ tz,\ q^{\frac{1}{2}}t \\ t\tau,\ 0 \end{matrix}\ ;\ q,\ q\right], \tag{5.1.12}$$

studied by P.I.Pastro [32]. Observe that the second line in (5.1.12) follows from the Jackson transformation formula [27]

$${}_2\phi_1(q^{-n},\ b;\ c;\ q,\ z) = \frac{(c/b;q)_n}{(c;q)_n}\ {}_3\phi_2\left[\begin{matrix} q^{-n},\ b,\ bz/cq^n \\ bq^{1-n}/c,\ 0 \end{matrix}\ ;\ q,\ q\right] \tag{5.1.13}$$

for terminating basic hypergeometric series ${}_2\phi_1$, upon employing the relation

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}\left(\frac{a}{b}\right)^n. \tag{5.1.14}$$

The normalization of polynomials (5.1.12) is chosen in such a way, that when $t = \tau = 0$ they coincide with the Rogers-Szegő polynomials (5.1.1), *i.e.*,

$$p_n(z;0,0;q) = H_n(q^{-\frac{1}{2}}z;q)\,. \tag{5.1.15}$$

This formula follows from the readily verified limit relation

$$\lim_{c\to\infty} {}_2\phi_1(q^{-n},a;c;q,cz) = {}_2\phi_0(q^{-n},a;q,z)\,. \tag{5.1.16}$$

An explicit evaluation of the Ramanujan-type biorthogonality integral

$$\int_{-\infty}^{\infty} p_m(-e^{2i\kappa x};\, t,\tau;q)\, p_n(-e^{-2i\kappa x};\tau,\, t;\, q)\, e_q(-te^{2i\kappa x})\, e_q(-\tau e^{-2i\kappa x})\, e^{-x^2}\, dx =$$

$$\sqrt{\pi}\,\frac{(q;q)_n}{q^n}\, e_q(q^n t\tau)\, E_q(-q^{\frac{1}{2}}t)\, E_q(-q^{\frac{1}{2}}\tau)\, \delta_{mn} \tag{5.1.17}$$

is given in the section 5.3.

Multiplying both sides of the integral (5.1.17) by a factor $t_3^m\, t_4^n\,(q;q)_m^{-1}\,(q;q)_n^{-1}$ and summing over indices m and n from zero to infinity with the aid of the generating function [3]

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\, p_n(z;a,b;q) = \frac{(atz, q^{-\frac{1}{2}}bt;q)_\infty}{(q^{-\frac{1}{2}}tz, t;q)_\infty} = \frac{e_q(t)\, e_q(q^{-\frac{1}{2}}tz)}{e_q(q^{-\frac{1}{2}}bt)\, e_q(atz)} \tag{5.1.18}$$

for the polynomials (5.1.12), we get $(t = t_1, \tau = t_2)$

$$\int_{-\infty}^{\infty} e^{-x^2}\, E_q(q^{\frac{1}{2}}t_1t_3e^{2i\kappa x})\, E_q(q^{\frac{1}{2}}t_2t_4e^{-2i\kappa x})\,\prod_{j=1}^{4} e_q(-t_je^{(-)^{j+1}2i\kappa x})\, dx =$$

$$\sqrt{\pi}\, e_q(t_1t_2)\ e_q(t_2t_3)\, e_q(t_3t_4)\, e_q(t_1t_4)\, E_q(-t_1t_2t_3t_4)\,\prod_{j=1}^{4} E_q(-q^{\frac{1}{2}}t_j)\,. \tag{5.1.19}$$

The integrand in (5.1.19) represents a weight function for the biorthogonality relation

$$\int_{-\infty}^{\infty} r_m(-e^{2i\kappa x};t_1,t_2,t_3,t_4;q)\, r_n(-e^{-2i\kappa x};t_2,t_1,t_4,t_3;q)\, e^{-x^2}\, E_q(q^{\frac{1}{2}}t_1t_3e^{2i\kappa x})\times$$

$$E_q(q^{\frac{1}{2}}t_2t_4e^{-2i\kappa x})\,\prod_{j=1}^{4} e_q(-t_je^{(-)^{j+1}2i\kappa x})\, dx = \sqrt{\pi}\,\frac{(q,t_1t_2,t_3t_4,t_1t_2t_3t_4q^{n-1};q)_n}{q^n\,(t_1t_2t_3t_4;q)_{2n}}\times$$

