1.22 c) $F(x,y,z) = 3xyz - z^3 - 1 = 0$ encontrar el plano tang. $A(0,1,-1)$

$z(x,y)$ $F(0,1,-1) = 0$ $F'_z = 3xy - 3z^2 \Big|_A = -3 \neq 0$

$F \in C^\infty(\mathbb{R}^3)$

T.F.I.T. $\exists\, z(x,y)$ def. U - entorno de A. $z \in C^\infty(U)$

$\vec{n} = (z_x, z_y, -1)$, $z_x = -\dfrac{F'_x}{F'_z}\Big|_A = -\dfrac{3yz}{3xy - 3z^2}\Big|_A = -\dfrac{-3}{-3} = -1$

$z_y = -\dfrac{F'_y}{F'_z}\Big|_A = -\dfrac{3xz}{-3}\Big|_A = 0$

$\vec{n} = (-1, 0, -1) \Rightarrow (1, 0, 1)$

$\vec{n} \cdot (x, y-1, z+1) = 0 \Rightarrow x + z + 1 = 0$

$$f(x,y,z) = x^3 + y^3 + z^3 - 2z(x+y) - 2x + y - 2z + 1 = 0$$ $A(0,0,1)$ $z(x,y)$

$f(0,0,1) = 1 - 2 + 1 = 0$ ✓ $f \in C^\infty(\mathbb{R}^3)$

$f'_z(0,0,1) = 3z^2 - 2(x+y) - 2 \Big|_A = 1 \neq 0$

$\Rightarrow$ Por el T.F.I. $\exists\, z(x,y)$ en todo un entorno $U(A)$ t.q.

$z(x,y) \in C^\infty(U(A))$

$z_x = -\dfrac{f'_x}{f'_z} = -\dfrac{3x^2 - 2z - 2}{3z^2 - 2(x+y) - 2}\Big|_A = 4$

$z_y = -\dfrac{f'_y}{f'_z} = -\dfrac{3y^2 - 2z + 1}{1}\Big|_A = 1$

$\rightarrow$ $\boxed{z(x,y)}$

$f'_x = 0 = 3x^2 + 3z^2 z_x - 2z_x(x+y) - 2z - 2 - 2z_x = 0$

$z_x(3z^2 - 2(x+y) - 2) + (3x^2 - 2z - 2) = 0$ Sust $A \Rightarrow z_x - 4 = 0$

$f'_y = 0$ $3y^2 + 3z^2 z_y - 2z_y(x+y) - 2z + 1 - 2z_y = 0$

$z_y(3z^2 - 2(x+y) - 2) + (3y^2 - 2z + 1) = 0$ Sust $A \Rightarrow z_y - 1 = 0$

$f''_{xx} = 0 = z_{xx}(3z^2 - 2(x+y) - 2) + z_x(6zz_x - 2) + 6x - 2z_x = 0$

Sust A. $\boxed{z_x = 4}, \boxed{z_y = 1}$ $\quad z_{xx} + 4(24 - 2) - 8 = 0 \Rightarrow \boxed{z_{xx} = -80}$

$f''_{yy} = 0 = z_{yy}(3z^2 - 2(x+1) - 2) + z_y(6zz_y - 2) + 6y - 2z_y = 0$

Sust. $\quad z_{yy} + 4 - 2 = 0 \Rightarrow \boxed{z_{yy} = -2}$

$f''_{xy} = z_{xy}(3z^2 - 2(x+y) - 2) + z_y(6zz_x - 2) - 2z_x = 0$

sust. $\quad z_{xy} + 22 - 8 = 0 \Rightarrow \boxed{z_{xy} = -14}$

$$P_2(x,y) = z(0,0) + Dz(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} D^2 z(0,0)\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= 1 + (4 \quad 1)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x \; y)\begin{pmatrix} -80 & -14 \\ -14 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= 1 + 4x + y - 40x^2 - y^2 - 14xy$$

