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ON FACTORIZATION AND ALGEBRAIZATION OF DIFFERENCE EQUATIONS OF HYPERGEOMETRIC TYPE

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Abstract

The "connection" between the approach of Nikiforov, Suslov and Uvarov and the factorization method of Infeld and Hull to find explicit solutions of finite difference equations of hypergeometric type is discussed. The group theoretical interpretation of the dynamical $su(2)$ algebra for Kravchuk functions is described.

1 Introduction

In Ref. [1]-[2] two questions were arised concerning to the theory of finite difference equations of the hypergeometric type [3]-[4].

[1]-[2] Using the case of Kravchuk polynomials $k_n^p(s,N)$ as an example it was shown that the corresponding finite difference equation is equivalent to the eigenvalue problem

$$H\psi_n = -\lambda_n \psi_n \tag{1}$$

where the "hamiltonian' H is factorized

$$H = -a^+ a \tag{2}$$

and a^+, a are constructed from standard first order difference operators

$$T_\pm = \exp(\pm \partial/\partial s). \tag{3}$$

This fact allows to apply to such type of equations the methods elaborated for the solution of equations with factorized hamiltonians [5] as it was suggested at first in the paper [6].

[3]-[4] Besides a^+ and a two other operators A^+, A were constructed in [1]-[2] which form a closed algebra together with the Hamiltonian H.

They satisfy the commutation relations

$$\begin{aligned} [H, A^+] &= A^+, [H, A] = -A, \\ [A^+, A] &= H, \end{aligned} \tag{4}$$

i.e., they generate the SU(2) algebra. Thus in the space of Kravchuk functions

$$\psi_n(s) = \sqrt{\rho(s)} k_n^p(s,N), \; n = 0, 1, \ldots, N, \tag{5}$$

where $\rho(s)$ is the usual weight function, the irreducible representation D^j of the SU(2) algebra is realized whose dimension $2j+1$ is equal to $N+1$. Therefore in [1]-[2] the dynamical algebra SU(2) for Kravchuk functions was found. It seems to be the first example of dynamical algebras for finite difference equations.

Since Ref. [1]-[2] concerns only Kravchuk polynomials (functions) it is interesting to study the problem of factorization and algebraization for other types of classical orthogonal polynomials of a discrete variable. In this paper we want to demonstrate (in sections 2 and 3) that the factorization found in [1]-[2] takes place also for all classical orthogonal polynomials of a discrete variable and a close connection exists between the approach to the finite difference equations of the hypergeometric type, developed in Refs. [3]-[4], and the factorization method [5] for the solution of second order differential equations. For simplicity we restrict ourselves in these sections by the discussion of finite difference equations on the uniform lattice. In the section 4 the group theoretical interpretation of the dynamical SU(2) algebra for Kravchuk functions is suggested.

It should be noted that in the general case the dynamical algebra does not coincide with any usual Lie algebra, it can be of more general type (quadratic or higher power algebra) [7].

2 Factorization of the finite difference equations of the hypergeometric type

The finite difference equation of the hypergeometric type is of the form [3]-[4]

$$\Big(\sigma(s)\nabla\Delta + \tau(s)\Delta + \lambda_n\Big)y_n(s) = 0. \tag{1}$$

Here $y_n(s)$ is of the polynomial of the power n with respect to the discrete variable s, $\sigma(s)$ and $\tau(s)$ are the polynomials of the second and first degree, respectively, and

$$\Delta = T_+ - 1 \quad \text{and} \quad \nabla = 1 - T_-. \tag{2}$$

As it was mentioned above we consider here only the case of the uniform lattice $s = 0, 1, \dots, N$, with a finite number of nodes $-N+1-$. The results obtained below can be extended easily on the case of infinite grid $s = 0, 1, 2, \dots$ (Meixner and Charlier polynomials).

