On the Krall-type polynomials

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Abstract

Using a general and simple algebraic approach, some results on Krall-type orthogonal polynomials and some of their extensions are obtained.

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1 Introduction

Let $\mathfrak{u} : \mathbb{P} \mapsto \mathbb{R}$ be a quasi-definite linear functional in the vector space $\mathbb{P}$ of polynomials with complex coefficients, then there exists a sequence of monic polynomials $(P_n)_n$ with $\deg P_n = n$, such that [11]

$$\langle \mathfrak{u}, P_n P_m \rangle = k_n \delta_{n,m}, \qquad k_n \neq 0.$$

Special cases of quasi-definite linear functionals are the classical ones (those of Jacobi, Laguerre, Hermite, and Bessel).

In the last years perturbations of a quasi-definite linear functional $\mathfrak{u}$ via the addition of Dirac delta functionals and their derivatives have been extensively studied (see e.g. [1, 5, 14, 15, 16, 17, 18, 19, 23] and references therein), i.e., the linear functional

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + \sum_{i=1}^{M} A_i \delta(x - a_i) - \sum_{j=1}^{N} B_j \delta'(x - b_j), \tag{1}$$

where $(A_i)_{i=1}^M$ and $(B_j)_{j=1}^N$ are real numbers, $\delta(x - y)$ and $\delta'(x - y)$ mean the Dirac linear functional and its derivative, respectively, defined by $\langle \delta(x - y), p(x) \rangle = p(y)$, $\langle \delta'(x - y), p(x) \rangle = -p'(y)$, $\forall p \in \mathbb{P}$. If at most one B_i is nonzero and $A_i = 0$, $i = 1, \ldots, m$, then the analysis of the quasi-definiteness of the linear functional $\widetilde{\mathfrak{u}}$ is presented in [9].

Special important examples appear when $\mathfrak{u}$ is a classical linear functional and the mass points are located at the ends of the interval or orthogonality. In this case we have studied the corresponding sequences of orthogonal polynomials in several papers: For the Laguerre linear functional see [2, 3], for the Bessel linear functional see [6], for the Jacobi linear functional see [7] and for the Hermite case see [1]. In particular the quasi-definiteness of $\widetilde{\mathfrak{u}}$, relative asymptotics, the representation as a hypergeometric function, and the location of their zeros have been obtained.

This kind of perturbations can be seen as the simplest (lower order) discrete-continuous version of the "bispectral" property. The continuous "bispectral problem" consists of describing all Schrödinger type differential operators of second order such that their eigenfunctions should satisfy a differential equation (of arbitrary finite order) in the spectral variable. The basic tools are based in the Darboux factorization method [13]. In the discrete-continuous case, this factorization is related to the LU and UL factorization of the Jacobi matrix associated with the sequence of orthogonal polynomials which are the eigenfunctions of a second order linear differential operator of hypergeometric type. Thus the so-called Krall polynomials appear in a natural way [15, 16, 17, 18, 19].

Very recently, in [12] the study of sequences of orthogonal polynomials and their associated Hankel determinants when we add an extra term to a weight function is considered using the Heine formula. In particular the authors analyze separately the addition of $A\delta(x - a)$ and $A\delta'(x - a)$. Notice that they do not discuss the quasi-definite character of the linear functional

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$\widetilde{\mathfrak{u}}$ given either by

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + A\delta(x-a) \quad \text{or} \quad \widetilde{\mathfrak{u}} = \mathfrak{u} - A\delta'(x-a).$$

This is a a very important gap, specially when derivatives of Dirac delta functionals are considered, because necessary and sufficient conditions for the existence of the corresponding sequence of orthogonal polynomials are needed. They obtain particular cases of our previous results [2, 3, 4, 5, 7] using a different approach based in the above mentioned Heine formula.