Sea $f(x,y,z,t) = (x^3z + y^3t^2 - 1, 2zt^3 + xy^2)$. Probar que f define una función implícita de clase $\mathcal{C}^\infty$, $(z,t) = (h_1(x,y), h_2(x,y))$ en un entorno de $(0,1,0,1)$. Calcula $Dh(0,1)$.

$$F(x,y,z,t) = \begin{pmatrix} x^3z + y^3t^2 - 1 \\ 2zt^3 + xy^2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \qquad F(0,1,0,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark$$

$A(0,1,0,1)$

$$F'_y = \begin{pmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial t} \end{pmatrix} = \begin{pmatrix} x^3 & 2y^3t \\ 2t^3 & 6zt^2 \end{pmatrix}\Bigg|_A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\det F'_y = -4 \neq 0$$

$$(F_y')^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}^{-1} \qquad \frac{1}{-4}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$F'_x = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2z & 3y^2t^2 \\ y^2 & 2xy \end{pmatrix}$$

1º $F(A)=0$ $F'_y(A)$ es invertible $F\in C^\infty(\mathbb{R}^4)$ $\Rightarrow$ T.F.I. $\Rightarrow$

$\exists$ $z(x,y)$, $t(x,y)$ $h(x,y)=\begin{pmatrix} z(x,y) \\ t(x,y) \end{pmatrix}$ $h:\mathbb{R}^2\to\mathbb{R}^2$ $h\in C^\infty(U(A))$

$$Dh(0,1) = -\left(F'_y\right)^{-1}\cdot F'_x =$$

$$= -\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & -3/2 \end{pmatrix}$$

$$e^{z^2-1} + (xe^{y^2} + e^{x^2}y)z - 1 = 0.$$

$A=(0,0,a)$ $\qquad z(x,y)$

$F(x,y,z) \in C^{\infty}(\mathbb{R}^3)$ $\qquad F(0,0,a) = e^{a^2-1} - 1 = 0$ $\qquad a^2-1 = 0 \begin{cases} a=1 \\ a=-1 \end{cases}$

$$F'_z(0,0,a) = 2ze^{z^2-1} + (\underbrace{xe^{y^2} + \cancel{e^{x^2}y}}_{0})\Big|_A = 2ae^{a^2-1} \begin{cases} \nearrow 2 \\ \searrow -2 \end{cases} \neq 0$$

$A(0,0,1)$ $\quad B(0,0,-1)$ define a $z(x,y)$ en un entorno de A y B

T.F.I $\rightarrow$ $z \in C^{\infty}(U_A)$ $\quad z \in C^{\infty}(U_B)$.

$$z_x = -\frac{F'_x}{F'_z}\Big|_{A,B} = -\frac{e^{y^2}z + \cancel{2yxe^{x^2}z}}{2ze^{z^2-1} + (xe^{y^2} + \cancel{e^{x^2}y})}\Big|_{A,B} = -\frac{\cancel{a}}{2\cancel{a}} = -\frac{1}{2}$$

$$z_y = -\frac{F'_y}{F'_z}\Big|_{A,B} = -\frac{z(\cancel{2xye^{y^2}} + e^{x^2})}{2ze^{z^2-1} + (xe^{y^2} + \cancel{ye^{x^2}})}\Big|_{A,B} = -\frac{1}{2} \quad \begin{cases} A(0,0,1) \\ B(0,0,-1) \end{cases}$$

$\vec{n} = (2x, 2y, -1)$

$= (1/2, 1/2, 1)$

A) $0 = \vec{n}\,(x, y, z-1) \Rightarrow \frac{x}{2} + \frac{y}{2} + z - 1 = 0$

B) $0 = \vec{n}\,(x, y, z+1) \Rightarrow \frac{x}{2} + \frac{y}{2} + z + 1 = 0$

$x^2 + y^3 + xy + x^3 + ay = 0$ en $A(0,0)$ define una $y(x)$.

y $y(x)$ tiene un extremo en $x=0$.