The Eq. (2.1) is equivalent to the recurrent relation.

$$A_s y_n(s+1) + B_s y_n(s) + C_s y_n(s-1) + \lambda_n y_n(s) = 0 \tag{3}$$

where

$$A_s = \sigma(s) + \tau(s), \quad C_s = \sigma(s), \quad B(s) = -(A_s + C_s). \tag{4}$$

Therefore the solution of the Eq. (2.1) corresponds to the diagonalization of the tridiagonal matrix $\boldsymbol{\mathcal{H}}^{(0)}$ with the following nonvanishing matrix elements

$$\boldsymbol{\mathcal{H}}^{(0)}_{ss} = B_s, \quad \boldsymbol{\mathcal{H}}^{(0)}_{s+1,s} = A_s, \quad \boldsymbol{\mathcal{H}}^{(0)}_{s-1,s} = C_s, \quad s = 0, 1, \dots, N. \tag{5}$$

This matrix is nonhermitian. In order to operate with the hermitian "Hamiltonian" $H^{(0)}$ instead of $\mathcal{H}^{(0)}$ it is reasonable to go, following to [1]-[2], from polynomial $y_n^{(0)}(s) \equiv y_n(s)$ to "functions"

$$\psi_n^{(0)}(s) = S_0 y_n^{(0)}(s) \tag{6}$$

using the similarity transformation

$$S_0 = \sqrt{\rho(s)}. \tag{7}$$

where $\rho(s)$ is an usual weight function for polynomials $y_n(s)$. Therefore the orthogonality relations for the functions (2.6) is of the form

$$\langle\psi_{n'}|\psi_n\rangle \equiv \sum_{s=0}^{N} \psi_n(s)\psi_{n'}(s) = \delta_{nn'} d_n^2. \tag{8}$$

As a consequence, the matrix $\mathcal{H}^{(0)}$ transforms into the hermitian tridiagonal matrix

$$H^{(0)} = S_0 \boldsymbol{\mathcal{H}}^{(0)} S_0^{-1} \tag{9}$$

with nonvanishing matrix elements

$$H^{(0)}_{ss} = B_s,\ H^{(0)}_{s+1s} = H^{(0)}_{ss+1} = \sqrt{A_s C_{s+1}}. \tag{10}$$

The recurrent relation (2.3) takes the form

$$\sqrt{A_s C_{s+1}}\psi^{(0)}_n(s+1) + B_s\psi^{(0)}_n(s) + \sqrt{A_{s-1}C_s}\psi^{(0)}_n(s-1) + \lambda^{(0)}_n\psi^{(0)}_n(s) = 0. \tag{11}$$

We can rewrite equation (2.12) as the eigenvalue problem

$$H^{(0)}\psi^{(0)}_n(s) = -\lambda^{(0)}_n\psi^{(0)}_n(s), \tag{12}$$

for the Hamiltonian

$$H^{(0)} = \sqrt{A_s C_{s+1}}T_+ + B_s + T_-\sqrt{A_s C_{s+1}}. \tag{13}$$

The above Hamiltonian (2.13) is hermitian if the following conjugation rule

$$(T_-)^\dagger = T_+ \tag{14}$$

is accepted for the finite difference operators T_+ and T_-.

Let us turn now to the factorization of the Hamiltonian $H^{(0)}$. As it was mentioned in [3]-[4] the initial operator $\boldsymbol{\mathcal{H}}^{(0)}$ can be easily factorized

$$\boldsymbol{\mathcal{H}}^{(0)} = -b^+ b, \tag{15}$$

where

$$b = \Delta, \qquad\qquad b^+ = -(\sigma(s)\nabla + \tau(s)) = C_s T_- - A_s. \tag{16}$$

