Sea $f(x,y,z) = x^2y + e^x + z$. Probar que existe una función g diferenciable, definida en un entorno del punto $(1,-1)$ tal que $g(1,-1) = 0$ y $f(g(y,z),y,z) = 0$. Calcula $Dg(1,-1)$.

$f(x,y,z) = 0 \qquad x = g(y,z) \qquad f \in C^\infty(\mathbb{R}^3) \qquad f(0,1,-1) = 0+1-1=0$

$f_x(0,-1,-1) = 2xy + e^x \Big|_{(0,1,-1)} = 1 \neq 0$

1º $\exists\ x = g(y,z) \qquad g \in C^\infty(U(1,-1))$.

$Dg(y,z) = \begin{pmatrix} 0 & 1 \end{pmatrix}$

$g_y = -\dfrac{f'_y}{f'_x} = -\dfrac{x^2}{2xy + e^x}\Big|_{(0,1,-1)} = 0$

$g_z = -\dfrac{f'_z}{f'_x} = -\dfrac{1}{2xy+e^x}\Big|_{(0,1,-1)} = -1$

$F = z^3 + 2(x+y)^2 z + e^{z-1} - 4 = 0.$ punto $(0,-1,1)$ $\quad z = f(x,y)$

1) $F(0,-1,1) = 1+2+1-4 = 0$ $\quad$ 2) $F \in C^\infty(\mathbb{R}^3)$ $\quad$ 3) $F'_z = 3z^2 + 2(x+y)^2 + e^{z-1} \Big|_A = 6 \neq 0$

T.F.I $\Rightarrow$ $\exists\, z = f(x,y) \in C^\infty(U(A))$

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = -\frac{4(x+y)z}{3z^2+2(x+y)^2+e^{z-1}}\Bigg|_A = -\frac{-4}{6} = \frac{2}{3}$$

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} = -\frac{4(x+y)z}{3z^2+2(x+y)^2+e^{z-1}}\Bigg|_A = \frac{2}{3} \qquad Df(0,-1) = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$D_u f(0,-1) \quad u = (1,+1) \quad \|u\| = \sqrt{2} \quad D_u f = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{4}{3}\frac{1}{\sqrt{2}}$

$$P_1(x,y) = f(0,-1) + Df(0,-1)\begin{pmatrix} x \\ y+1 \end{pmatrix} = 1 + \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} x \\ y+1 \end{pmatrix} = 1 + \frac{2}{3}x + \frac{2}{3}(y+1)$$

Sea la ecuación $z^3 - xyz + y^2 = 16$.

a) Prueba que dicha ecuación define una función $z = f(x,y)$ en cierto entorno U de $(1,4,2)$ y que dicha función f es $C^{(p)}(U)$ para todo $p \in \mathbb{N}$.
b) Calcula la expresión formal de $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$ en un punto (x,y) de U.
c) Calcula los valores numéricos de las derivadas parciales $\frac{\partial f}{\partial x}(1,4)$ y $\frac{\partial f}{\partial y}(1,4)$. ¿Cuánto vale la derivada direccional de f en la dirección $(1,-2)$ en dicho punto $(1,4)$?
d) Calcula el valor de $\frac{\partial^2 f}{\partial x^2}(x,y)$ y $\frac{\partial^2 f}{\partial x^2}(1,4)$

$F = z^3 - xyz + y^2 - 16 = 0$ $\quad A(1,4,2)$ $\quad F \in C^\infty(\mathbb{R}^3)$

$F(1,4,2) = 8 - 8 + 16 - 16 = 0$ $\quad F'_z(1,4,2) = 3z^2 - xy\Big|_A = 12 - 4 = 8 \neq 0$

$z_x = \frac{\partial f}{\partial x} = -\frac{F'_x}{F'_z} = \frac{yz}{3z^2 - xy}\Big|_A = \frac{8}{8} = 1$

Debemos $\Rightarrow$ z_{xx} usando $z(x,y)$

$z_y = \frac{\partial f}{\partial y} = -\frac{F'_y}{F'_z} = -\frac{2y - xz}{3z^2 - xy}\Big|_A = -\frac{8-2}{8} = -\frac{6}{8} = -\frac{3}{4}$

$$D_u f(1,4) = \begin{pmatrix} 1 & -3/4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \frac{1}{\sqrt{5}} = \left(1 + \frac{3}{2}\right) \frac{1}{\sqrt{5}} = \frac{5}{2} \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{2}$$

