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POSITION AND MOMENTUM INFORMATION ENTROPIES OF THE HARMONIC OSCILLATOR AND LOGARITHMIC POTENTIAL OF HERMITE POLYNOMIALS

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Abstract

The exact analytical values of the position and momentum information entropies for the stationary states of the one-dimensional quantum harmonic oscillator are only known for the ground ($n = 0$) and first excited ($n = 1$) states. In the general case, the problem of calculating these entropies reduces to the evaluation of the logarithmic potential $V_n(t) = -\int_{-\infty}^{\infty}(H_n(x))^2 \ln|x-t|e^{-x^2}dx$ at $t = x_{n,i}$ $(i = 1,\dots,n)$, where $x_{n,i}$ is the i-th zero of the Hermite polynomial $H_n(x)$. Here, a closed analytical formula for $V_n(t)$ in terms of ${}_1F_1$ and ${}_2F_2$ hypergeometric functions is obtained, which in turn provides us with analytical expressions for the entropies when the exact location of the zeros of $H_n(x)$ is known. The complete asymptotic expansion of $V_n(t)$ for $|t| \gg 1$, which is expressed in terms of ${}_2F_0$ and ${}_3F_1$ (divergent) hypergeometric series, is also obtained. Finally, it is shown that the exact formula for $V_n(t)$ can be written as an infinite series involving the Gauss (${}_2F_1$) hypergeometric function, which allows the entropies to be expressed in terms of the even-order spectral moments of Hermite polynomials, $\mu_{2k}(n) = n^{-1}\sum_{i=1}^{n}(x_{n,i})^{2k}$. The asymptotic ($n \to \infty$) limit of this alternative expression for the entropies is discussed.

1 Introduction

For the n-th eigenstate of the one-dimensional harmonic oscillator Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \ , \tag{1}$$

the probability densities for position and momentum are expressed in terms of the orthogonal Hermite polynomials $H_n(x)$,

$$\rho(x) = \frac{\alpha}{2^n n!\sqrt{\pi}}(H_n(\alpha x))^2 e^{-\alpha^2 x^2} \ , \quad \gamma(p) = \frac{1}{2^n n!\sqrt{\pi}\alpha}(H_n(p/\alpha))^2 e^{-p^2/\alpha^2} \ , \tag{2}$$

where $\alpha \equiv (m\omega)^{1/2}$ (we choose units such that $\hbar = 1$). The corresponding Boltzmann-Shannon entropies of position and momentum [4, 5, 11, 12] can be written as

$$S_\rho = -\ln\alpha + S_n \ , \quad S_\gamma = \ln\alpha + S_n \ , \tag{3}$$

where

$$S_n = \ln(2^n n!\sqrt{\pi}\,) + n + \frac{1}{2} + \frac{1}{2^n n!\sqrt{\pi}}E_n(H) \ , \tag{4}$$

is given in terms of $E_n(H)$, the so-called entropy of Hermite polynomials, whose expression is

$$E_n(H) = -\int_{-\infty}^{\infty} e^{-x^2}(H_n(x))^2 \ln(H_n(x))^2\, dx \ . \tag{5}$$

The values of S_n have been numerically calculated up to $n = 500$ [11]. For $n \gg 1$, the L^p-norm-based approach to the calculation of the entropies of orthogonal polynomials leads to the asymptotic formula [1, 5, 11]

$$S_n \sim \ln\left(\pi\sqrt{2n}\,\right) - 1\ , \tag{6}$$

which can also be derived from the semiclassical (WKB) approximation for general one-dimensional quantum systems [9]. However, the exact analytical value of S_n is only known in the simplest cases $n = 0$ and $n = 1$ [4, 5, 11, 12],

$$S_0 = \ln\left(\sqrt{\pi}\,\right) + \frac{1}{2}\ , \quad S_1 = \ln\left(2\sqrt{\pi}\,\right) - \frac{1}{2} + \gamma\ , \tag{7}$$

where γ is Euler's constant. The main aim of the present work is to find the generalization of these results to arbitrary values of n by applying in the Hermite case the potential theory approach to the calculation of the entropies, discussed in Ref. [5] for general orthogonal polynomials.