To factorize the Hamiltonian $H^{(0)}$ we apply to the expression (2.15) the similarity transformation

$$H^{(0)} = -S_0 b^+ S_1^{-1} S_1 b S_0^{-1} \tag{17}$$

with

$$S_1 = \sqrt{\rho_1(s)}. \tag{18}$$

Then, we obtain

$$H^{(0)} = -a^+ a, \tag{19}$$

where

$$\begin{aligned} a &= S_1 b S_0^{-1} = \sqrt{\frac{\rho_1(s)}{\rho(s+1)}}T_+ - \sqrt{\frac{\rho_1(s)}{\rho(s)}}, \\ a^+ &= S_0 b^+ S_1^{-1} = \sqrt{\frac{\rho(s)}{\rho_1(s-1)}}C_s T_- - \sqrt{\frac{\rho(s)}{\rho_1(s)}}A_s. \end{aligned} \tag{20}$$

Taking account the relation [3]-[4]

$$\frac{\rho(s+1)}{\rho(s)} = \frac{A_s}{C_{s+1}}, \tag{21}$$

$$\rho_1(s) = \rho(s+1)C_{s+1} \tag{22}$$

we can rewrite the formulas (2.20) as follows

$$a = \sqrt{C_{s+1}}T_+ - \sqrt{A_s}, \qquad a^+ = \sqrt{C_s}T_- - \sqrt{A_s}. \tag{23}$$

These operators are an hermitian conjugate pair in the sense of Eq. (2.14). The results (2.19) and (2.23) are equivalent to ones from [1]-[2] in the case of Kravchuk functions. Thus the factorization mentioned in [1]-[2] is valid for all finite difference equations of the hypergeometric type on the uniform grid.

Now we shall demonstrate that the method developed in [3]-[4] to solve such type of equations has a deep parallelism with the factorization method [5] if we work with the functions (2.6) instead of polynomials $y_n(s)$.

3 Factorization method and classical orthogonal polynomials of a discrete variable

To begin with we remind the main points of the factorization method (FM) [5] adopted to our problem.

In the FM has been considered a finite chain of hermitian Hamiltonians

$$H^{(0)}, H^{(1)}, \dots, H^{(m)}, \dots, H^{(l)}. \tag{1}$$

Each of these Hamiltonians is factorized as follows

$$H^{(m)} = -a^+(m, m+1)a(m+1, m). \tag{2}$$

Let the initial Hamiltonian $H^{(0)}$ has a number (finite or infinite) of discrete levels with eigenfunctions $\psi_n^{(0)}$

$$H^{(0)}\psi_n^{(0)} = \lambda_n^{(0)}\psi_n^{(0)}. \tag{3}$$

We select one of them with the function $\psi_l^{(0)} (l = 0, 1, 2, \dots)$ and form the chain of functions

$$\psi_l^{(0)}, \psi_{l-1}^{(1)}, \dots, \psi^{(l-m)}, \dots, \psi_0^{(l)} \tag{4}$$

where $\psi_k^{(m)}$ is an eigenfunction of the Hamiltonian

$$H^{(m)}\psi_k^{(m)} = \lambda_k^{(m)}\psi_k^{(m)}. \tag{5}$$

The neighbouring functions in the chain (3.4) are related to each other by the following expressions involving the operators a^+, a (3.2)

$$\begin{aligned} a(m+1, m)\psi_{l-m}^{(m)} &= \sqrt{\lambda_{l-m}^{(m)}}\psi_{l-m-1}^{(m+1)}, \\ a^+(m, m+1)\psi_{l-m-1}^{(m+1)} &= \sqrt{\lambda_{l-m}^{(m)}}\psi_{l-m}^{(m)}. \end{aligned} \tag{6}$$

The last function in (3.4) has the sense of a "vacuum" vector because it is annihilated by the lowering operator a

$$a(l+1, l)\psi_0^{(l)} = 0. \tag{7}$$

It is clear from (3.6)-(3.7) that any function from the chain (3.4) can be obtained from the vacuum vector using the raising operator

$$\psi_{l-m}^{(m)} = \frac{1}{\sqrt{\lambda_0^{(l)}\lambda_1^{(l-1)}\dots\lambda_{l-m}^{(m)}}}a^+(m, m+1)\dots a^+(l-1, l)\psi_0^{(l)}. \tag{8}$$