The aim of the present contribution is to cover the above gap about the quasidefiniteness when we add two extra terms to a linear functional $\mathfrak{u}$ in such a way that the perturbed linear functional is given by

$$\widetilde{\mathfrak{u}} = \mathfrak{u} + A\delta(x-a) - B\delta'(x-b), \qquad a \neq b. \tag{2}$$

The case $a = b$ has been considered in some previous work by the authors and coworkers [2, 3, 5, 6].

We use an innovative and algebraic approach taking into account the orthogonality properties of the sequence of polynomials.

Section 2 deals with the analysis of necessary and sufficient conditions for the quasidefiniteness of $\widetilde{\mathfrak{u}}$ given by (2). In such a case, the coefficients of the three-term recurrence relation that the corresponding sequence of monic orthogonal polynomials satisfies is obtained. Furthermore, if we assume the $\mathfrak{u}$ is a semiclassical linear functional, i.e. it satisfies a Pearson equation $D[\phi\mathfrak{u}] = \Psi\mathfrak{u}$, with ϕ, ψ polynomials such that $\deg(\psi) \geq 1$, see [24], then the linear functional $\widetilde{\mathfrak{u}}$ is also semiclassical. This can be read in terms of the holonomic second order linear differential equation that such polynomials. The explicit expression for their coefficients is given. Finally, as an application, the WKB method is used for the the analysis of the location of their zeros.

2 General theory

For the sake of simplicity we will consider a linear functional $\widetilde{\mathfrak{u}}$ given by $\widetilde{\mathfrak{u}} = \mathfrak{u} + A\delta(x-a) - B\delta'(x-b)$. Here we assume a different from b. When $a = b$ this situation has been extensively considered in the literature and yields a simplification of the computation in the representation formulas (see e.g. [1]). Thus we will recover the result in [12] in a very simple way (for the general case we refer the reader to [1]).

2.1 Representation formulas

If $\widetilde{\mathfrak{u}}$ is quasi-definite then there exists a sequence of monic polynomials $(\widetilde{P}_n)_n$ orthogonal with respect to $\widetilde{\mathfrak{u}}$ and therefore we can consider the Fourier expansion

$$\widetilde{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} \lambda_{n,k} P_k(x). \tag{3}$$

Thus, for $0 \le k \le n-1$,

$$\lambda_{n,k} = \frac{\langle \mathfrak{u}, \widetilde{P}_n(x) P_k(x) \rangle}{\langle \mathfrak{u}, P_k^2(x) \rangle} = -\frac{A\widetilde{P}_n(a)P_k(a) + B\widetilde{P}_n(b)P_k'(b) + B\widetilde{P}_n'(b)P_k(b)}{\langle \mathfrak{u}, P_k^2(x) \rangle}$$

$$= A\widetilde{P}_n(a)\frac{P_k(a)}{\langle \mathfrak{u}, P_k^2(x) \rangle} - B\widetilde{P}_n(b)\frac{P_k'(b)}{\langle \mathfrak{u}, P_k^2(x) \rangle} - B\widetilde{P}_n'(b)\frac{P_k(b)}{\langle \mathfrak{u}, P_k^2(x) \rangle}.$$

As a consequence, (3) becomes

$$\begin{aligned}
\widetilde{P}_n(x) = \ & P_n(x) - A\widetilde{P}_n(a)\mathrm{K}_{n-1}(x,a) - B\widetilde{P}_n(b)\mathrm{K}_{n-1}^{(0,1)}(x,b) \\
& - B\widetilde{P}_n'(b)\mathrm{K}_{n-1}(x,b),
\end{aligned} \tag{4}$$

where, as usual,

$$\mathrm{K}_n^{(i,j)}(x,y) = \sum_{l=0}^{n} \frac{P_l^{(i)}(x) P_l^{(j)}(y)}{\langle \mathfrak{u}, P_l^2(x) \rangle}, \qquad P_l^{(i)}(x) := \frac{d^i}{dx^i} P_l(x), \quad i,j \in \mathbb{N}.$$

