$$f(x,y,z) = e^{xy} + z\cos x.$$

$$\frac{\partial f}{\partial x} = y e^{xy} - z \operatorname{sen} x \,, \quad \frac{\partial f}{\partial y} = x e^{xy} \,, \quad \frac{\partial f}{\partial z} = \cos x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y e^{xy} - z \operatorname{sen} x\right) = e^{xy} + yx e^{xy}$$

$$\parallel$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x e^{xy}\right) = e^{xy} + xy e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial z} = -\operatorname{sen} x \quad = \quad \frac{\partial^2 f}{\partial z \partial x} = -\operatorname{sen} x$$

$$\frac{\partial^2 f}{\partial y \partial z} = 0 \quad = \quad \frac{\partial^2 f}{\partial z \partial y} = 0$$

$$f(x,y)=\begin{cases} x^2 \operatorname{artg} y/x - y^2 \operatorname{artg} x/y & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

$(a,0)$ $(0,a)$

$$\frac{\partial f}{\partial x}(x,y) = 2x \operatorname{atan}(y/x) - y \qquad \frac{\partial f}{\partial y} = -2y \operatorname{atan}(x/y) + x$$

$$\frac{\partial f}{\partial x}(a,0) = \lim_{x\to 0} \frac{\cancel{f(a+x,0)}^{0} - \cancel{f(a,0)}^{0}}{x} = 0$$

$$\frac{\partial f}{\partial y}(a,0) = \lim_{y\to 0} \frac{f(a,y) - \cancel{f(a,0)}^{0}}{y} = \lim_{y\to 0} \frac{\overbrace{a^2 \operatorname{artg} y/a}^{\sim y/a} - y^2 \operatorname{artg} a/y}{y} = a \quad a\neq 0$$

$$\frac{\partial f}{\partial y}(0,a) = 0 \qquad \frac{\partial f}{\partial x}(0,a) = -a$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{\cancel{f(0,y)}^{0} - \cancel{f(0,0)}^{0}}{y} = 0$$

$$\frac{\partial f}{\partial x}(0,0) = 0$$

$(x,y) \neq (a,0), (0,a)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[-2y \operatorname{artg}(x/y) + x\right] = \frac{x^2-y^2}{x^2+y^2} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial x \partial y}(a,0) = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}(a,0)\right] = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(a+x,0) - \frac{\partial f}{\partial y}(a,0)}{x} = \lim_{x\to 0} \frac{a+x-a}{x} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(a,0) = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}(a,0)\right] = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(a,y) - \cancelto{0}{\frac{\partial f}{\partial x}(a,0)}}{y} = \lim_{y\to 0} \frac{2a \operatorname{arctg} y/a - y}{y} = 1$$

$$= \begin{cases} 1 & a \neq 0 \\ -1 & a = 0 \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0) \qquad \left\{ \frac{\partial^2 f}{\partial x \partial y}(0,a) = \frac{\partial^2 f}{\partial y \partial x}(0,a) \quad \text{si } a \neq 0 \right.$$

Teorema de Schwarz: 1^o Probarlo $\mathbb{R}^2 \to \mathbb{R}$ $\quad \dfrac{\partial^2 f}{\partial x_i \partial x_j}$

Sin pérdida de generalidad nos restringimos a $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad x_0 = (a,b) \in A$

$u(x) = f(x, b+k) - f(x,b)$ $\quad u: [a, a+h] \subset A$ $\quad U = \{(x,y),\ x \in [a,a+h],\ y \in [b, b+k]\} \in A$

$\Delta f(a,b) = \underbrace{u(a+h) - u(a)} = [f(a+h, b+k) - f(a+h, b)] - [f(a, b+k) - f(a,b)]$

$[a, a+h]$ cont $\checkmark$ $\quad$ dif $(a, a+h)$ $\quad x \in (a, a+h)$ $\quad$ T.V.M. Lag. $\quad \exists \dfrac{\partial^2 f}{\partial y \partial x}$

$\Delta f(a,b) = u'(x) h = \left[\dfrac{\partial f}{\partial x}(x, b+k) - \dfrac{\partial f}{\partial x}(x,b) \right] h = $ T.V.M.

$= \dfrac{\partial^2}{\partial y \partial x} f(x,y)\, h \cdot k$ $\quad y \in (b, b+k)$, $x \in (a, a+h)$ $\quad \forall h,k$ t.q. $U \in A$

$\dfrac{\partial f}{\partial x}(x,y)$ en $\underline{\underline{y}}$ es cont en $[b, b+k]$ y der. en $(b, b+k)$

Sup $\frac{\partial^2}{\partial y \partial x} f(x,y)$ es cont. en $x_0=(a,b)$

$\forall \varepsilon>0 \quad \exists \delta>0 \quad t.q. \quad \forall (x,y) \in B_{x_0}(\delta)$ $\quad \left\| \frac{\partial^2}{\partial y \partial x} f(x,y) - \frac{\partial^2}{\partial y \partial x} f(a,b) \right\| < \varepsilon/2$

$\forall h,k$ en t.q. $U \in B_{x_0}(\delta) \subset A$

$$\left| \frac{\Delta f(a,b)}{hk} - \frac{\partial^2 f(a,b)}{\partial y \partial x} \right| < \varepsilon/2$$

$$\lim_{k\to 0} \left\{ \frac{\Delta f(a,b)}{hk} = \frac{1}{h} \left\{ \frac{[f(a+h,b+k) - f(a+h,b)]}{k} - \frac{[f(a,b+k) - f(a,b)]}{k} \right\} \right\} =$$

$$= \frac{1}{h} \left[\frac{\partial f(a+h,b)}{\partial y} - \frac{\partial f}{\partial y}(a,b) \right]$$

$$\lim_{k\to 0} \left| \lim_{k\to 0} \frac{\Delta f(a,b)}{h} - \frac{\partial^2 f(a,b)}{\partial y \partial x} \right| \leq \varepsilon/2 < \varepsilon$$

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall (x,y) \in B_{x_0}(\delta) \quad \Rightarrow$

$$\left| \frac{\frac{\partial f}{\partial y}(a+h, b) - \frac{\partial f}{\partial y}(a,b)}{h} - \frac{\partial^2 f(a,b)}{\partial y \partial x} \right| < \varepsilon$$

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(a+h, b) - \frac{\partial f}{\partial y}(a,b)}{h} = \frac{\partial^2 f(a,b)}{\partial y \partial x}$$

$$\| \qquad\qquad\qquad \|$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} f(a,b)\right] = \frac{\partial^2}{\partial x \partial y} f(a,b)$$