This function will be normalized to 1 in the sense that

$$\langle \psi^m_{l-m} | \psi^{(m)}_{l-m} \rangle = 1, \tag{9}$$

if the vacuum vector is correctly normalized, i.e.,

$$\langle \psi^l_0 | \psi^{(l)}_0 \rangle = 1. \tag{10}$$

The authors [5] have remarked that the main advantage of the FM is the fact that it allows to calculate in a systematic way the eigenvalues and the normalized eigenfunctions of various physical systems whose Hamiltonian can be factorized as described above. In the original formulation the operators a^+, a were first order differential operators as well as the Hamiltonians $H^{(m)}$ were a second order differential operator Now we apply this approach to the first order difference operators (2.16) and (2.23) and to the Hamiltonians given in (2.15) or (3.2). Notice that the algorithm developed in [3]-[4] allows us to find the solution $y_n^{(0)}(s)$ of the difference equation (2.1) and also their finite difference derivatives defined by

$$y_n^m(s) = \Delta^m y_n^{(0}(s), \tag{11}$$

which satisfy the following second order difference equation

$$\boldsymbol{\mathcal{H}}^{(m)} y_k^{(m)} \equiv [\sigma(s)\Delta\nabla + \tau_m(s)\Delta]\, y_k^{(m)}(s) = -\lambda_k^{(m)} y_k^{(m)}(s), \tag{12}$$

where

$$\begin{aligned} \tau_m(s) &= \tau(s+m) + \sigma(s+m) - \sigma(s), & (13)\\ \lambda_k^{(m)} &= \lambda_k^{(0)} + m\tau' + \frac{m(m-1)}{2}\sigma''. & (14) \end{aligned}$$

The constants τ' and σ'' can be found from the expressions

$$\tau(s) = \tau' s + \tau(0), \qquad \sigma(s) = \tfrac{1}{2}\sigma'' s^2 + \sigma'(0)s + \sigma(0). \tag{15}$$

The polynomials $y_k^{(m)}(s)$ are defined at $s = 0, 1, 2, \ldots, N-m$. They are orthogonal with the weight function

$$\rho_m(s) = \rho(s+m) \prod_{t=1}^{m} \sigma(s+t). \tag{16}$$

The operators b, b^+ play the role of lowering and raising operators for these polynomials. In fact

$$\begin{aligned} \Delta y_k^{(m)} &= y_{k-1}^{(m+1)} & (17)\\ \Big(\sigma(s)\nabla &+ \tau_m(s)\Big) y_{k-1}^{(m+1)} = \lambda_k^m y_k^{(m)}. & (18) \end{aligned}$$

Now, we can introduce, instead of polynomials (3.12), the corresponding functions

$$\psi_k^{(m)} = S_m y_k^{(m)}, \qquad S_m = \sqrt{\rho_m(s)}. \tag{19}$$

They are normalized by the relation

$$\langle \psi_k^{(m)} | \psi_{k'}^{(m)} \rangle = \delta_{kk'} (d_k^m)^2. \tag{20}$$

where

$$(d_k^m)^2 = (d_k^{m+1})^2 / \lambda_k^{(m)}. \tag{21}$$

Similarly to Eqs. (2.17)-(2.23) we can introduce the hermitian Hamiltonians

$$H^{(m)} = S_m \boldsymbol{\mathcal{H}}^{(m)} S_m^{-1} = \sqrt{A_s^m C_{s+1}^m}T_+ + B_s^m + T_-\sqrt{A_s^m C_{s+1}^m}. \tag{22}$$

where

$$A_s^m = \sigma(s) + \tau(s), \qquad C_s^m = C_s = \sigma(s)$$

They can be factorized in the form

$$H^{(m)} = -a^+(m, m+1)a(m+1, m) \tag{23}$$

with

$$\begin{aligned} a^+(m, m+1) &= \sqrt{C_s^m}T_- - \sqrt{A_s^m}, \\ a(m+1, m) &= T_+\sqrt{C_s^m} - \sqrt{A_s^m}, \end{aligned} \tag{24}$$

