$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x+y} & x+y \neq 0 \\ (0,0) & x+y=0 \end{cases}$$

$(0,0)$ [ dobles $(a,-a)$ $a \neq 0$

$$\frac{x^2+y^2}{x+y} \qquad x_n = \frac{1}{n}, \; y_n = -\frac{1}{n} + \frac{1}{n^4}$$

no T.L en $(0,0)$

$$\frac{x_n^2+y_n^2}{x_n+y_n} = \frac{1/n^2 + \left(-1/n + 1/n^4\right)^2}{\frac{1}{n^4}} \sim \frac{\frac{2}{n^2}}{\frac{1}{n^4}} \sim 2n^2$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - \cancel{f(0,0)}^{0}}{x} = \lim_{x\to 0} \frac{x}{x} = 1 = \frac{\partial f}{\partial y}(0,0)$$

$$D_u f(0,0) = \cancel{Df(0,0)(u)}$$

No es dif. pues no es cont.

$u_x^2 + u_y^2 = 1$

$u_x + u_y \neq 0$

$$D_u f(0,0) = \lim_{\lambda \to 0} \frac{f(\lambda u_x, \lambda u_y) - \cancel{f(0,0)}^{0}}{\lambda} = \lim_{\lambda \to 0} \frac{u_x^2\lambda^2 + u_y^2\lambda^2}{(\lambda u_x + \lambda u_y)\lambda} =$$

$$= \frac{u_x^2+u_y^2}{u_x+u_y} = \frac{1}{u_x+u_y}$$

$$f(x,y) = \begin{cases} \dfrac{|xy|^\alpha}{x^2-xy+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases} \qquad \alpha>0$$

cont. en $\mathbb{R}^2$

$$x^2+y^2-xy=0 \iff x=y=0$$

$$x^2-xy+\frac{y^2}{4}+\frac{3y^2}{4}=0$$

$$\left(x-\frac{y}{2}\right)^2+\frac{3y^2}{4}=0 \qquad y=0, \quad x=\frac{y}{2}=0$$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\ ?$$

$$|\pm xy| \le \frac{x^2+y^2}{2}$$

$$-\frac{x^2+y^2}{2} \le \pm xy \le \frac{x^2+y^2}{2}$$

$$\frac{1}{2}(x^2+y^2) = (x^2+y^2)-\frac{x^2+y^2}{2} \le x^2+y^2-xy \le \frac{x^2+y^2}{2}+x^2+y^2 = \frac{3}{2}(x^2+y^2)$$

$$\frac{1}{x^2+y^2-xy} \le \frac{2}{x^2+y^2} \Rightarrow \left|\frac{|xy|^\alpha}{x^2+y^2}\right| \le 2\,\frac{|xy|^\alpha}{x^2+y^2}$$

$$\frac{|xy|^\alpha}{x^2+y^2} \le \begin{cases} \dfrac{|x|^\alpha|y|^\alpha}{x^2} = |x|^{\alpha-2}|y|^\alpha \to 0 & \alpha\ge 2 \\ \dfrac{|x|^\alpha|y|^\alpha}{y^2} = |y|^{\alpha-2}|x|^\alpha \to 0 & \alpha\ge 2 \end{cases}$$

$$\underbrace{\frac{\overbrace{|xy|}^{\le 1/2}}{x^2+y^2}}\,|xy|^{\alpha-1} \le \frac{1}{2}\,|xy|^{\alpha-1} \to 0 \qquad \alpha>1.$$

$$¿\alpha=1? \qquad \frac{|xy|}{x^2+y^2-xy} \qquad y=mx. \qquad \Rightarrow \qquad \text{N.T.L.}$$

$f(x,y)$ ¿Diferenciable? $\quad \alpha>1$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-\cancelto{0}{f(0,0)}}{x} = \lim_{x\to 0}\frac{0}{x} = 0 = \frac{\partial f}{\partial y}(0,0)$$

$$\lim_{(x,y)\to 0} \frac{f(x,y)-\cancelto{0}{f(0,0)}-\cancel{(0\ \ 0)}\begin{pmatrix}x\\y\end{pmatrix}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to 0} \frac{|xy|^{\alpha}}{(x^2+y^2-xy)\sqrt{x^2+y^2}}$$

$$\frac{|xy|^{\alpha}}{(x^2+y^2-xy)\sqrt{x^2+y^2}} \le \frac{2\,|xy|^{\alpha}}{(x^2+y^2)\sqrt{x^2+y^2}} \xrightarrow{?} 0$$

$$\frac{|xy|^{\alpha}}{(x^2+y^2)^{3/2}} \leq \begin{cases} |x|^{\alpha} |y|^{\alpha-3} & \alpha \geq 3 \\ |y|^{\alpha} |x|^{\alpha-3} & \\ \left(\frac{|xy|}{x^2+y^2}\right)^{3/2} |xy|^{\alpha-3/2} & \alpha > 3/2 \end{cases}$$

$\alpha = 3/2$

$$\frac{|xy|^{3/2}}{(x^2+y^2-xm)\sqrt{x^2+y^2}}$$

$y = mx \Rightarrow$ lim dep. de $m \Rightarrow$ N.T.L

Diferenciable en $(0,0) \iff \alpha > 3/2$

- $$f(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2+xy} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

1) cont. en $(0,0)$.