As a convention $\mathrm{K}_n^{(0,0)}(x,y) := \mathrm{K}_n(x,y)$ denotes the reproducing kernel associated with the linear functional $\mathfrak{u}$. It is very well known (reproducing property of the kernels) that $\langle \mathfrak{u}, \mathrm{K}_n(x,y)p(x) \rangle = p(y)$ for every polynomial $p(x)$ of degree less than or equal to n. Evaluating (4) for $x = a$, we get

$$\begin{aligned}
\widetilde{P}_n(a) = \ & P_n(a) - A\widetilde{P}_n(a)\mathrm{K}_{n-1}(a,a) - B\widetilde{P}_n(b)\mathrm{K}_{n-1}^{(0,1)}(a,b) \\
& - B\widetilde{P}_n'(b)\mathrm{K}_{n-1}(a,b).
\end{aligned} \tag{5}$$

A similar evaluation in (4) for $x = b$ gives

$$\begin{aligned}
\widetilde{P}_n(b) = \ & P_n(b) - A\widetilde{P}_n(a)\mathrm{K}_{n-1}(b,a) - B\widetilde{P}_n(b)\mathrm{K}_{n-1}^{(0,1)}(b,b) \\
& - B\widetilde{P}_n'(b)\mathrm{K}_{n-1}(b,b).
\end{aligned} \tag{6}$$

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Finally, taking derivatives in (4) and evaluating the resulting expression for $x = b$, we obtain

$$\widetilde{P}'_n(b) = \quad P'_n(b) - A\widetilde{P}_n(a)\mathrm{K}^{(1,0)}_{n-1}(b,a) - B\widetilde{P}_n(b)\mathrm{K}^{(1,1)}_{n-1}(b,b)$$
$$- B\widetilde{P}'_n(b)\mathrm{K}^{(1,0)}_{n-1}(b,b). \tag{7}$$

Thus we get a system of three linear equations in the variables $\widetilde{P}_n(a)$, $\widetilde{P}_n(b)$, and $\widetilde{P}'_n(b)$

$$\begin{pmatrix} \widetilde{P}_n(a) \\ \widetilde{P}_n(b) \\ \widetilde{P}'_n(b) \end{pmatrix} = \begin{pmatrix} P_n(a) \\ P_n(b) \\ P'_n(b) \end{pmatrix} - \mathbb{K}_{n-1}\mathrm{D} \begin{pmatrix} \widetilde{P}_n(a) \\ \widetilde{P}_n(b) \\ \widetilde{P}'_n(b) \end{pmatrix}, \tag{8}$$

where

$$\mathrm{D} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix}, \qquad \mathbb{K}_{n-1} = \begin{pmatrix} \mathrm{K}_{n-1}(a,a) & \mathrm{K}_{n-1}(a,b) & \mathrm{K}^{(0,1)}_{n-1}(a,b) \\ \mathrm{K}_{n-1}(b,a) & \mathrm{K}_{n-1}(b,b) & \mathrm{K}^{(0,1)}_{n-1}(b,b) \\ \mathrm{K}^{(1,0)}_{n-1}(b,a) & \mathrm{K}^{(0,1)}_{n-1}(b,b) & \mathrm{K}^{(1,1)}_{n-1}(b,b) \end{pmatrix},$$

Notice that if the matrix $\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D}$, where I is the identity matrix, is non singular, then we get the existence and uniqueness for the solution of (8). Therefore, (4) becomes

$$\widetilde{P}_n(x) = \quad P_n(x) - \big(\mathrm{K}_{n-1}(x,a), \mathrm{K}_{n-1}(x,b), \mathrm{K}^{(0,1)}_{n-1}(x,a) \big) \times$$
$$\mathrm{D}(\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D})^{-1} \left(P_n(a), P_n(b), P'_n(b) \right)^T . \tag{9}$$

