$Df(a) \quad \mathbb{R}^n \to \mathbb{R}^m$ $\qquad Df(a)\ \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

$Df \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

$$Df(a+h) - Df(a) - L(a)(h) = o(\|h\|)$$

$$L : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$L = D^2 f(a) \qquad D^2 f \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$

$$D^k f(a)(h) = \underbrace{\left(h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_n \frac{\partial}{\partial x_n} \right)}_{L}^k f$$

$k=1$.

$$Df(a)(h) = \left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n} \right) f = h_1 \frac{\partial f}{\partial x_1} + \cdots + h_n \frac{\partial f}{\partial x_n} =$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = Df(a)(h)$$

$$D^2 f(a)(h) = \left(h_1 \frac{\partial}{\partial x} + \cdots + h_n \frac{\partial}{\partial x_n} \right)^2 f(x) \Big|_{x=a}$$

$$= \left(h_1 \frac{\partial}{\partial x_1} + \cdots h_n \frac{\partial}{\partial x_n} \right) \left[h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n} \right] f =$$

$$= \left(h_1 \frac{\partial}{\partial x_1} + \cdots h_n \frac{\partial}{\partial x_n} \right) \left(h_1 \frac{\partial f}{\partial x_1} + \cdots + h_n \frac{\partial f}{\partial x_n} \right) =$$

$$= h_1 \frac{\partial}{\partial x_1} \left(h_1 \frac{\partial f}{\partial x_1} + \cdots + h_n \frac{\partial f}{\partial x_n} \right) + \cdots + h_n \frac{\partial}{\partial x_n} \left(\qquad \right)$$

$$D^k f(a)(h) = L^k f = L(L^{k-1} f)$$

$$L, \ L^2, \ L^3 \ldots$$

$l : \mathbb{R} \to \mathbb{R}^n$

$f(x) \in C^k(A)$ $\quad l = x + ht \quad h \in \mathbb{R}^n, \; x \in \mathbb{R}^n \quad t \in \mathbb{R}$

$$\phi(t) = (f \circ l)(t) \Rightarrow \phi(t) = f(x+ht)$$

$\left\{ \; Dl(t) = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = h \right.$

$$\phi'(t) = Df(x+ht) \cdot h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x+ht) \cdot h_i \qquad k \geq 1$$

$$\phi''(t) = \frac{d}{dt} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x+ht) \cdot h_i = \sum_{i=1}^{n} h_i \left[\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}(x+ht) \, h_j \right] =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x+ht) \cdot h_i h_j = \left(h_1 \frac{\partial}{\partial x_1} + \dots + h_n \frac{\partial}{\partial x_n} \right)^2 f(x+ht)$$

$$\phi'''(t) = \sum_{i,j,l} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_l}(x+ht) \, h_i h_j h_l = \left(\qquad \right)^3 f(x+ht) \dots$$

$\dots$

Teorema de Taylor: $f \in C^k(A)$ $\quad [a, a+h] \subset A$

$\phi(t) = f(a+ht)$ $\quad t \in [0,1]$ $\quad \phi(0) = f(a)$ $\quad \phi(1) = f(a+h)$

$l : \mathbb{R} \to \mathbb{R}^n$ $\quad l(t) = a+ht$ $\Rightarrow$ T.R.C. $\Rightarrow$ $f \circ l \in C^k(A)$.

$\phi'(t) = Df(a+ht)(h)$ $\quad \phi''(t) = D^2 f(a+ht)(h)$ $\quad \dots \quad \phi^{(k)}(t) = D^k f(a+ht)(h)$

$\phi \in C^k[0,1]$

$$= \sum_{i_1 \dots i_k = 1}^{n} \frac{\partial^k f(a+ht)}{\partial x_{i_1} \dots \partial x_{i_k}} h_{i_1} \dots h_{i_k}$$

Usar el T. de Taylor de una variable a $\phi(t)$.