$F(0,0)=0 \quad F \in C^{\infty}(\mathbb{R}^2)$

$F(x,y) = x^2+y^3+xy+x^3+ay = 0$, $F'_y = 3y^2+x+a\Big|_{(0,0)} = a \neq 0$

Si $a \neq 0$ T.F.I $\Rightarrow \exists\, y(x)$ $y \in C^{\infty}(U(A))$

$F'_x = 2x + 3y^2y' + y + xy' + 3x^2 + ay' = 0$

$\underline{\underline{\exists y(x)}}$

$y'(3y^2+x+a) + (3x^2+2x+y) = 0$

Sust. $ay' + 0 = 0 \Rightarrow a \neq 0$ entonces $y'(0) = 0$

$F''_{xx} = 0 = y''(3y^2+x+a) + y'(\cancel{6yy'+1}) + 6x+2+y' = 0$

sust. $ay'' + 2 = 0 \Rightarrow y'' = -\frac{2}{a} \Rightarrow \begin{cases} a>0 & y''<0 \quad \text{máx local} \\ a<0 & y''>0 \quad \text{mín local.} \end{cases}$

$F(x,y)=-\log(x)+xy+\log(y)=0$. Prueba que $y=f(x)$ tiene un extremo. $x>0,\ y>0$

$y(x)$ t.q. $F(x,y(x))=0 \quad \forall x>0$

$\forall x>0 \quad \exists y$ t.q. $F(x,y)=0 \qquad F(x,y)=-\log(x)+xy+\log(y) \quad \forall x_0$.

$F'_y = x+\frac{1}{y}>0 \quad \forall x,y>0$

x_0 cualquiera.

$F(x_0,y)$ si $y\to 0 \quad F(x_0,y)<0$

$F(x_0,y)$ si $y\to +\infty \quad F(x_0,y)>0$

$\forall x_0>0 \quad \exists y_0>0$ t.q. $F(x_0,y_0)=0$.

$\exists\ y=f(x)$ definida en $\mathbb{R}\setminus\{0\}$.

1º $\exists (x_0,y_0) \quad F(x_0,y_0)=0$ 2º $F'_y(x_0,y_0)=x_0+\frac{1}{y_0}\neq 0 \ (>0)$

$F\in C^{\infty}(A) \quad A=\{(x,y),\ x>0,\ y>0\}$.

$\forall (x_0,y_0)\subset A$ El T.F.I. $\Rightarrow \exists\ y(x) \quad y(x)\in C^{\infty}(U(x_0,y_0))$

$F(x,y(x)) = -\log x + xy + \log y = 0$

$-\log x + 1 + \log\frac{1}{x} = 0$

$1 - 2\log x = 0 \Rightarrow x = e^{1/2}$

$x = \frac{1}{y}$

$xy > 0$

$F'_x = 0 = -\frac{1}{x} + y + xy' + \frac{y'}{y} = 0$

$-\frac{y}{x} + y^2 + xyy' + y' = 0$ $\qquad$ $y'\underbrace{(1+xy)}_{\neq 0} + \left(y^2 - \frac{y}{x}\right) = 0$

$y' = \frac{xy^2 - y}{x(1+xy)} = \frac{y(xy-1)}{x(xy+1)} = 0 \Rightarrow \cancel{y=0}\ ,\ xy - 1 = 0$

$\Downarrow$

$x = \frac{1}{y}$

$(e^{1/2}, e^{-1/2})$ donde $y'(e^{1/2}) = 0$

$F''_{xx} = 0 = -\frac{\cancel{y'}x - y}{x^2} + y''(1 + x\overset{=1}{y}) + \cancel{y'(y + xy')} = 0$

sust.

