Ejemplo: Sea la ecuación $z^3 + 2(x+y)^2 z + e^{z-1} - 4 = 0$.

1. Prueba que la ecuación anterior define una función $z = f(x,y)$ en el entorno U del punto $(0,-1,1)$ y que dicha función es una función $C^{(\infty)}(U)$ en dicho U.

1º $F(x,y,z) = z^3 + 2(x+y)^2 z + e^{z-1} - 4$ $\qquad A = (0,-1,1)$

$$\begin{cases} F(0,-1,1) = 1 + 2 + 1 - 4 = 0 \\ \dfrac{\partial F}{\partial z}(0,-1,1) = 3z^2 + 2(x+y)^2 + e^{z-1}\Big|_A = 3 + 2 + 1 = 6 \neq 0 \\ F \in C^{\infty}(\mathbb{R}^3) \end{cases}$$

$\exists\ z = f(x,y) \in C^{\infty}(U)$

$$Df(0,-1) = \left(\frac{\partial f}{\partial x}(0,-1) \quad \frac{\partial f}{\partial x}(0,-1) \right)$$

$F(x,y,z) = z^3 + 2(x+y)^2 z + e^{z-1} - 4$

$$\frac{\partial f}{\partial x} = -\frac{\frac{\partial F}{\partial x}(0,-1,1)}{\frac{\partial F}{\partial z}(0,-1,1)} = - \frac{4(x+y)z}{3z^2+2(x+y)^2+e^{z-1}}\Bigg|_A = - \frac{-4}{6} = \frac{2}{3}$$

$$\frac{\partial f}{\partial y}(0,-1) = -\frac{\frac{\partial F}{\partial y}(0,-1,1)}{\frac{\partial F}{\partial z}(0,-1,1)} = - \frac{4(x+y)z}{3z^2+2(x+y)^2+e^{z-1}}\Bigg|_A = \frac{2}{3}$$

$D_u f(0,-1) \qquad u = (1,2) \qquad \|u\| = \sqrt{5}$

$$D_u f(0,-1) = Df(0,-1)\begin{pmatrix}1\\2\end{pmatrix}\frac{1}{\sqrt{5}} = \begin{pmatrix}\frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Plan tang a la sup $F(x,y,z)=0$ en $(0,-1,1) \Rightarrow z = f(x,y)$

$\vec{n} = \left(\frac{2}{3} \quad \frac{2}{3} \quad -1\right) \qquad \vec{n}\cdot(x, y+1, z-1) = 0$

$$\frac{2}{3}x + \frac{2}{3}(y+1) - z + 1 = 0 \qquad \frac{2}{3}x + \frac{2}{3}y - z + \frac{5}{3} = 0$$

Calculo el $P_2(x,y)$ de f en $(0,-1)$ $\quad P_2 = f(0,-1) + Df(0,-1)\begin{pmatrix} x \\ y+1 \end{pmatrix} + \frac{1}{2} D^2f(0,-1)\begin{pmatrix} x \\ y+1 \end{pmatrix}$

$F(x,y,z) = z^3 + 2(x+y)^2 z + e^{z-1} - 4 = 0 \qquad z = f(x,y) \qquad z(x,y)$

Necesito z_x, z_y $\quad z_{xx}$ $\quad z_{yy}$ $\quad z_{xy}$ $\qquad A(0,-1,1)$ $\quad F(A)=0$ $\quad \frac{\partial F}{\partial z}(A) \neq 0$

$$\frac{\partial F}{\partial x}(x,y,z) = \frac{\partial F}{\partial x}(x,y,\boxed{z(x,y)}) \quad \Bigg| \; z_x = \frac{\partial z}{\partial x} \neq 0 \; \Bigg| \; z_y = \frac{\partial z}{\partial y} \neq 0 \; \Bigg|$$

$$0 = F_x = 3z^2 z_x + 4(x+y)z + 2(x+y)^2 z_x + e^{z-1} z_x \Big|_A = 0$$

$$z_x\left(3z^2 + 2(x+y)^2 + e^{z-1}\right) + 4(x+y)z = 0 \qquad \text{sust. en } A$$

$$z_x \cdot 6 - 4 = 0 \;,\quad \Rightarrow \quad z_x = 2/3$$

$$\frac{\partial F}{\partial x}(x,y,z(x,y)) = \frac{\partial F}{\partial x}\underbrace{\cancel{\frac{\partial x}{\partial x}}}^{1} + \frac{\partial F}{\partial y}\underbrace{\cancel{\frac{\partial y}{\partial x}}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \underbrace{\frac{\partial F}{\partial z}}_{\neq 0} z_x = 0$$

$$F_y = 3z^2 z_y + 4(x+y)z + 2(x+y)^2 z_y + e^{z-1} z_y \Big/_{A} = 0$$

$$3z_y - 4 + 2z_y + z_y = 0 \Rightarrow \quad 6z_y = 4 \Rightarrow z_y = \frac{2}{3}$$

$\underline{\underline{z_{xx}}}$

$$\frac{\partial}{\partial x} z_x = \frac{\partial}{\partial x}\left(-\frac{4(x+y)z}{3z^2 + 2(x+y)^2 + e^{z-1}}\right) = \ldots$$

$$F_x = 3z^2 z_x + 4(x+y)z + 2(x+y)^2 z_x + e^{z-1} z_x = 0 \qquad z(x,y)$$

$$z_x(x,y)$$

$$F_{xx} = \frac{\partial}{\partial x}\left(z_x\left(3z^2 + 2(x+y)^2 + e^{z-1}\right) + 4(x+y)z\right) = 0$$

$$0 = z_{xx}\left(3z^2 + \underset{\substack{\shortparallel \\ 6}}{2(x+y)^2} + e^{z-1}\right) + z_x\left(6zz_x + 4(x+y) + e^{z-1}z_x\right) + 4z + 4(x+y)z_x$$

evaluar en A.

$$6z_{xx} + \frac{2}{3}\left(\cancel{6}^{\,2}\frac{2}{\cancel{3}} - \cancel{4} + \frac{2}{3}\right) + 4 - \frac{8}{3} = 0$$

$$6z_{xx} + \frac{4}{9} + 4 - \frac{8}{3} \Rightarrow \quad 6z_{xx} = -\frac{16}{9} \qquad z_{xx} = -\frac{\cancel{16}^{8}}{9 \cdot \cancel{6}_{3}} = -\frac{8}{27}$$

$$\frac{4 + 36 - 24}{9} = \frac{16}{9}$$

$$F_{yy} = \frac{\partial}{\partial y}\left(3z^2 z_y + 4(x+y)z + 2(x+y)^2 z_y + e^{z-1} z_y\right) = 0$$

$$F_{xy} = \frac{\partial}{\partial x}\left(3z^2 z_y + 4(x+y)z + 2(x+y)^2 z_y + e^{z-1} z_y\right) = 0$$