$$f(a+h)-f(a)-Df(a)(h)=o(\|h\|) \qquad Df(a)(h)=\left(\frac{\partial f}{\partial x_1}(a)\ \dots\ \frac{\partial f}{\partial x_n}(a)\right)$$

Sin pérdida de generalidad $\quad f:\mathbb{R}^n\to\mathbb{R}$

$$f(a_1+h_1,a_2+h_2,\dots,a_n+h_n)-f(a_1,a_2,\dots a_n)=$$

$$[f(a_1+h_1,a_2+h_2,\dots a_n+h_n)-f(a_1,a_2+h_2,\dots,a_n+h_n)]+[f(a_1,a_2+h_2,\dots,a_n+h_n)-f(a_1,a_2,a_3+h_3,\dots a_n+h_n)]+[f(a_1,a_2,a_3+h_3,\dots,a_n+h_n)-\dots]\dots$$

$$+[f(a_1,a_2,\dots,a_{n-1},a_n+h_n)-f(a_1,a_2,\dots a_n)]=$$

$$\left\{\begin{array}{l}\xi_1\in(a_1,a_1+h_1)\\ \xi_2\in(a_2,a_2+h_2)\\ \vdots\\ \xi_n\in(a_n,a_n+h_n)\end{array}\right.$$

$$=\frac{\partial f}{\partial x_1}\underbrace{(\xi_1,a_2+h_2,\dots,a_n+h_n)}_{\tilde\xi_1}h_1+\frac{\partial f}{\partial x_2}\underbrace{(a_1,\xi_2,a_3+h_3,\dots a_n+h_n)}_{\tilde\xi_2}h_2+\dots$$

$$+\frac{\partial f}{\partial x_n}\underbrace{(a_1,a_2,\dots,a_{n-1},\xi_n)}_{\tilde\xi_n}h_n$$

$\tilde\xi_k\to a \qquad h\to 0$

$$f(a+h)-f(a)-Df(a)(h)=\frac{\partial f}{\partial x_1}(\tilde\xi_1)h_1+\frac{\partial f}{\partial x_2}(\tilde\xi_2)h_2+\dots+\frac{\partial f}{\partial x_n}(\xi_n)h_n-\left(\frac{\partial f}{\partial x_1}(a)h_1+\frac{\partial f}{\partial x_2}(a)h_2+\dots+\frac{\partial f}{\partial x_n}(a)h_n\right)$$

$$=\left[\frac{\partial f}{\partial x_1}(\tilde\xi_1)-\frac{\partial f}{\partial x_1}(a)\right]h_1+\left[\frac{\partial f}{\partial x_2}(\tilde\xi_2)-\frac{\partial f}{\partial x_2}(a)\right]h_2+\left[\frac{\partial f}{\partial x_n}(\tilde\xi_n)-\frac{\partial f}{\partial x_n}(a)\right]h_n$$

$f: \mathbb{R}^n \to \mathbb{R}$

$$\frac{\| f(a+h) - f(a) - Df(a)(h) \|}{\|h\|} \xrightarrow[\|h\| \to 0]{?} 0$$

$$\frac{\| \qquad -\|}{\|h\|} \le \left\| \frac{\partial f}{\partial x_1}(\hat{\xi}_1) - \frac{\partial f}{\partial x_1}(a) \right\| \underbrace{\frac{|h_1|}{\|h\|}}_{\le 1} + \left\| \frac{\partial f}{\partial x_2}(\hat{\xi}_2) - \frac{\partial f}{\partial x_2}(a) \right\| \underbrace{\frac{|h_2|}{\|h\|}}_{\le 1} + \cdots + \left\| \frac{\partial f}{\partial x_n}(\xi_n) - \frac{\partial f}{\partial x_n}(a) \right\| \underbrace{\frac{|h_n|}{\|h\|}}_{\le 1}$$

