$$(g \circ f)(x) = g(f(x))$$

$$\underbrace{D(g \circ f)(x_0)}_{\mathbb{R}^n \to \mathbb{R}^k} = \underbrace{Dg(f(x_0))}_{\mathbb{R}^m \to \mathbb{R}^k} \cdot \underbrace{Df(x_0)}_{\mathbb{R}^n \to \mathbb{R}^m}$$

$$g(f(x_0+h)) - g(f(x_0)) = L(h) + o(\|h\|) \qquad y_0 = f(x_0) \quad y = f(x_0+h)$$

$$g(f(x_0+h)) - g(f(x_0)) = g(y) - g(y_0) = Dg(y_0)(y-y_0) + o(\|y-y_0\|)$$

$$= Dg(f(x_0))\,(f(x_0+h) - f(x_0)) + o(\|f(x_0+h) - f(x_0)\|) =$$

$$= Dg(f(x_0))\left[Df(x_0)(h) + o(\|h\|)\right] + o(\|f(x_0+h) - f(x_0)\|)$$

$$= \left[Dg(f(x_0)) \circ Df(x_0)\right](h) + \underbrace{Dg(f(x_0))}_{\mathrm{I}}(o(\|h\|)) + o(\underbrace{\|f(x_0+h) - f(x_0)\|}_{\mathrm{II}})$$

I) $$\frac{\|Dg(f(x_0))(o(\|h\|))\|}{\|h\|} \le \|Dg(f(x_0))\| \, \cancelto{0}{\frac{\|o(\|h\|)\|}{\|h\|}} \longrightarrow 0 \qquad \mathrm{I} = o(\|h\|)$$

II) $\mathrm{II} = o(\|h\|)$

$$\cancelto{0}{\frac{o(\|f(x_0+h) - f(x_0)\|)}{\|f(x_0+h) - f(x_0)\|}} \; \underbrace{\frac{\|f(x_0+h) - f(x_0)\|}{\|h\|}}_{\text{acotado}} \longrightarrow 0$$

$$\|f(x_0+h) - f(x_0)\| = \|Df(x_0)(h) + o(\|h\|)\| \le \|Df(x_0)(h)\| + o(\|h\|)$$

$\leq \|Df(x_0)\| \, \|h\| + o(\|h\|)$

$$\Rightarrow \quad \frac{\|f(x_0+h) - f(x)\|}{\|h\|} \leq \|Df(x_0)\| + \frac{o(\|h\|)}{\|h\|} \xrightarrow{} 0 \quad \leq M \to 0$$

Ejemplo 1 $\quad f: \mathbb{R} \to \mathbb{R}^3 \qquad f(t) = \begin{pmatrix} a \operatorname{sen} t \\ a \cos t \\ \alpha t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad a, \alpha \neq 0$

$g: \mathbb{R}^3 \to \mathbb{R} \qquad g(x,y,z) = x^2 + y^2 + z^2$

$h: \mathbb{R} \to \mathbb{R} \qquad h(t) = (g \circ f)(t) = g(f(t))$

$h(t) = f_1^2 + f_2^2 + f_3^2 = a^2 \operatorname{sen}^2 t + a^2 \cos^2 t + \alpha^2 t^2 = a^2 + \alpha^2 t^2$

$Dh(t) = h'(t) = 2\alpha^2 t.$

$Dh(t) = \left(\frac{\partial h}{\partial t} \right) = Dg(f(t)) \cdot Df(t) \quad , \quad Df(t) = \begin{pmatrix} \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial t} \\ \frac{\partial f_3}{\partial t} \end{pmatrix} = \begin{pmatrix} a \cos t \\ -a \operatorname{sen} t \\ \alpha \end{pmatrix}$