$$\delta_{mn}\, e_q(t_1t_2)\ e_q(t_2t_3)\, e_q(t_3t_4)\, e_q(t_1t_4)\, E_q(-t_1t_2t_3t_4)\,\prod_{l=1}^{4} E_q(-q^{\frac{1}{2}}t_l)\,, \tag{5.1.20}$$

which is satisfied by ${}_4\phi_3$ rational functions of Al-Salam and Ismail [3]

$$r_n(z;t_1,t_2,t_3,t_4;q) = \frac{(t_1t_2;q)_n}{(q^{\frac{1}{2}}t_1)^n}\,{}_4\phi_3\left[\begin{matrix} q^{-n},\ t_1z,\ q^{\frac{1}{2}}t_1,\ t_1t_2t_3t_4q^{n-1} \\ q^{\frac{1}{2}}t_1t_3z,\ t_1t_2,\ t_1t_4\end{matrix};q,q\right]. \tag{5.1.21}$$

The Ramanujan-type biorthogonality relation (5.1.20) was derived by a different method in [14], but it can be also directly evaluated in exactly the same way as the integral (5.1.17) (see section 5.3), namely, by using the definition (5.1.21), the representation (5.1.19) and the Pfaff-Saalschütz (5.3.7) and Chu-Vandermonde (5.3.11) q-sums. The normalization in (5.1.21) is chosen in such a way that for $t_3 = t_4 = 0$ these rational functions and the biorthogonality relation (5.1.20) reduce to the biorthogonal polynomials (5.1.12) and the relation (5.1.17), respectively. Consequently, for $t_j = 0,\ 1 \le j \le 4$, the rational functions (5.1.21) coincide with the Rogers-Szegő polynomials (5.1.1), satisfying the Ramanujan-type orthogonality relation (5.1.5).

5.2 The Stieltjes-Wigert ladder $(q > 1)$

The Stieltjes-Wigert polynomials [27]

$$S_n(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} z^k = {}_1\phi_1(q^{-n};\, 0;\, q,\, zq^{n+1}) \tag{5.2.1}$$

represent the lower level in another ladder of ${}_4\phi_3$ biorthogonal rational functions [3], corresponding to the values $q > 1$ of the parameter q. They are related to the Rogers-Szegő polynomials (5.1.1) by the Fourier transformation [18]

$$S_n(\alpha\, e^{2\kappa x};\, q)\, e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(\alpha\, e^{-2i\kappa y};\, q)\, e^{ixy-\frac{y^2}{2}}\, dy\,, \tag{5.2.2}$$

where α is an arbitrary complex number. It is interesting to compare (5.2.2) with the relation

$$S_n(x;q) = \frac{1}{\pi} \int_{0}^{\pi} H_n(x\, e^{i\varphi};q)\, \vartheta_3(\varphi,q)\, d\varphi\,, \tag{5.2.3}$$

derived by L.Carlitz in Ref. [25].

From (5.1.5) and (5.2.2) follows that the Stieltjes-Wigert polynomials (5.2.1) satisfy the Ramanujan-type orthogonality relation of the form [18]

$$\int_{-\infty}^{\infty} S_m(-q^{-\frac{1}{2}}\, e^{2\kappa x};\, q)\, S_n(-q^{-\frac{1}{2}}\, e^{2\kappa x};\, q)\, e^{-x^2}\, dx = \sqrt{\pi}\, \frac{(q;q)_m}{q^m}\, \delta_{mn}\,. \tag{5.2.4}$$

This can also be readily verified by a direct evaluation of (5.2.4) with the aid of the explicit representation (5.2.1), the formula (*cf.* (5.1.6))

$$\int_{-\infty}^{\infty} e^{2xy-x^2}\, dx = \sqrt{\pi}\, e^{y^2}\,, \tag{5.2.5}$$

and the identity (4.1.7).

As in the previous section for the Rogers-Szegő ladder we could have started with the orthogonality relation (5.2.4) and the generating function [28]

