$$f(x,y) = \frac{x^2 \operatorname{sen} y + y^2 \operatorname{sen} x}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \operatorname{sen} y + y^2 \operatorname{sen} x}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 y + y^2 x}{x^2+y^2}$$

$$= \lim \frac{xy}{x^2+y^2}(x+y) = 0$$

$$\underbrace{\left|\frac{xy}{x^2+y^2}\right|}\,|x+y| \leq \frac{1}{2}\big(|x|+|y|\big) \longrightarrow 0$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\,\frac{x^2 \operatorname{sen} y + y^2 \operatorname{sen} x}{x^2+y^2} = \frac{(2x \operatorname{sen} y + y^2 \cos x)(x^2+y^2) - (x^2 \operatorname{sen} y + y^2 \operatorname{sen} x)\,2x}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - \cancel{f(0,0)}}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\cancel{f(0,h)} - \cancel{f(0,0)}}{h} = 0$$

$$D_u f(0,0) = \lim_{\lambda\to 0} \frac{f(\lambda u_x, \lambda u_y) - \cancel{f(0,0)}}{\lambda} = \lim_{\lambda\to 0} \frac{\lambda^2 u_x^2 \operatorname{sen} \lambda u_y + \lambda^2 u_y^2 \operatorname{sen} \lambda u_x}{(\lambda^2 u_x^2 + \lambda^2 u_y^2)\,\lambda}$$

$$= \lim_{\lambda\to 0} \frac{u_x^2 u_y + u_y^2 u_x}{\underbrace{u_x^2 + u_y^2}_{1}} = u_x^2 u_y + u_y^2 u_x = u_x u_y (u_x + u_y)$$

$\|u\| = 1$

$D_u f(0,0)$ N. es aplicación lineal en $u$ → $D_u f$. No es lineal!

$$\lim_{(x,y)\to 0} \frac{f(x,y) - \cancel{f(0,0)} - \cancel{D f(0,0)} \begin{pmatrix} x \\ y \end{pmatrix}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to 0} \frac{x^2 \operatorname{sen} y + y^2 \operatorname{sen} x}{(x^2+y^2)\sqrt{x^2+y^2}}$$

N.T.L.

$y = mx$ $\qquad \dfrac{x^2 \operatorname{sen} mx + m^2 x^2 \operatorname{sen} x}{(x^2 + m^2 x^2)^{3/2}} \sim \dfrac{x^3 m + x^3 m^2}{(1+m^2)^{3/2}\, x^3} = \dfrac{m + m^2}{(1+m^2)^{3/2}}$

depende de $m$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad f(x,y) = \begin{pmatrix} x^2 + y^2 \\ \dfrac{2x+y}{x^2+y^2+1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $(0,0)$

$$\lim_{(x,y)\to 0} f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial f_1}{\partial x}(0,0) = \lim_{h\to 0} \frac{f_1(h,0) - \cancel{f_1(0,0)}^{0}}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$Df_1(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$\frac{\partial f_1}{\partial y}(0,0) = \lim_{h\to 0} \frac{f_1(0,h) - \cancel{f_1(0,0)}^{0}}{h} = 0$$

$$\frac{\partial f_2}{\partial x}(0,0) = \lim_{h\to 0} \frac{f_2(h,0) - \overset{0}{\cancel{f_2(0,0)}}}{h} = \lim_{h\to 0} \frac{2h}{(h^2+1)h} = 2$$

$$Df_2(0,0) = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

$$\frac{\partial f_2}{\partial y}(0,0) = \lim_{h\to 0} \frac{f_2(0,h) - f_2(0,0)}{h} = \lim_{h\to 0} \frac{h}{(h^2+1)h} = 1$$

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\lim_{(x,y)\to 0} \frac{f(x,y) - f(0,0) - Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix}}{\sqrt{x^2+y^2}} = 0$$

$$\frac{1}{\sqrt{x^2+y^2}}\left[\begin{pmatrix} x^2+y^2 \\ \frac{2x+y}{x^2+y^2+1} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] \xrightarrow{(x,y)\to 0} 0$$

$$F = \begin{pmatrix} \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} \\ \dfrac{2x+y}{(x^2+y^2+1)\sqrt{x^2+y^2}} - \dfrac{(2x+y)}{\sqrt{x^2+y^2}} \end{pmatrix}$$

$\|F\| \to 0$

$F_1 = \sqrt{x^2+y^2} \to 0 \quad (x,y) \to 0$

$$F_2 = \frac{(2x+y) - (1+x^2+y^2)(2x+y)}{(x^2+y^2+1)\sqrt{x^2+y^2}} = \frac{(2x+y)}{1+x^2+y^2} \frac{\left[(-1)\cancel{(x^2+y^2)}\right]^{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}} \to 0$$

$$f(x,y) = \begin{cases} x^2 \operatorname{artg} y/x - y^2 \operatorname{artg} \frac{x}{y} & x \neq 0, y \neq 0 \\ 0 \end{cases}$$

$(a,0) \quad (0,a)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = 0 \qquad \frac{\partial}{\partial y} f(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = 0$$

$$\lim_{(x,y)\to 0} \frac{f(x,y) - f(0,0) - Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix}}{\sqrt{x^2+y^2}} \overset{?}{=} 0$$

$$\left| \frac{x^2 \operatorname{artg} y/x - y^2 \operatorname{artg} x/y}{\sqrt{x^2+y^2}} \right| \leq \frac{x^2 \overbrace{|\operatorname{artg} y/x|}^{\leq M}}{\sqrt{x^2+\cancel{y^2}}} + \frac{y^2 \overbrace{|\operatorname{artg} x/y|}^{\leq M}}{\sqrt{\cancel{x^2}+y^2}}$$

$$\leq M \frac{x^2}{\sqrt{x^2}} + M \frac{y^2}{\sqrt{y^2}} = M(|x| + |y|) \to 0$$

—ω—

Ejercicio: $f(x,y) = \dfrac{\operatorname{sen} x + \operatorname{sen} y}{x+y}$ $\qquad x \neq -y$