The Hermite polynomial $H_n(x)$ has n real and simple zeros, and is of the form $H_n(x) = 2^n x^n + O(x^{n-1})$ (see, e.g., Ref. [6]), so that it can be factorized as

$$H_n(x) = 2^n \prod_{i=1}^{n} (x - x_{n,i})\ ,$$

where $x_{n,i}$ $(i = 1, 2, \ldots, n)$ is the i-th root of $H_n(x)$. Introducing this expression into the logarithmic function in (5), and taking into account the normalization integral for Hermite polynomials,

$$\int_{-\infty}^{\infty} e^{-x^2} (H_n(x))^2\, dx = 2^n n! \sqrt{\pi}\ ,$$

we see that $E_n(H)$ can be written in the form

$$E_n(H) = -2^n n! \sqrt{\pi}\, \ln\left(2^{2n}\right) + 2 \sum_{i=1}^{n} V_n(x_{n,i})\ , \tag{8}$$

where $V_n(t)$ is the logarithmic potential of Hermite polynomials,

$$V_n(t) = -\int_{-\infty}^{\infty} (H_n(x))^2 \ln|x - t|\, e^{-x^2}\, dx\ . \tag{9}$$

The n (real and simple) zeros of $H_n(x)$ are symmetrically distributed around the origin, since $H_n(-x) = (-1)^n H_n(x)$ [6]. Therefore, it is readily seen that $V_n(-t) = V_n(t)$, and Eq. (8) can also be written as

$$E_n(H) = -2^n n! \sqrt{\pi}\, \ln\left(2^{2n}\right) + 2\epsilon V_n(0) + 4 \sum_{i=1}^{m} V_n(x_{n,i})\ , \tag{10}$$

where we have introduced the convenient notations

$$m \equiv \left[\frac{n}{2}\right], \quad \epsilon \equiv n - 2m \tag{11}$$

(i.e., ϵ is equal to 0 or 1 according to whether n is even or odd), and $x_{n,i}$ $(i = 1, 2, \ldots, m)$ denotes the i-th *positive* root of $H_n(x)$.

Eqs. (8) and (10) show that the problem of calculating $E_n(H)$, and hence S_n, reduces to the evaluation of $V_n(t)$ at the zeros of $H_n(x)$. In Section 2 below we obtain a closed analytical expression for $V_n(t)$ in terms of ${}_1F_1$ and ${}_2F_2$ hypergeometric functions, which, unlike the recursive formula previously derived by Dehesa and co-workers [4, 5], provides us with analytical expressions for $E_n(H)$ and S_n when the exact location of the zeros of $H_n(x)$ is known. In Section 3, we make use of the general theory of asymptotic approximations for the ${}_pF_p$ hypergeometric functions to obtain the complete asymptotic expansion of $V_n(t)$ when $|t| \gg 1$, which is expressed in terms of ${}_2F_0$ and ${}_3F_1$ (divergent) hypergeometric series. Finally, in Section 4, it is shown that the exact formula for $V_n(t)$ can be written as an infinite series involving the Gauss (${}_2F_1$) hypergeometric function, which allows $E_n(H)$ and S_n to be expressed in terms of the even-order spectral moments of Hermite polynomials $\mu_{2k}(n)$, and this alternative expression for S_n is used to discuss the relation between our exact results and the asymptotic formula (6).

2 Calculation of the logarithmic potential

To calculate $V_n(t)$, we first make use of the multiplication formula for Hermite polynomials (see, e.g., Ref. [7]),

$$H_m(x)H_n(x) = \sum_{j=0}^{\min(m,n)} \frac{m!\,n!\,2^j}{(m-j)!\,(n-j)!\,j!} H_{m+n-2j}(x) \ ,$$

which in the particular case $m = n$ gives

$$(H_n(x))^2 = \sum_{j=0}^{n} \frac{(n!)^2\,2^j}{((n-j)!)^2 j!} H_{2n-2j}(x) = 2^n\,n! \sum_{k=0}^{n} \binom{n}{k} \frac{H_{2k}(x)}{2^k k!} \ .$$

Substituting this equation in the expression (9) of the logarithmic potential $V_n(t)$, we find

$$V_n(t) = 2^n n! \sum_{k=0}^{n} \binom{n}{k} \frac{W_{2k}(t)}{2^k k!} \ , \quad W_{2k}(t) = -\int_{-\infty}^{\infty} H_{2k}(x) \ln|x-t|\, e^{-x^2}\, dx \ . \tag{12}$$

Now we are faced with the problem of calculating the integrals $W_{2k}(t)$, which can also be considered as logarithmic potentials for Hermite polynomials and thus have independent interest. To achieve this goal, we consider the Taylor series expansion

$$W_{2k}(t) = \sum_{r=0}^{\infty} \frac{W_{2k}^{(r)}(0)}{r!} t^r \ . \tag{13}$$

Making the change of variables $x = y + t$ in Eq. (12), we have

$$W_{2k}(t) = -\int_{-\infty}^{\infty} H_{2k}(y+t) \ln|y|\, e^{-(y+t)^2}\, dy \ .$$