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n}\, S_n(z;q) = e_q(t)\, {}_0\phi_1(0;q,qzt) \tag{5.2.6}$$

for the Stieltjes-Wigert polynomials (5.2.1). But, for the approach under consideration, it is important that generating functions are expressed in terms of the q-exponential functions (5.1.10) and (5.1.14), because biorthogonality relations at every next level follow from their properties (5.1.15) and (5.3.5). Fortunately, there exist along with (5.2.4) another orthogonality relation (*cf.* Ref.[25])

$$\int_{-\infty}^{\infty} S_m(-q^{\frac{1}{2}-m}\, e^{2\kappa x};\, q)\, S_n(-q^{\frac{1}{2}-n}\, e^{-2\kappa x};\, q)\, e^{-x^2}\, dx =$$

$$= \sqrt{\pi}\, (-1)^n\, q^{-\frac{n(n-1)}{2}}\, (q;\, q)_n\, \delta_{mn}\,, \tag{5.2.7}$$

since the Stieltjes and Hamburger moment problems [35] for the Stieltjes-Wigert polynomials (5.2.1) are indeterminate. This is a direct consequence of (5.1.5) and the identity

$$H_n(z;q^{-1}) = S_n(q^{-n}z;q)\,, \tag{5.2.8}$$

which follows from the definitions (5.1.1) and (5.2.1) upon using the transformation property of the q-binomial coefficients [27]

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{5.2.9}$$

In analogy with the q^{-1}-Hermite polynomials $h_n(x|q) = i^{-n} H_n(ix|q^{-1})$ [5] it is convenient to consider Stieltjes-Wigert polynomials of the form

$$s_n(z;\, q) = i^{-n} H_n(z; q^{-1}) = i^{-n} S_n(q^{-n} z; q) . \tag{5.2.10}$$

To distinguish them from the standard Stieltjes-Wigert polynomials (5.2.1), they will be denoted by the small letter s. From (5.2.7) it is evident that

$$\int_{-\infty}^{\infty} s_m(-q^{\frac{1}{2}} e^{2\kappa x};\, q)\, s_n(-q^{\frac{1}{2}} e^{-2\kappa x};\, q)\, e^{-x^2}\, dx = \sqrt{\pi}\, q^{-\frac{n(n-1)}{2}} (q;q)_n\, \delta_{mn}. \tag{5.2.11}$$

A generating function for the polynomials (5.2.10) is [25]

$$\sum_{n=0}^{\infty} \frac{(-it)^n}{(q;q)_n}\, q^{\frac{n(n-1)}{2}}\, s_n(z;q) = E_q(-t)\, E_q(-zt) . \tag{5.2.12}$$

This can be easily checked by using the well-known Cauchy multiplication rule

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k\, b_{n-k} = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} b_k$$

for the product of two sequences $\{a_n\}$ and $\{b_k\}$, together with the definition of the q-exponential function (5.1.14). Combining (5.2.12) with (5.2.11) leads to the integral

$$\int_{-\infty}^{\infty} E_q(q^{\frac{1}{2}}\, t\, e^{2\kappa x})\, E_q(q^{\frac{1}{2}}\, \tau\, e^{-2\kappa x})\, e^{-x^2}\, dx = \sqrt{\pi}\, e_q(t)\, e_q(\tau)\, E_q(-t\tau) , \tag{5.2.13}$$

which is a particular case of Ramanujan's identity (5.1.10) with the vanishing parameter m.

The integrand in (5.2.13) represents a weight function for the biorthogonal polynomials

$$\widetilde{p}_n(z;t,\tau;q) = i^{-n}\, p_n(z;\, t,\, \tau;\, q^{-1}) = i^{-n}\, (q^{1/2}\, \tau;\, q^{-1})_n\, {}_2\phi_1(q^{-n},\, q^{\frac{1}{2}} t^{-1};\, \tau\, q^{\frac{3}{2}-n};\, q,\, qtz) , \tag{5.2.14}$$

where $p_n(z;\, t, \tau;\, q)$ are defined by (5.1.12). Indeed, it is not difficult to verify by a direct evaluation (actually repeating the same line of reasoning *mutatis mutandis* as for the case of $0 < q < 1$, considered in the section 5.1), that the following biorthogonality relation holds

$$\int_{-\infty}^{\infty} \widetilde{p}_m(-e^{2\kappa x};\, t,\, \tau;\, q)\, \widetilde{p}_n(-e^{-2\kappa x};\, \tau,\, t;\, q)\, E_q(qte^{2\kappa x})\, E_q(q\tau e^{-2\kappa x})\, e^{-x^2}\, dx =$$

$$\sqrt{\pi}\, q^{-\frac{n(n-1)}{2}} (q;q)_n\, E_q(-q^{1-n}\, t\tau)\, e_q(q^{\frac{1}{2}} t)\, e_q(q^{\frac{1}{2}} \tau)\, \delta_{mn} . \tag{5.2.15}$$