All the above lead to the following

Theorem 1 *The linear functional $\widetilde{\mathbf{u}}$ is a quasi-definite linear functional if and only if*

(i) *The matrix $\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D}$ is nonsingular for every $n \in \mathbb{N}$.*

(ii) $\langle \mathbf{u}, P^2_n(x) \rangle + \big(P_n(a), P_n(b), P'_n(b) \big) \mathrm{D} \left(\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D} \right)^{-1} \begin{pmatrix} P_n(a) \\ P_n(b) \\ P'_n(b) \end{pmatrix} \neq 0$, *for every $n \in \mathbb{N}$.*

(iii) *The norm $\langle \widetilde{\mathbf{u}}, \widetilde{P}^2_n(x) \rangle$ is given by*

$$\langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x) \rangle = \langle \mathbf{u}, P_n^2(x) \rangle + (P_n(a), P_n(b), P_n'(b)) \times$$
$$\mathrm{D}\,(\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D})^{-1}\,(P_n(a), P_n(b), P_n'(b))^T \neq 0. \tag{10}$$

In such a case, the sequence $(\widetilde{P}_n)_n$ of monic orthogonal polynomials with respect to $\widetilde{\mathbf{u}}$ is given by (9).

The formula (10) constitutes the first representation formula for the polynomials $(\widetilde{P}_n)_n$.

Remark 2 *If the entries of the matrix* D *are nonzero, then* D *is an nonsingular hermitian matrix, and*

$$\mathrm{D}(\mathrm{I} + \mathbb{K}_{n-1}\mathrm{D})^{-1} = (\mathrm{D}^{-1} + \mathbb{K}_{n-1})^{-1} = \mathrm{M}_{n-1},$$

where $\mathrm{D}^{-1} + \mathbb{K}_{n-1}$ *is an hermitian matrix. Therefore (10) means that*

$$1 + \varepsilon_n\,(\widehat{P}_n(a), \widehat{P}_n(b), \widehat{P}_n'(b))\,\mathrm{M}_{n-1} \begin{pmatrix} \widehat{P}_n(a) \\ \widehat{P}_n(b) \\ \widehat{P}_n'(b) \end{pmatrix} \neq 0, \quad \widehat{P}_n(a_i) = \frac{P_n(a_i)}{\sqrt{|\langle \mathbf{u}, P_n^2(x) \rangle|}},$$

$a_i = a, b,$ *and* $\varepsilon_n = \mathrm{sign}\,(\langle \mathbf{u}, P_n^2(x) \rangle).$

Let us now obtain another representation formula. Taking into account the Christoffel-Darboux formula

$$\mathrm{K}_{n-1}(x, y) = \frac{1}{k_n} \left[\frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x - y} \right], \qquad k_n = \langle \mathbf{u}, P_n^2(x) \rangle,$$

as well as

$$\mathrm{K}_{n-1}^{(0,1)}(x, y) = \frac{1}{k_n} \left[\frac{P_n(x)P_{n-1}'(y) - P_n'(y)P_{n-1}(x)}{x - y} \right.$$
$$\left. + \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{(x - y)^2} \right],$$

inserting these two expressions in (9) and denoting $\phi(x) = (x - a)(x - b)^2$, we deduce

$$\phi(x)\widetilde{P}_n(x) = A(x; n)P_n(x) + B(x; n)P_{n-1}(x), \tag{11}$$

where $A(x; n)$ and $B(x; n)$ are polynomials of degree independent of n and at most 3 and 2, respectively. On the other hand, from the three-term recurrence relation that the sequence $(P_n)_n$ satisfies

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \qquad \gamma_n \neq 0, \quad \forall n \in \mathbb{N}, \quad (12)$$

and taking into account (11), for $n \geq 1$ we get

$$\phi(x) \widetilde{P}_n(x) = C(x; n) P_{n+1}(x) + D(x; n) P_n(x),$$

$$C(x; n) = -\frac{B(x; n)}{\gamma_n}, \qquad D(x; n) = A(x; n) + \frac{x - \beta_n}{\gamma_n} B(x; n). \tag{13}$$