By repeated application of Leibniz's rule for differentiating under the integral sign, and taking into account that, from Rodrigues' formula for Hermite polynomials,

$$\frac{d^r}{dz^r}\left(e^{-z^2} H_n(z)\right) = (-1)^r e^{-z^2} H_{n+r}(z) \ ,$$

we get

$$W_{2k}^{(r)}(t) = (-1)^r W_{2k+r}(t) \ ,$$

so that Eq. (13) then reads

$$W_{2k}(t) = \sum_{r=0}^{\infty} \frac{(-1)^r W_{2k+r}(0)}{r!} t^r \ . \tag{14}$$

The parity property $H_n(-x) = (-1)^n H_n(x)$ implies that $W_{2k+r}(0) = 0$ if r is odd, and Eq. (14) simplifies to

$$W_{2k}(t) = \sum_{r=0}^{\infty} \frac{W_{2k+2r}(0)}{(2r)!} t^{2r} \ , \quad W_{2k+2r}(0) = -2\int_0^{\infty} H_{2k+2r}(x)\, e^{-x^2} \ln x\, dx \ . \tag{15}$$

The integrals $W_{2k+2r}(0)$ may be evaluated by means of the following result [6],

$$\begin{aligned}\int_0^\infty H_{2n}(x)\,x^\nu e^{-2\alpha x^2}\,dx &= (-1)^n \frac{2^{2n-(\nu+3)/2}}{\sqrt{\pi}\,\alpha^{(\nu+1)/2}}\,\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\\ &\quad\times F\left(-n,\frac{\nu+1}{2};\frac{1}{2};\frac{1}{2\alpha}\right),\end{aligned}$$

where $F(a,b;c;z) = {}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function, which is valid for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > -1$. In our case, with $\alpha = \frac{1}{2}$ and $n = k+r$, we get

$$\begin{aligned}\int_0^\infty H_{2k+2r}(x)\,x^\nu e^{-x^2}\,dx &= (-1)^{k+r}\frac{2^{2k+2r-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(k+r+\frac{1}{2}\right)\\ &\quad\times F\left(-k-r,\frac{\nu+1}{2};\frac{1}{2};1\right).\end{aligned} \tag{16}$$

The hypergeometric function of unit argument in the right-hand side can be simplified using the property [8],

$$F(-n,b;c;1) = \frac{(c-b)_n}{(c)_n},$$

where n is a positive integer or zero, c is not a negative integer or zero, and $(z)_n$ is Pochhammer's symbol,

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = (-1)^n\frac{\Gamma(1-z)}{\Gamma(1-z-n)}. \tag{17}$$

We thus have

$$F\left(-k-r,\frac{\nu+1}{2};\frac{1}{2};1\right) = \frac{\left(-\frac{\nu}{2}\right)_{k+r}}{\left(\frac{1}{2}\right)_{k+r}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(k+r-\frac{\nu}{2}\right)}{\Gamma\left(k+r+\frac{1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right)},$$

and Eq. (16) then reads

$$\int_0^\infty H_{2k+2r}(x)\,x^\nu e^{-x^2}\,dx = (-1)^{k+r}\,2^{2k+2r-1}\,\Gamma\left(\frac{\nu+1}{2}\right)\frac{\Gamma\left(k+r-\frac{\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)}.$$

Differentiating this equation with respect to ν, we obtain

$$\begin{aligned}\int_0^\infty H_{2k+2r}(x)\,x^\nu e^{-x^2}\ln x\,dx &= (-1)^{k+r}\,2^{2k+2r-1}\,\Gamma\left(\frac{\nu+1}{2}\right)\frac{\Gamma\left(k+r-\frac{\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)}\\ &\quad\times\frac{1}{2}\left(\psi\left(\frac{\nu+1}{2}\right)-\psi\left(k+r-\frac{\nu}{2}\right)+\psi\left(-\frac{\nu}{2}\right)\right),\end{aligned} \tag{18}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Gamma function. In the case when $k+r=0$, this formula reduces to

$$\int_0^\infty H_0(x)\,x^\nu e^{-x^2}\ln x\,dx = \frac{1}{4}\Gamma\left(\frac{\nu+1}{2}\right)\psi\left(\frac{\nu+1}{2}\right),$$

so that we readily get

$$W_0(0) = -\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}(\gamma+2\ln 2). \tag{19}$$