Observe that for $t = \tau = 0$ the polynomials (5.2.14) reduce to the Stieltjes-Wigert polynomials (5.2.10), *i.e.*,

$$\widetilde{p}_n(z;0,0;q) = s_n(q^{\frac{1}{2}} z;q) . \tag{5.2.16}$$

This follows from (5.2.14), (5.2.1) and (5.2.10) upon using the limit relation

$$\lim_{a\to\infty} {}_2\phi_1(q^{-n}, a; c; q, z/a) = {}_1\phi_1(q^{-n}; c; q, z) . \tag{5.2.17}$$

A generating function for the polynomials (5.2.14)

$$\sum_{n=0}^{\infty} \frac{(-i\alpha)^n}{(q;q)_n}\, q^{\frac{n(n-1)}{2}}\, \widetilde{p}_n(z;t,\tau;q) = \frac{e_q(q^{\frac{1}{2}}\alpha\tau)\, e_q(\alpha t z)}{e_q(\alpha)\, e_q(q^{\frac{1}{2}}\alpha z)} \tag{5.2.18}$$

can be obtained either by transforming q into q^{-1} in (5.1.18), or by a direct evaluation of the left member of (5.2.18) with the aid of the q-binomial theorem (4.1.20). The biorthogonality relation (5.2.15) and the generating functtion (5.2.18) imply the integral representation

$$\int_{-\infty}^{\infty} e_q(-q^{\frac{1}{2}}t_1t_3e^{2\kappa x})\, e_q(-q^{\frac{1}{2}}t_2t_4e^{-2\kappa x}) \prod_{j=1}^{4} E_q(qt_je^{(-)^{j+1}2\kappa x})\, e^{-x^2}\, dx =$$

$$= \sqrt{\pi}\, E_q(-qt_1t_2)\, E_q(-qt_2t_3)\, E_q(-qt_3t_4)\, E_q(-qt_1t_4) e_q(qt_1t_2t_3t_4) \prod_{k=1}^{4} e_q(q^{\frac{1}{2}}t_k)\,. \tag{5.2.19}$$

The integrand in (5.2.19) represents a weight function for the biorthogonality relation [14]

$$\int_{-\infty}^{\infty} dx\, e^{-x^2}\, \widetilde{r}_m(-e^{2\kappa x};t_1,t_2,t_3,t_4;q)\, \widetilde{r}_n(-e^{-2\kappa x};t_2,t_1,t_4,t_3;q)\, e_q(-q^{\frac{1}{2}}t_1t_3e^{2\kappa x})\times$$

$$e_q(-q^{\frac{1}{2}}t_2t_4e^{-2\kappa x}) \prod_{j=1}^{4} E_q(qt_je^{(-)^{j+1}2\kappa x}) = \frac{\sqrt{\pi}\,(q;q)_n}{1-Tq^{1-2n}}\, q^{-\frac{n(n-1)}{2}} \times \tag{5.2.20}$$

$$E_q(-q^{1-n}t_1t_2)\, E_q(-q^{1-n}t_3t_4)\, E_q(-qt_1t_4)\, E_q(-qt_2t_3)\, e_q(q^{2-n}T) \prod_{k=1}^{4} e_q(q^{\frac{1}{2}}t_j)\,,$$

where $T \equiv t_1t_2t_3t_4 \neq q^k$, k is an arbitrary integer number, and q^{-1}-rational functions $\widetilde{r}_m(z)$ and $\widetilde{r}_n(z)$ of Al-Salam and Ismail are defined as (*cf.* (5.1.21))

$$\widetilde{r}_n(z;t_1,t_2,t_3,t_4;q) = i^{-n}\, r_n(z;t_1,t_2,t_3,t_4;q^{-1}) =$$

$$(iq)^n\, q^{-\frac{n^2}{2}}\, \frac{(\widetilde{t}_1\widetilde{t}_2;q)}{\widetilde{t}_2^{\,n}}\; {}_4\phi_3 \left[\begin{matrix} q^{-n},\, \widetilde{t}_1z^{-1},\, q^{\frac{1}{2}}\widetilde{t}_1,\, \widetilde{T}q^{n-1} \\ q^{\frac{1}{2}}\widetilde{t}_1\widetilde{t}_3z^{-1},\, \widetilde{t}_1\,\widetilde{t}_2,\, \widetilde{t}_1\widetilde{t}_4 \end{matrix} ;q,\, q \right] \tag{5.2.21}$$

with $\widetilde{t}_j = t_j^{-1}$, $j = 1,2,3,4$, and $\widetilde{T} = T^{-1}$. The biorthogonality relation (5.2.20) can be directly evaluated by using the definition (5.2.21) and the formulas (5.1.11), (5.3.5), (5.3.7) and (5.3.11).