$$\left|\frac{x^4-y^4}{x^2+y^2+xy}\right| \leq 2\,\frac{|x^4|+|y^4|}{[x^2+y^2]} \leq 2\,\frac{|x|^4}{x^2+\cancel{y^2}} + 2\,\frac{|y|^4}{\cancel{x^2}+y^2} \leq 2|x|^2+2|y|^2 \to 0$$

$$\wedge$$

$$2\,\frac{|x^4-y^4|}{x^2+y^2} = 2\,|x^2-y^2| \leq 2\,|x^2|+|y|^2$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{x^2}{x} = 0 \qquad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{\overset{=y^2}{f(0,y)}-\cancelto{0}{f(0,0)}}{y} = 0$$

$$\left|\frac{f(x,y)-\cancelto{0}{f(0,0)}-\cancel{Df(0,0)}\begin{pmatrix}x\\y\end{pmatrix}^{0}}{\sqrt{x^2+y^2}}\right| = \left|\frac{x^4-y^4}{x^2+y^2+xy}\right|\left|\frac{1}{\sqrt{x^2+y^2}}\right| \leq$$

$$\leq \frac{2(x^2+y^2)}{\sqrt{x^2+y^2}} = 2\sqrt{x^2+y^2} \to 0 \qquad \text{!! es diferenciable en } (0,0)$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{4x^3 \cdot (x^2+y^2+xy) - (x^4-y^4)(2x+y)}{(x^2+y^2+xy)^2} =$$

$$= \frac{4x^3}{x^2+y^2+xy} - \frac{(x^4-y^4)(2x+y)}{(x^2+y^2+xy)^2} \quad \checkmark \qquad (x,y) \neq 0$$

$$\frac{\partial f}{\partial y}(x,y) = -\frac{4y^3}{x^2+y^2+xy} + \frac{(x^4-y^4)(2y+x)}{(x^2+y^2+xy)^2} \quad \checkmark$$

?? cont. en $(0,0)$.

$$f(x,y) = \begin{cases} (x+y)\sqrt{x^2+y^2} \cdot \operatorname{sen} \frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

cont. $\mathbb{R}^2$

$$\lim_{(x,y)\to 0} f(x,y) = ?$$

$$\left| (x+y)\sqrt{x^2+y^2} \operatorname{sen} \frac{1}{\sqrt{x^2+y^2}} \right| \leq (|x|+|y|)\sqrt{x^2+y^2} \longrightarrow 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - \cancel{f(0,0)}^{0}}{x} = \lim_{x\to 0} \underbrace{\cancel{x}\,|x|}_{\downarrow\, 0} \operatorname{sen} \underbrace{\frac{1}{|x|}}_{\text{acotado}} \longrightarrow 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0 .$$

$$\frac{f(x,y) - \cancel{f(0,0)}^{0} - \cancel{Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix}}^{0}}{\sqrt{x^2+y^2}} = \underbrace{(x+y)}_{\downarrow\, 0} \underbrace{\operatorname{sen} \frac{1}{\sqrt{x^2+y^2}}}_{\text{acotado !}} \longrightarrow 0$$

$$f(x,y)=\begin{cases}\dfrac{x \operatorname{sen} y - y \operatorname{sen} x}{x^2+y^2} & (x,y)\neq 0\\[2mm] 0 & (x,y)=0\end{cases}$$

$\operatorname{sen} x \cancel{\sim} x \qquad \operatorname{sen} y \cancel{\sim} y$

$\operatorname{sen} x = x - x^3/6 + o(x^3)$

$$f(x,y)=\frac{x\left(y-y^3/6+o(y^3)\right)-y\left(x-x^3/6+o(x^5)\right)}{x^2+y^2}=$$

$$=\frac{-\frac{xy^3}{6}+\frac{yx^3}{6}}{x^2+y^2}+\frac{x\,o(y^3)+y\,o(x^3)}{x^2+y^2}\to 0$$

$$\frac{|x^2-y^2|}{|x^2+y^2|}\le\frac{x^2+y^2}{x^2+y^2}=1$$

$$|f(x,y)|\sim\left|\frac{yx\,(x^2-y^2)}{6\,(x^2+y^2)}\right|\le\frac{|xy|}{6}\longrightarrow 0$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{\overset{0}{\cancel{f(x,0)}}-\overset{0}{\cancel{f(0,0)}}}{x}=\lim_{x\to 0}\frac{0}{x}=0=\frac{\partial f}{\partial y}(0,0)\to 0 \quad\Rightarrow\quad Df(0,0)\overset{?}{=}(0,0)$$

$$\left| \frac{f(x,y) - \cancel{f(0,0)}^{0} - \cancel{Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix}}^{0}}{\sqrt{x^2+y^2}} \right| = \left| \frac{x \sin y - y \sin x}{(x^2+y^2)\sqrt{x^2+y^2}} \right| \xrightarrow{?} 0$$

$$\left| \quad \right| \sim \left| \frac{|xy|\,(x^2-y^2)}{\sqrt{x^2+y^2}\; x^2+y^2} \right| \le \sqrt{\underbrace{\frac{|xy|}{x^2+y^2}}_{\le \frac{1}{2}}}\; \underbrace{\frac{|x^2-y^2|}{x^2+y^2}}_{\le 1}\; \sqrt{|xy|} \le \frac{1}{2}\sqrt{|xy|} \to 0$$

$$\frac{\partial f}{\partial x}(x,y) = ? \qquad \frac{\partial f}{\partial y}(x,y) = ?$$