$\frac{y}{x^2} + 2y'' = 0$ $\qquad$ $y'' = -\frac{y}{2x^2} = -\frac{1}{2x^3} < 0$

$y''(0) < 0 \Rightarrow y$ tiene un máx. local en $x = e^{1/2}$

Probar que existen funciones f y g de clase $\mathcal{C}^\infty$ definidas en un entorno de $(1,1)$, tales que $f(1,1) = -1$, $g(1,1) = 0$ y verificando las ecuaciones:

$$f(x,y)^3 + xg(x,y)^2 + y = 0, \qquad g(x,y)^3 + yg(x,y) + f(x,y)^2 = x.$$

$$F(x,y,f,g) = \begin{pmatrix} f^3 + xg^2 + y \\ g^3 + yg + f^2 - x \end{pmatrix} = 0 \quad - \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$A(1,1,-1,0)$

$F \in C^\omega(\mathbb{R}^4)$ ✓

$$F(1,1,-1,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark$$

$$F'_{\underline{\underline{y}}} = \begin{pmatrix} \frac{\partial F_1}{\partial f} & \frac{\partial F_1}{\partial g} \\ \frac{\partial F_2}{\partial f} & \frac{\partial F_2}{\partial g} \end{pmatrix} = \begin{pmatrix} 3f^2 & 2xg \\ 2f & 3g^2 + y \end{pmatrix}_A = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$$

$\det F'_y = 3 \neq 0$ ✓

$\underline{\text{T.F.I}} \Rightarrow \exists\ h = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \in C^\infty(U(1,1))$

$$Dh(1,1) = -\left(F'_y\right)^{-1}\left(F'_x\right) = -\frac{1}{3}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1/3 \\ 1 & -2/3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$F'_{\underline{\underline{x}}} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} a^2 & 1 \\ -1 & y \end{pmatrix} \Bigg|_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$$

$$x^2+y^2+z^2=e^{x+y+z}-1$$ $z(x,y)$ en $A(0,0,0)$?

$F\in C^{\infty}(\mathbb{R}^3)$

$F(x,y,z)= x^2+y^2+z^2-e^{x+y+z}+1=0$

$F(0,0,0)= 0-1+1=0$ $F'_z(0,0,0)= 2z-e^{x+y+z}\Big|_A =1\neq 0$

$\Rightarrow$ El T.F.I. $\Rightarrow$ $\exists\ z(x,y)\in C^{\infty}(U(A))$

$F'_x = 2x+2z\,z_x-(1+z_x)\,e^{x+y+z}=0$ $\Big|$ sust A $\Rightarrow$ $z_x=-1$

($z(x,y)$)

$F'_y=0 \Rightarrow z'_y=-1$

$F''_{xx}= (\quad)\,z_{xx}+ \dots =0$

La ecuación $\operatorname{sen}(ax+by+cz)+e^{cxyz}+x+2y=1$ define z como función implícita de x e y en un entorno del origen. Calcular los valores de las constantes a, b y c que hacen que el desarrollo de Taylor en un entorno de $(0,0)$ de dicha función implícita tenga los tres primeros términos nulos.

$F(x,y,z) = \operatorname{sen}(ax+by+cz)+e^{cxyz}+x+2y-1=0$

$F \in C^{\infty}(\mathbb{R}^3)$ ✓

$F(0,0,0) = 0+1+0-1=0$ ✓

$F'_z = c\cos(ax+by+cz)+cxy\,e^{cxyz}\Big|_A = c \neq 0$ ✓

$\Rightarrow$ T.F.I. que $\exists\ z(x,y) \in C^{\infty}(U(A))$

$$-\frac{(\overset{b}{\cancel{\beta}}+2)\,y}{\Gamma}-\frac{(\overset{a}{\cancel{\alpha}}+1)\,x}{\Gamma} = P_2(x,y)=0 \iff \begin{array}{l} a=-1 \\ b=-2 \end{array} \quad \forall c\neq 0$$