Now, from any of the above expressions a very useful representation formula follows. In fact, if $\mathfrak{u}$ is a semiclassical linear functional [24], then there exist a polynomial $\psi(x)$ and two polynomials $M_1(x; n)$, $N_1(x; n)$ with degree independent of n, such that

$$\psi(x) P_n'(x) = M_1(x; n) P_n(x) + N_1(x; n) P_{n-1}(x), \tag{14}$$

or, equivalently, using the TTRR (12) we get

$$\psi(x) P_n'(x) = M_2(x; n) P_n(x) + N_2(x; n) P_{n+1}(x). \tag{15}$$

where again the degree of $M_2(x; n)$ and $N_2(x; n)$ are independent of n. Usually the formulas (14) and (15) represent the lowering and raising operators for the semiclassical family $(P_n)_n$.

Using (11) and (14) or (13) and (15) we obtain the following formula

$$q(x; n) \widetilde{P}_n(x) = a(x; n) P_n(x) + b(x; n) P_n'(x), \tag{16}$$

where a, b, and q are polynomials with degree independent of n. The last formula plays a fundamental role in order to find the second order linear differential equation that the polynomials $(\widetilde{P}_n)_n$ satisfy as it is shown in section 2.3.

2.2 Three-Term Recurrence Relation and Lowering and Raising Operators

In the following we assume that $\widetilde{\mathfrak{u}}$ is quasi-definite. Then, the sequence $(\widetilde{P}_n)_n$ of monic polynomials orthogonal with respect to $\widetilde{\mathfrak{u}}$ satisfies a three-term recurrence relation (TTRR)

$$x \widetilde{P}_n(x) = \widetilde{P}_{n+1}(x) + \widetilde{\beta}_n \widetilde{P}_n(x) + \widetilde{\gamma}_n \widetilde{P}_{n-1}(x), \qquad n \in \mathbb{N}, \tag{17}$$

with the initial conditions $\widetilde{P}_{-1}(x) = 0$, $\widetilde{P}_0(x) = 1$.

Here we will obtain the coefficients $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ of the TTRR (17) for the polynomials $\widetilde{P}_n$ orthogonal with respect to $\widetilde{\mathbf{u}}$, in terms of the coefficients β_n and γ_n of the TTRR (12) of the monic polynomials orthogonal with respect to $\mathbf{u}$. Since

$$\widetilde{\gamma}_n = \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x) \rangle}{\langle \widetilde{\mathbf{u}}, \widetilde{P}_{n-1}^2(x) \rangle}, \tag{18}$$

then, taking into account (10) as well as the remark 2 of theorem 1, for $n > 1$ we get

$$\widetilde{\gamma}_n = \gamma_n \frac{1 + \varepsilon_n (\widehat{P}_n(a), \widehat{P}_n(b), \widehat{P}'_n(b)) \mathrm{M}_{n-1} (\widehat{P}_n(a), \widehat{P}_n(b), \widehat{P}'_n(b))^T}{1 + \varepsilon_{n-1} (\widehat{P}_{n-1}(a), \widehat{P}_{n-1}(b), \widehat{P}'_{n-1}(b)) \mathrm{M}_{n-2} (\widehat{P}_{n-1}(a), \widehat{P}_{n-1}(b), \widehat{P}'_{n-1}(b))^T},$$

as well as for $n = 1$

$$\begin{aligned}
\widetilde{\gamma}_1 &= \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_1^2(x) \rangle}{\langle \widetilde{\mathbf{u}}, \widetilde{P}_0^2(x) \rangle} = \frac{\langle \widetilde{\mathbf{u}}, \widetilde{P}_1^2(x) \rangle}{\langle \mathbf{u}, \widetilde{P}_0^2(x) \rangle + \sum_{i=1}^M A_i} \\
&= \gamma_1 \frac{1 + \varepsilon_1 (\widehat{P}_1(a), \widehat{P}_1(b), \widehat{P}'_1(b)) \mathrm{M}_0 (\widehat{P}_1(a), \widehat{P}_1(b), \widehat{P}'_1(b))^T}{1 + A/u_0},
\end{aligned}$$

where $u_0 = \langle \mathbf{u}, 1 \rangle$ is the first moment of the linear functional $\mathbf{u}$.