$Dg(x) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$

$$Dg(f(t)) = \begin{pmatrix} 2a\,\text{sen}\,t & 2a\cos t & 2\alpha t \end{pmatrix}$$

$$Dh(t) = Dg(f(t)) \cdot Df(t) = \begin{pmatrix} 2a\,\text{sen}\,t & 2a\cos t & 2\alpha t \end{pmatrix} \begin{pmatrix} a\cos t \\ -a\,\text{sen}\,t \\ \alpha \end{pmatrix} =$$

$$= 2a\,\cancel{\text{sen}\,t\cos t} - 2a\,\cancel{\cos t\,\text{sen}\,t} + 2\alpha^2 t$$

Exemplo 2)

$$f(x,y) = \begin{pmatrix} x^2+y^2 \\ 2x+y \end{pmatrix} \qquad g(u,v) = \begin{pmatrix} u^2 \\ u+v \\ v^2 \end{pmatrix}$$

$\mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $\mathbb{R}^2 \to \mathbb{R}^3$

$$g(f(x,y)) = h(x,y) = g(\underbrace{x^2+y^2}_{u}, \underbrace{2x+y}_{v}) = \begin{pmatrix} (x^2+y^2)^2 \\ x^2+y^2+2x+y \\ (2x+y)^2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \quad Dh(x,y)$$

$h: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad$ $\mathbb{R}^2 \to \mathbb{R}^3$

$$Dh(x,y) = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_3}{\partial x} & \frac{\partial h_3}{\partial y} \end{pmatrix} = \begin{pmatrix} 2\cdot 2x\,(x^2+y^2) & 2\cdot 2y\,(x^2+y^2) \\ 2x+2 & 2y+1 \\ 4(2x+y) & 2(2x+y) \end{pmatrix}$$

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2 & 1 \end{pmatrix}$$

$$Dg(u,v) = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & 0 \\ 1 & 1 \\ 0 & 2v \end{pmatrix}$$

$$Dg\big(f(x,y)\big) = \begin{pmatrix} 2(x^2+y^2) & 0 \\ 1 & 1 \\ 0 & 2(2x+y) \end{pmatrix}$$

$$D(g\circ f)(x,y) = Dg(f(x,y))\cdot Df(x,y) =$$

$$\begin{pmatrix} 2(x^2+y^2) & 0 \\ 1 & 1 \\ 0 & 2(2x+y) \end{pmatrix}\begin{pmatrix} 2x & 2y \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4x(x^2+y^2) & 4y(x^2+y^2) \\ 2x+2 & 2y+1 \\ 4(2x+y) & 2(2x+y) \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ 2x+y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$g(u,v) = \begin{pmatrix} u^2 \\ u+v \\ v^2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \qquad g(f(x,y)) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$h_2(x,y) = g(x^2+y^2, 2x+y)$$

$$\frac{\partial h_2(x,y)}{\partial x} = \frac{\partial g_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g_2}{\partial v}\frac{\partial v}{\partial x} = 1 \cdot 2x + 1 \cdot 2 = 2x + 2$$

$$\frac{\partial h_3(x,y)}{\partial y} = \frac{\partial g_3}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g_3}{\partial v}\frac{\partial v}{\partial y} = 0 \cdot \cancel{2y} + 2v \cdot 1 = 2(2x+y) = 4x + 2y$$

$$x_1 = f_1$$
$$x_2 = f_2$$
$$\vdots$$
$$x_m = f_m$$

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial h_i}{\partial x_1}\frac{\partial x_1}{\partial x_j} + \frac{\partial h_i}{\partial x_2}\cdot\frac{\partial x_2}{\partial x_j} + \dots + \frac{\partial h_i}{\partial x_n}\frac{\partial x_n}{\partial x_j}$$

$$= \frac{\partial g_i(f(x))}{\partial x_1}\frac{\partial f_1}{\partial x_j} + \dots + \frac{\partial g_i(f(x))}{\partial x_m}\cdot\frac{\partial f_m}{\partial x_j}$$

$j = 1, \dots, m$

$i = 1, \dots, k$