5.3 Appendix

To evaluate the Ramanujan-type biorthogonality integral

$$I_{mn}(t,\tau;q) = \int_{-\infty}^{\infty} p_m(-e^{2i\kappa x};\, t,\, \tau;\, q)\, p_n(-e^{-2i\kappa x};\, \tau,\, t;\, q)\times$$

$$e_q(-te^{2i\kappa x})\, e_q(-\tau e^{-2i\kappa x})\, e^{-x^2}\, dx\,, \tag{5.3.1}$$

substitute the explicit expression

$$p_m(-e^{2i\kappa x};t,\,\tau;q) = (q^{-\frac{1}{2}}\tau;q)_m \sum_{j=0}^{m} \frac{(q^{-m},q^{\frac{1}{2}}t;q)_j}{(q,q^{\frac{3}{2}-m}/\tau;q)_j}\, (-\frac{q}{\tau}\, e^{2i\kappa x})^j \tag{5.3.2}$$

into (5.3.1) for both polynomials in it, which are defined by (5.1.12). This gives

$$I_{mn}(t,\tau;q) = (q^{-\frac{1}{2}}\tau;q)_m\,(q^{-\frac{1}{2}}t;q)_n \sum_{j=0}^{m} \frac{(q^{-m},q^{\frac{1}{2}}t;q)_j}{(q,q^{\frac{3}{2}-m}/\tau;q)_j}\,(-\frac{q}{\tau})^j \times$$

$$\sum_{k=0}^{n} \frac{(q^{-n},q^{\frac{1}{2}}\tau;q)_k}{(q,q^{\frac{3}{2}-n}/t;q)_k}\,(-\frac{q}{t})^k \int_{-\infty}^{\infty} e^{2i\kappa(j-k)x-x^2}\,e_q(-te^{2i\kappa x})\,e_q(-\tau e^{-2i\kappa x})\,dx\,. \tag{5.3.3}$$

The integration over the variable x in (5.3.3) is performed by (5.1.9) with $m = 2i\kappa(j-k)$, $\alpha = -t$ and $\beta = -\tau$, *i.e.*,

$$I_{mn}(t,\tau;q) = \sqrt{\pi}\,e_q(t\tau)\,(q^{-\frac{1}{2}}\tau;q)_m\,(q^{-\frac{1}{2}}t;q)_n \sum_{j=0}^{m} \frac{(q^{-m},q^{\frac{1}{2}}t;q)_j}{(q,q^{\frac{3}{2}-m}/\tau;q)_j} \left(-\frac{q}{\tau}\right)^j \times$$

$$\sum_{k=0}^{n} \frac{(q^{-n},q^{\frac{1}{2}}\tau;q)_k}{(q,q^{\frac{3}{2}-n}/t;q)_k} \left(-\frac{q}{t}\right)^k q^{\frac{1}{2}(j-k)^2}\,E_q(-t\,q^{j-k+\frac{1}{2}})\,E_q(-\tau\,q^{k-j+\frac{1}{2}})\,. \tag{5.3.4}$$

The formulas (5.1.11) and

$$E_q(z) = (-z;q)_k\,E_q(zq^k) \tag{5.3.5}$$

enable the right side of (5.3.4) to be written as

$$I_{mn}(t,\tau;q) = \sqrt{\pi}\,e_q(t\tau)\,E_q(-q^{\frac{1}{2}}t)\,E_q(-q^{\frac{1}{2}}\tau)\,(q^{-\frac{1}{2}}\tau;q)_m\,(q^{-\frac{1}{2}}t;q)_n \times$$

$$\sum_{j=0}^{m} \frac{(q^{-m},q^{\frac{1}{2}}/\tau;q)_j}{(q,q^{\frac{3}{2}-m}/\tau;q)_j}\,q^j \sum_{k=0}^{n} \frac{(q^{-n},\,q^{\frac{1}{2}}\tau,\,q^{\frac{1}{2}-j}/t;\,q)_k}{(q,\,q^{\frac{1}{2}-j}\tau,\,q^{\frac{3}{2}-n}/t;\,q)_k}\,q^k\,. \tag{5.3.6}$$