On the other hand, $\widetilde{\beta}_n = \langle \widetilde{\mathbf{u}}, x\widetilde{P}_n^2(x) \rangle / \langle \widetilde{\mathbf{u}}, \widetilde{P}_n^2(x) \rangle$. Nevertheless, it is more convenient to compute $\widetilde{\beta}_n$ in a different way. If $\widetilde{b}_n$ denote the coefficient of x^{n-1} for $\widetilde{P}_n$ and b_n the corresponding coefficient of x^{n-1} for P_n then we have, $\widetilde{\beta}_n = \widetilde{b}_n - \widetilde{b}_{n+1}$. To obtain $\widetilde{b}_n$ we use the Eq. (9)

$$\widetilde{b}_n = b_n - \varepsilon_n \varepsilon_{n-1} |\gamma_n|^{1/2} (\widehat{P}_{n-1}(a), \widehat{P}_{n-1}(b), \widehat{P}'_{n-1}(b)) \mathrm{M}_{n-1} (\widehat{P}_n(a), \widehat{P}_n(b), \widehat{P}'_n(b))^T.$$

Thus, for $n \geq 1$,

$$\begin{aligned}
\widetilde{\beta}_n = \beta_n &+ \varepsilon_n \varepsilon_{n+1} |\gamma_{n+1}|^{1/2} (\widehat{P}_n(a), \widehat{P}_n(b), \widehat{P}'_n(b)) \mathrm{M}_n (\widehat{P}_{n+1}(a), \widehat{P}_{n+1}(b), \widehat{P}'_{n+1}(b))^T \\
&- \varepsilon_n \varepsilon_{n-1} |\gamma_n|^{1/2} (\widehat{P}_{n-1}(a), \widehat{P}_{n-1}(b), \widehat{P}'_{n-1}(b)) \mathrm{M}_{n-1} (\widehat{P}_n(a), \widehat{P}_n(b), \widehat{P}'_n(b))^T.
\end{aligned}$$

Finally, for $n = 0$ we have

$$\widetilde{\beta}_0 = \frac{\langle \widetilde{\mathbf{u}}, x \rangle}{\langle \widetilde{\mathbf{u}}, 1 \rangle} = \frac{u_1 + aA + B}{u_0 + A}.$$

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Notice that in order to compute the values $\widetilde{b}_n$ we can also use (16).

Following the ideas in [1] it is easy to show that in this case the polynomials $\widetilde{P}_n$ admit both lowering and raising operators of the form

$$\phi^2(x)\psi(x)\widetilde{P}'_n(x) = \widetilde{M}_1(x;n)\widetilde{P}_n(x) + \widetilde{N}_1(x;n)\widetilde{P}_{n-1}(x), \qquad (19)$$

$$\phi^2(x)\psi(x)\widetilde{P}'_n(x) = \widetilde{M}_2(x;n)\widetilde{P}_n(x) + \widetilde{N}_2(x;n)\widetilde{P}_{n+1}(x), \qquad (20)$$

where $\widetilde{M}_1(x;n)$, $\widetilde{N}_1(x;n)$, $\widetilde{M}_2(x;n)$, and $\widetilde{M}_2(x;n)$ are polynomials depending on n and of bounded degree independent of n, respectively. These two formulas are very useful. In particular from them we can easily derive the second order linear differential equation that the polynomials $\widetilde{P}_n$ satisfy (see [1] for more details). Nevertheless, from formula (16) it is possible to find it by a simpler method as we will show in the next section. In particular this leads to the second order linear differential equation that the polynomials considered in [2, 3, 6, 7] and therefore [12] satisfy.