In accordance with (3.17)-(3.19) we obtain

$$\begin{aligned} a(m+1, m)\psi_k^{(m)} &= \sqrt{\lambda_k^{(m)}}\psi_{k-1}^{(m+1)}, \\ a^+(m, m+1)\psi_{k-1}^{(m)} &= \sqrt{\lambda_k^{(m)}}\psi_k^{(m)}. \end{aligned} \tag{25}$$

Therefore,

$$H^{(m)}\psi_k^{(m)} = -a^+(m, m+1)a(m+1, m)\psi_k^{(m)} = -\lambda_k^{(m)}\psi_k^{(m)}, \tag{26}$$

as it should be in agreement with (3.12).

Now it is possible to compare the approach given [3]-[4] with the FM [5]. For this aim we identify the finite difference operators (3.23) with the Hamiltonian (3.2) from the FM, the function (2.6) with the initial function $\psi_l^{(0)}$ in the chain (3.4) and the function (3.19) at $k = l - m$ with the rest elements of (3.4). Notice that the operators $a_\pm$ having the properties (3.25), play the same role as the lowering and raising operators (3.6) in the FM. Thus we can consider the approach to finite difference equations of the hypergeometric type developed in [3]-[4] as an equivalent of the FM, realized in terms of the finite difference operators. The explicit expression for the function $\psi_l^{(0)}$ can be found using the Eq. (3.7) of the FM approach which is an equivalent of the Rodrigues formula for the solution of (2.1) or (3.12) [3]-[4]. The normalization factor for $\psi_l^{(0)}$ can be also calculated because all the necessary eigenvalues (3.14) are known. Therefore, all is reduced to the calculation of the normalization factor for the vacuum function

$$(d_0^l)^2 = \sum_{s=0}^{N-l} \rho_n(s), \tag{27}$$

which should be evaluated directly.

4 Dynamical algebra for Kravchuk functions

In this section we want to give the group theoretical interpretation of the dynamical SU(2) algebra for the Kravchuk functions, found in Ref. [1]-[2]. It is well known [3,8] that normalized Kravchuk function

$$\psi_n^p(s, N) = (d_n)^{-1}\sqrt{\rho(s)}k_n^p(s, N) \tag{1}$$

is proportional to the Wigner d-function

$$\psi_n^p(s, N) = (-1)^{m-m'}d_{mm'}^j(\beta). \tag{2}$$

Here $j = N/2, n = j - m, s = j - m', p = \sin^2 \beta/2, q = 1 - p = \cos^2 \beta/2, d_n^2 = N!(pq)^n/n!(N-n)!$ The difference equation for d-functions can be obtained when we are seeking the eigenfunctions ϕ_{jm} of the operator

$$J_{z'} = \cos\beta J_z + \sin\beta J_x \tag{3}$$

in terms of functions $|jm'>$, quantized along the axis z forming the angle β with the axis z'.

Thus we solve the problem

$$J_{z'}\phi_{jm} = m\phi_{jm} \tag{4}$$

where

$$\phi_m = \sum_{m'} t_{m'}|jm'\rangle. \tag{5}$$

Substituting (4.5) into (4.4) and taking the scalar product of the resulting expression by $< jm'|$ we obtain the recurrent relation for coefficients $t_{m'}$

$$\tfrac{1}{2}\sin\beta\langle jm'|J_+|jm'-1\rangle t_{m'-1} + \tfrac{1}{2}\sin\beta\langle jm'|J_-|jm'+1\rangle t_{m'+1} + \cos\beta m' t_{m'} = m t_{m'}. \tag{6}$$