The sum with respect to the index k in (5.3.6) is a particular case of the Pfaff-Saalschütz q-sum

$${}_3\phi_2\left[\begin{matrix} q^{-n},\,a,\,b \\ c,\,abc^{-1}q^{1-n}\end{matrix};q,\,q\right] = \frac{(c/a,\,c/b;\,q)_n}{(c,\,c/ab;\,q)_n} \tag{5.3.7}$$

with $a = q^{\frac{1}{2}}\tau$, $b = q^{\frac{1}{2}-j}/t$ and $c = \tau q^{\frac{1}{2}-j}$, therefore it equals to

$$\frac{(q^{-j},\,t\tau;\,q)_n}{(q^{\frac{1}{2}-j}\tau,q^{-\frac{1}{2}}t;\,q)_n}\,.$$

Consequently,

$$I_{mn}(t,\tau;q) = \sqrt{\pi}\,(q^{-\frac{1}{2}}\tau;q)_m\,e_q(t\tau q^n)\,E_q(-q^{\frac{1}{2}}t)\,E_q(-q^{\frac{1}{2}}\tau)\times$$

$$\sum_{j=0}^{m} \frac{(q^{-m},\,q^{1/2}/\tau;\,q)_j}{(q,q^{\frac{3}{2}-m}/\tau;q)_j}\,q^j\,\frac{(q^{-j};\,q)_n}{(q^{\frac{1}{2}-j}\,\tau;q)_n}\,. \tag{5.3.8}$$

Due to the factor $(q^{-j};q)_n$ a sum over the index j in (5.3.8) is equal to zero for $n > m$, while for $n \le m$ it starts with $j = n$. Therefore it can be written as

$$\sum_{j=0}^{m} \frac{(q^{-m},\,q^{1/2}/\tau;\,q)_j}{(q,q^{\frac{3}{2}-m}/\tau;q)_j}\,q^j\,\frac{(q^{-j};\,q)_n}{(q^{\frac{1}{2}-j}\,\tau;q)_n} =$$

$$\frac{(q^{-m};q)_n}{(q^{\frac{3}{2}-m}/\tau;q)_n}\left(\frac{q^{\frac{1}{2}}}{\tau}\right)^n \sum_{l=0}^{N} \frac{(q^{-N},q^{\frac{1}{2}}/\tau;q)_l}{(q,\,q^{\frac{3}{2}-N}/\tau;q)_l}\,q^l\,, \tag{5.3.9}$$

where $N = m - n$ and we have used the formulas (5.1.14) and (see, for example, [27])

$$(a;q)_{n+k} = (a;q)_n\,(aq^n;\,q)_k\,. \tag{5.3.10}$$

The remaining sum over l is calculated by the Chu-Vandermonde formula

$$_2\phi_1(a, q^{-n}; c; q, q) = \frac{(c/a;q)_n}{(c;q)_n}\,a^n \tag{5.3.11}$$

and gives

$$\frac{(q^{1-N};q)_N}{(q^{\frac{3}{2}-N}/\tau;q)_N}\left(\frac{q^{\frac{1}{2}}}{\tau}\right)^N = \delta_{N0} = \delta_{mn}\,,$$

because of the evident relation $(q^{1-N};q)_N = \delta_{N0}$, valid for any nonnegative integer number N. Hence, the entire sum over j in (5.3.8) is equal to

$$\frac{(q^{-n};q)_n}{(q^{\frac{3}{2}-n}/\tau;q)_n}\left(\frac{q^{\frac{1}{2}}}{\tau}\right)^n \delta_{mn} = \frac{(q;q)_n}{q^n\,(q^{-\frac{1}{2}}\tau;q)_n}\,\delta_{mn}\,, \tag{5.3.12}$$

where we have again used (5.1.14) with $a = 1$ and $b = q^{\frac{3}{2}}/\tau$. Substituting (5.3.12) into (5.3.8), we obtain the following expression for the biorthogonality integral (5.3.1)

$$I_{mn}(t,\tau;q) = \sqrt{\pi}\,\frac{(q;q)_n}{q^n}\,e_q(q^n t\tau)\,E_q(-q^{\frac{1}{2}}t)\,E_q(-q^{\frac{1}{2}}\tau)\,\delta_{mn}$$

to complete the proof of (5.1.17).

Acknowledgments

I would like to thank Francisco Marcellań for inviting me to this workshop in honour of André Ronveaux. I am very grateful to Renato Álvarez-Nodarse, who has carefully read the manuscript and helped to correct some misprints.

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