2.3 Second order linear differential equation for $(\widetilde{P}_n)_n$.

Here we will describe an algorithm for finding the second order differential equations (SODE) which satisfy the polynomials $\widetilde{P}_n(x)$ defined by (16)

$$q(x;n)\widetilde{P}_n(x) = a(x;n)P_n(x) + b(x;n)P'_n(x), \qquad (21)$$

where $(P_n)_n$ is a solution of the SODE

$$\sigma(x)P''_n(x) + \tau(x)P'_n(x) + \lambda(x)P_n(x) = 0. \qquad (22)$$

A special case is when $(P_n)_n$ constitutes a classical or semiclassical orthogonal family so (22) takes place. Taking derivatives in (21) we get

$$\begin{aligned} q'(x;n)\widetilde{P}_n(x) + q(x;n)\widetilde{P}'_n(x) = \;\; & a'(x;n)P_n(x) + a(x;n)P'_n(x) \\ & +b'(x;n)P'_n(x) + b(x;n)P''_n(x). \end{aligned}$$

Now, multiplying the above formula by $\sigma(x)$ and using the SODE for the polynomials $(P_n)_n$, we can rewrite $b(x;n)P''_n(x)$ as a combination of $P_n(x)$ and $P'_n(x)$, with polynomial coeffcients. To eliminate the term $q'(x;n)\widetilde{P}_n(x)$ we can multiply by q and use (21). These two operations yield

$$r(x;n)\widetilde{P}'_n(x) = c(x;n)P_n(x) + d(x;n)P'_n(x), \qquad (23)$$

where

$$r(x;n) = \sigma(x)q^2(x;n),$$
$$c(x;n) = q(x;n)\sigma(x)a'(x) - b(x;n)q(x;n)\lambda(x) - \sigma(x)a(x;n)q'(x;n),$$
$$d(x;n) = q(x;n)\sigma(x)a(x;n) + q(x;n)\sigma(x)b'(x)$$
$$-b(x;n)q(x;n)\tau(x) - \sigma(x)b(x;n)q'(x;n). \tag{24}$$

Analogously, taking derivatives in (23) and using the SODE (22), we obtain

$$s(x;n)\widetilde{P}_n''(x) = e(x;n)P_n(x) + f(x;n)P_n'(x), \tag{25}$$

with

$$s(x;n) = \sigma(x)r^2(x;n),$$
$$e(x;n) = r(x;n)\sigma(x)c'(x) - d(x;n)r(x;n)\lambda(x) - \sigma(x)c(x;n)r'(x;n)],$$
$$f(x;n) = r(x;n)\sigma(x)c(x;n) + r(x;n)\sigma(x)d'(x) - d(x;n)r(x)\tau(x)$$
$$-\sigma(x)d(x;n)r'(x;n). \tag{26}$$

In formulas (23), (25) $r, s, c, d, e,$ and f are polynomials of bounded degree with coefficients depending, in general, on n (they depend only on the functions $\sigma(x)$, $\tau(x)$, $\lambda(x)$, $q(x;n)$, $a(x;n)$, and $b(x;n)$ which are polynomials on x). The expressions (21), (23), and (25) yield

$$\begin{vmatrix} q(x;n)\widetilde{P}_n(x) & a(x;n) & b(x;n) \\ r(x;n)\widetilde{P}_n'(x) & c(x;n) & d(x;n) \\ s(x;n)\widetilde{P}_n''(x) & e(x;n) & f(x;n) \end{vmatrix} = 0, \tag{27}$$