$$\|h\| = \sqrt{|h_1|^2 + \cdots + |h_n|^2} \Rightarrow |h_k| \le \|h\| \quad \forall k = 1, \dots, n$$

$$\frac{\| f(a+h) - f(a) - Df(a)(h) \|}{\|h\|} \le \underbrace{\left\| \frac{\partial f}{\partial x_1}(\hat{\xi}_1) - \frac{\partial f}{\partial x_1}(a) \right\|}_{\downarrow\, 0} + \cdots + \underbrace{\left\| \frac{\partial f(\hat{\xi}_n)}{\partial x_n} - \frac{\partial f(a)}{\partial x_n} \right\|}_{\downarrow\, 0} \xrightarrow{h \to 0} 0$$

$h \to 0$ $(\|h\| \to 0)$

$\tilde{\xi}_1 \to a$ $\quad \tilde{\xi}_2 \to a$ $\quad \cdots \quad \hat{\xi}_n \to a$ $\qquad \frac{\partial f}{\partial x_k}(x)$ cont. en $x = a$

$$\Rightarrow f(a+h) - f(a) - Df(a)(h) = o(\|h\|)$$

$$f(x,y) = \begin{cases} (x^2+y^2)\,\operatorname{sen}\dfrac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$|f(x,y)| \leq (x^2+y^2)\left|\operatorname{sen}\frac{1}{x^2+y^2}\right| \leq x^2+y^2 \xrightarrow[(x,y)\to 0]{} 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - \cancelto{0}{f(0,0)}}{x} = \lim_{x\to 0} \frac{x^2 \operatorname{sen} 1/x^2}{x} = \lim_{x\to 0} x \operatorname{sen}\frac{1}{x^2} = 0.$$

$$\left|x \operatorname{sen}\frac{1}{x^2}\right| \leq |x| \to 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - \cancelto{0}{f(0,0)}}{y} = \lim_{y\to 0} \frac{y^2 \operatorname{sen}\frac{1}{y^2}}{y} = 0$$

$$Df(0,0) = (0 \quad 0)$$

$$\frac{f(x,y) - \cancelto{0}{f(0,0)} - \cancelto{0}{(0\ \ 0)\begin{pmatrix} x \\ y \end{pmatrix}}}{\sqrt{x^2+y^2}} \xrightarrow[(x,y)\to 0]{?} 0$$

$$\left| \sqrt{x^2+y^2}\, \underbrace{\operatorname{sen} \frac{1}{x^2+y^2}}_{\text{acotada}} \right| \leq \sqrt{x^2+y^2} \to 0$$

La función es dif. en $(0,0)$

$$f(x,y) = (x^2+y^2) \operatorname{sen} \frac{1}{x^2+y^2}$$

$$\frac{\partial f(x,y)}{\partial x} = 2x \operatorname{sen} \frac{1}{x^2+y^2} + (x^2+y^2) \cdot \left(\cos \frac{1}{x^2+y^2} \right) \frac{(-1)\, 2x}{(x^2+y^2)^2}$$

$$= 2x \operatorname{sen} \frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2} \cdot \cos \frac{1}{x^2+y^2}$$

$$\left| \frac{\partial f}{\partial x}(0,0) = 0 \right.$$

$$\lim_{(x,y)\to 0} \frac{\partial f(x,y)}{\partial x} = \lim_{(x,y)\to 0} \cancel{2x \operatorname{sen} \frac{1}{x^2+y^2}}^{\to 0} - \frac{2x}{x^2+y^2} \underbrace{\cos \frac{1}{x^2+y^2}}_{\text{oscila !!!}}$$

$$\frac{2x}{x^2+y^2} \qquad y = mx \quad \Rightarrow \quad \text{N.T.L.}$$

$$f(x,y) = (x^2+y^2) \operatorname{sen} \frac{1}{\sqrt{x^2+y^2}} \quad (x,y) \neq 0 \quad , \quad 0 \quad \text{si} \quad (x,y) = (0,0)$$