$$\phi(t) = \phi(0) + \phi'(0)t + \phi''(0)\frac{t^2}{2} + \dots + \phi^{(k-1)}(0)\frac{t^{k-1}}{(k-1)!} + \frac{1}{k!}\phi^{(k)}(\xi)t^k$$

$\xi \in (0,t)$

$\phi(1) = f(a+h) \qquad \phi(0) = f(a) \qquad \phi'(0) = Df(a)(h) \quad , \ldots , \quad \phi^{(k-1)}(0) = D^{k-1} f(a)(h)$

$$\phi^{(k)}(\xi) = D^k f(a+h\xi)(h)$$

$\xi \in (0,1)$

Cor. $\qquad f(a+h) = f(a) = \sum_{l=1}^{k-1} \frac{D^l f(a)(h)}{l!} + \frac{1}{k!} D^k f(a+\xi h)(h) \qquad h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$

$$f(a+h) = f(a) + \sum_{l=1}^{k-1} \frac{D^l f(a)}{l!}(h) + \frac{1}{k!} D^k f(a)(h) + o(\|h\|^k)$$

$$\lim_{h\to 0} \frac{\frac{1}{k!} D^k f(a+\xi h)(h) - \frac{1}{k!} D^k f(a)(h)}{\|h\|^k} = 0$$

$$\frac{1}{k!}\left[\sum_{i_1 i_2 \ldots i_k = 1}^{n} \left(\frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} f(a+\xi h) - \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} f(a) \right) \right] \frac{h_{i_1}}{\|h\|} \cdots \frac{h_{i_k}}{\|h\|} \to 0 \qquad \forall i \ \frac{|h_i|}{\|h\|} \le 1$$

$\to 0$

$h \to 0 \qquad f \in C^k(A) \qquad \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} f(a+\xi h) \xrightarrow{h\to 0} \frac{\partial^k f(a)}{\partial x_{i_1} \ldots \partial x_{i_k}}$

$$\frac{\left\| \frac{1}{k!} D^k f(a+\xi h)(h) - \frac{1}{k!} D^k f(a)(h) \right\|}{\|h\|^k} \leq$$

$$\leq \frac{1}{k!} \left[\sum_{i_1 \ldots i_1 = 1}^{n} \left\| \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} f(a+\xi h) - \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} f(a) \right\| \frac{|h_{i_1}|}{\|h\|} \ldots \frac{|h_{i_k}|}{\|h\|} \right.$$

$\leq 1 \qquad \leq 1$

0

Prob. $f(x,y) = e^{ax+ay}$ el polinomio de Taylor de orden 2 en $(0,0)$

$f(0,0) = 1$, $\quad \frac{\partial f}{\partial x}(0,0) = a \qquad \frac{\partial f}{\partial y}(0,0) = a$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = a^2$, $\quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = a^2 = f_{yx} \qquad f_{yy} = a^2$

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$h = \begin{pmatrix} x \\ y \end{pmatrix}$

$$P_2(x,y;0,0) = f(0,0) + Df(0,0)(h) + \frac{D^2 f(0,0)(h)}{2} =$$

$$= 1 + \begin{pmatrix} a & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a^2 & a^2 \\ a^2 & a^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a^2x + a^2y \\ a^2x + a^2y \end{pmatrix}$$

$$a^2x^2 + a^2xy + a^2xy + a^2y^2$$

$$= 1 + a(x+y) + \frac{1}{2}a^2(x+y)^2$$

$$t = ax + ay = a(x+y)$$

$$e^{ax+ay} = e^t = 1 + t + \frac{t^2}{2} + t^3/6 + \cdots + \frac{t^k}{k!} + o(\|h\|^k) \qquad h = (x,y)$$

$$= 1 + a(x+y) + \frac{a^2(x+y)^2}{2} + \frac{a^3(x+y)^3}{3!} + \cdots + \frac{a^k(x+y)^k}{k!} + o\left((x^2+y^2)^{k/2}\right)$$

Ejercicio: Usa la fórmula de sumatoria para $D^3f(0,0)(h)$ y $D^4f(0,0)(h)$