$u = (1,-2)$

$\|u\| = \sqrt{5}$

$$z^3 - xyz + y^2 - 16 = 0$$

$z(x,y)$

$$3z^2 z_x - yz - xy z_x = 0$$

$$z_x(3z^2 - xy) - yz = 0$$

$$z_x = \frac{yz}{3z^2 - xy}$$

$z_x(1,4) = 1$

$A(1,4,2)$

$$z_{xx}(3z^2 - xy) + z_x(6z z_x - y) - y z_x = 0$$

$$z_{xx} = \frac{z_x(2y - 6z z_x)}{3z^2 - xy} = \dots$$

$$z_{xx}(1,4) = \frac{1(8-12)}{8} = -\frac{4}{8} = -\frac{1}{2}$$

$$f(x,y) = (x+y)\exp(-x^2-y^2).$$

$f \in C^2(\mathbb{R}^2)$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2} + (x+y)(-2x)e^{-x^2-y^2} = e^{-x^2-y^2}\big(1-2(x+y)x\big) = 0$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2} + (x+y)(-2y)e^{-x^2-y^2} = e^{-x^2-y^2}\big(1-2(x+y)y\big) = 0$$

$$\begin{cases} 2x(x+y)-1=0 & (1) \\ 2y(x+y)-1=0 & (2) \end{cases}$$

$(1)-(2) \Rightarrow (2x-2y)(x+y)=0$

$2(x-y)(x+y)=0 \quad \begin{cases} x=y \\ x=-y \end{cases}$

Si $x=y$: $\quad 2x\cdot 2x-1=0 \Rightarrow 4x^2=1 \quad x^2=1/4 \quad x=\pm 1/2$

$A(1/2, 1/2) \quad B(-1/2, -1/2)$

Si $x=-y$ $\quad 2x\cdot 0=-1 \Rightarrow 0=-1$??

MAXIMA

$$f(x,y) = xy\exp(-x^2-y^2),\ f(x,y) = xy\exp(x^2+y^2).$$

$$\frac{\partial f}{\partial x} = y e^{-x^2-y^2} + xy(-2x)e^{-x^2-y^2} = y(1-2x^2)e^{-x^2-y^2} = 0$$

$$\frac{\partial f}{\partial y} = x e^{-x^2-y^2} + xy(-2y)e^{-x^2-y^2} = x(1-2y^2)e^{-x^2-y^2} = 0$$

$y(1-2x^2) = 0$ $\quad A(0,0) \quad B_1(1/\sqrt{2}, 1/\sqrt{2})\ B_2(-1/\sqrt{2}, -1/\sqrt{2})$

$x(1-2y^2) = 0$ $\quad C_1(-1/\sqrt{2}, 1/\sqrt{2})\ C_2(1/\sqrt{2}, -1/\sqrt{2})$

~~1) $y=0 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$ 2)~~ $\Rightarrow$ maxima !!

~~2) $x=0$ 1) $x = \pm 1/\sqrt{2}$~~

$$f(x,y) = x^3 - 9xy + y^3 + 27.$$

$$\frac{\partial f}{\partial x} = 3x^2 - 9y = 0$$

$$\frac{\partial f}{\partial y} = 3y^2 - 9x = 0$$

$(0,0)$

$(3,3)$

$x^2 - 3y = 0 \quad (1)$

$y^2 - 3x = 0 \quad (2)$

$(1) \to y = x^2/3 \xrightarrow{(2)} \frac{x^4}{9} - 3x = 0$

$x(x^3 - 27) = 0$

$x = 0$

$x = 3$

$x = y^2/3$

$\frac{y^4}{9} - 3y = 0$

$(x^2 - y^2) - 3y + 3x = 0$

$x = y$

$(x-y)(x+y+3) = 0$

$x + y = -3$

$y = -3 - x$

$(1)\ x = y \Rightarrow x^2 - 3x = 0 \quad x(x-3) = 0 \Rightarrow x = 0,\ x = 3$

$y = 3 - x \quad (1) \to x^2 + 3(3+x) = 0 \quad x^2 + 9 + 3x = 0 \quad x^2 + 3x - 9 = 0$

$$x = -\frac{3}{2} \pm \sqrt{\underbrace{\frac{9}{4} \cancel{+} 9}_{<0}} = -\frac{3}{2} \pm \sqrt{\cancel{\frac{45}{4}}}$$