where the functions $q(x;n)$, $a(x;n)$, and $b(x;n)$ are known from expression (21), and $c(x;n)$, $d(x;n)$, $e(x;n)$, $f(x;n)$, $r(x)$, and $s(x;n)$ can be found from (24) and (26). Expanding the determinant in (27) by the first column

$$\widetilde{\sigma}_n(x)\widetilde{P}_n''(x) + \widetilde{\tau}_n(x)\widetilde{P}_n'(x) + \widetilde{\lambda}_n(x)\widetilde{P}_n(x) = 0, \tag{28}$$

where
$$\widetilde{\sigma}_n(x) = s(x;n)\left[a(x;n)d(x;n) - c(x;n)b(x;n)\right],$$
$$\widetilde{\tau}_n(x) = r(x;n)[e(x;n)b(x;n) - a(x;n)f(x;n)],$$
$$\widetilde{\lambda}_n(x) = q(x;n)[c(x;n)f(x;n) - e(x;n)d(x;n)]. \tag{29}$$

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A straightforward calculation shows us that all coefficients in (28) have a common factor $q^4(x; n)\sigma^2(x)$. Then (29) can be rewriten as

$$\widetilde{\sigma}_n(x) = s(x; n) \left[\frac{a(x; n)d(x; n) - c(x; n)b(x; n)}{q^4(x; n)\sigma^2(x)} \right],$$

$$\widetilde{\tau}_n(x) = r(x; n) \left[\frac{e(x; n)b(x; n) - a(x; n)f(x; n)}{q^4(x; n)\sigma^2(x)} \right], \qquad (30)$$

$$\widetilde{\lambda}_n(x) = q(x; n) \left[\frac{c(x; n)f(x; n) - e(x; n)d(x; n)}{q^4(x)\sigma^2(x)} \right].$$

To conclude this section let us point out that special cases of the above algorithm have been extensively used for getting the differential equations of the Krall-type polynomials, Sobolev-type polynomials, etc. (see e.g., [2, 4, 5, 6, 7, 20, 21, 22] and references therein).

2.4   Density of zeros

From the SODE (28) the density of the distribution of zeros immediately follows. In fact, using the WKB approximation (for more details about this approach we refer to the papers [8, 25] and references therein) we have the following

Theorem 3 *Let S and ϵ be such that*

$$S(x) = \frac{1}{4\widetilde{\sigma}_n^2(x)} \{ 2\widetilde{\sigma}_n(x)[2\widetilde{\lambda}_n(x) - \widetilde{\tau}_n'(x)] + \widetilde{\tau}_n(x)[2\widetilde{\sigma}_n'(x) - \widetilde{\tau}_n(x)] \}, \qquad (31)$$

$$\epsilon(x) = \frac{1}{4[S(x)]^2} \left\{ \frac{5[S'(x)]^2}{4[S(x)]} - S''(x) \right\} = \frac{P(x, n)}{Q(x, n)}, \qquad (32)$$

where $P(x, n)$ and $Q(x, n)$ are polynomials in x as well as in n. If the condition $\epsilon(x) << 1$ holds, then the semiclassical or WKB density of zeros of the solutions of (28) is

$$\rho_{WKB}(x) = \frac{1}{\pi} \sqrt{S(x)}, \quad x \in I \subseteq \mathbb{R}, \qquad (33)$$

in every interval I where the function S is positive.

The above theorem have been extensively used in the study of the asymptotic distribution of zeros of the Krall-type polynomials [5] as well as their

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extensions [3, 7]. Let us mention also that in all cases under consideration, for n large enough, we get $\epsilon(x) \sim n^{-1}$. Then, from the above Theorem the corresponding WKB density of zeros of the polynomials $(\widetilde{P}_n)_n$ follows. The computations are very hard and cumbersome so the use of the symbolic package MATHEMATICA is again a very useful tool (see [3, 5, 7]). Finally, let us point out that from the SODE the moments of the zero distribution easily follows using the approach described in [10].

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