It is well known [3,8] that coefficients $t_{m'}$ in (4.5) are identical with $d^j_{mm'}(\beta)$. Thus (4.6) gives the finite difference equation for d-functions or, taking into account the relation (4.2), one for Kravchuk functions. As usually, the operators $J_\pm = J_x \pm iJ_y$. Their matrix elements, appearing in (4.6), are well known

$$\langle jm \pm 1|J_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}. \tag{7}$$

We are interesting in the dynamical algebra for the eigenvalue problem (4.4). Therefore, we need the operators which transform the solution ϕ_{jm} into $\phi_{jm\pm1}$. Obviously the operators

$$J'_\pm = J_{x'} \pm iJ_{y'} = \sin\beta J_z + \tfrac{1}{2}(\cos\beta \pm 1)J_+ + \tfrac{1}{2}(\cos\beta \mp 1)J_-, \tag{8}$$

which are, together with $J_{z'}$, the generators of the SU(2) algebra, can be taken as such operators because

$$J'_\pm\phi_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}\phi_{jm\pm1}. \tag{9}$$

Thus the dynamical algebra for the eigenvalue problem (4.4) is found. In order to transform it into the dynamical algebra for d-functions or Kravchuk functions we should multiply the equation (4.9) by the vector $< jm'|$. In such a manner we obtain

$$\begin{aligned}&\sqrt{(j \mp m)(j \pm m + 1)}d^j_{m\pm1m'}(\beta) = m'\sin\beta d^j_{mm'}(\beta) + \\ &+\tfrac{1}{2}(\cos\beta \pm 1)\langle jm'|J_+|jm'-1\rangle d^j_{mm'-1}(\beta) + \tfrac{1}{2}(\cos\beta \mp 1)\langle jm'|J_-|jm'+1\rangle d^j_{mm'+1}(\beta).\end{aligned} \tag{10}$$

The corresponding relations for the Kravchuk functions can be written in the form

$$A_+\psi^p_n(s,N) = \sqrt{(n+1)(N-n)}\psi^p_{n+1}(s,N), \tag{11}$$

$$A_-\psi^p_n(s,N) = \sqrt{n(N-n+1)}\psi^p_{n-1}(s,N), \tag{12}$$

where

$$A_+ = 2\sqrt{pq}(\frac{N}{2} - s) - p\sqrt{(s+1)(N-s)}e^{-\partial_s} + q\sqrt{s(N-s+1)}e^{\partial_s}, \tag{13}$$

$$A_- = 2\sqrt{pq}(\frac{N}{2} - s) + qe^{-\partial_s}\sqrt{s(N-s+1)} - pe^{\partial_s}\sqrt{(s+1)(N-s)}. \tag{14}$$

These operators together with

$$A_0 = (q-p)(\frac{N}{2} - s) - \sqrt{pq}e^{\partial_s}\sqrt{(N-s)(s+1)} - \sqrt{pq}e^{-\partial_s}\sqrt{s(N-s)}, \tag{15}$$

are the generators of the dynamical SU(2) algebra found in [1]-[2]. It is easy to verify that they satisfy the commutation relations

$$[A_0, A_\pm] = \pm A_\pm, \; [A_+, A_-] = 2A_0, \tag{16}$$

which is a consequence from the fact that the operators $A_\pm$ and A_0, represent the matrix realization of SU(2) generators $-J_{z'}, J'_-$ and J'_+ considered above. The Kravchuk functions (4.1) belong to the irreducible representation $D^j, j = N/2$ of this algebra.

5 Conclusion

It was shown that the approach to the solution of the finite difference equations of hypergeometric type developed by Nikiforov, Suslov and Uvarov [3]-[4], becomes equivalent to the factorization method of Infeld and Hull [5] after transition from the classical orthogonal polynomials to the corresponding functions introduced in [1]-[2].

The factorization of finite difference equations found in [1]-[2] for the Kravchuk functions can be extended on all classical orthogonal polynomials of a discrete variable. The group theoretical interpretation of the SU(2) dynamical algebra, constructed in [1]-[2] for Kravchuk functions, is suggested.

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