On the other hand, both $\Gamma(z)$ and $\psi(z)$ have simple poles for $z=0$, with residues 1 and -1 respectively. Therefore, when $k+r>0$, we can take the limit $\nu\to 0$ in Eq. (18), to obtain

$$\int_0^\infty H_{2k+2r}(x)\,e^{-x^2}\ln x\,dx = -\frac{\sqrt{\pi}}{4}(-1)^{k+r}\,2^{2k+2r}\,\Gamma(k+r), \quad k+r>0,$$

which in turn leads to

$$W_{2k+2r}(0) = \frac{\sqrt{\pi}}{2}(-1)^{k+r}\, 2^{2k+2r}\, \Gamma(k+r)\ , \quad k+r>0\ . \tag{20}$$

We can evaluate $W_{2k}(t)$ by substituting Eqs. (19) and (20) into (15). In the case $k=0$, we have

$$W_0(t) = W_0(0) + \sum_{r=1}^{\infty} \frac{W_{2r}(0)}{(2r)!} t^{2r} = \frac{\sqrt{\pi}}{2}\left(\gamma + 2\ln 2 + \sum_{r=1}^{\infty} \frac{(-1)^r 2^{2r}\Gamma(r)}{\Gamma(2r+1)} t^{2r}\right). \tag{21}$$

Using the recurrence and duplication formulas for the Gamma function [6, 8],

$$\Gamma(z+1) = z\Gamma(z)\ , \quad \Gamma(2z) = 2^{2z-1}\left(\frac{1}{2}\right)_z \Gamma(z)\ ,$$

together with (17), and shifting the summation index to $s=r-1$, Eq. (21) can be written in terms of a ${}_2F_2$ hypergeometric function,

$$W_0(t) = \frac{\sqrt{\pi}}{2}\left(\gamma + 2\ln 2 - 2t^2\, {}_2F_2\left(1,1;\frac{3}{2},2;-t^2\right)\right). \tag{22}$$

On the other hand, for $k>0$ we have

$$W_{2k}(t) = \sum_{r=0}^{\infty} \frac{W_{2k+2r}(0)}{(2r)!} t^{2r} = \frac{\sqrt{\pi}}{2}(-1)^k 2^{2k} \sum_{r=0}^{\infty} \frac{(-1)^r 2^{2r}\Gamma(k+r)}{\Gamma(2r+1)} t^{2r}\ , \tag{23}$$

and use of the duplication formula for the Gamma function and Eq. (17) leads to

$$W_{2k}(t) = \frac{\sqrt{\pi}}{2}(-1)^k 2^{2k}(k-1)!\, M\left(k;\frac{1}{2};-t^2\right), \tag{24}$$

where $M(a;c;z) = {}_1F_1(a;c;z)$ is Kummer's confluent hypergeometric function. Substituting Eqs. (22) and (24) into (12), we finally obtain

$$\begin{aligned} V_n(t) &= 2^n n! \sqrt{\pi}\left(\frac{\gamma}{2} + \ln 2 - t^2\, {}_2F_2\left(1,1;\frac{3}{2},2;-t^2\right)\right. \\ &\quad \left. + \frac{1}{2}\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k 2^k}{k} M\left(k;\frac{1}{2};-t^2\right)\right), \end{aligned} \tag{25}$$

which is the sought for closed analytical expression for the logarithmic potential $V_n(t)$ defined by Eq. (9).

In the particular case $t=0$, using the identity [10]

$$\gamma + 2\ln 2 + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k 2^k}{k} = -\psi\left(m+\epsilon+\frac{1}{2}\right), \tag{26}$$

where m and ϵ are defined by Eq. (11), Eq. (25) reduces to

$$V_n(0) = -2^{n-1} n! \sqrt{\pi}\, \psi\left(m+\epsilon+\frac{1}{2}\right). \tag{27}$$

A closed formula for the entropy of Hermite polynomials $E_n(H)$ can be obtained by combining Eqs. (8) and (25),

$$\begin{aligned} E_n(H) &= 2^n n! \sqrt{\pi}\left(n\gamma - 2\sum_{i=1}^{n} x_{n,i}^2\, {}_2F_2\left(1,1;\frac{3}{2},2;-x_{n,i}^2\right)\right. \\ &\quad \left. + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k 2^k}{k} \sum_{i=1}^{n} M\left(k;\frac{1}{2};-x_{n,i}^2\right)\right). \end{aligned} \tag{28}$$

Alternatively, using Eq. (10) instead of (8), and taking into account Eq. (27), together with [6, 8]

$$\psi\left(n+\frac{1}{2}\right) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1}\,,$$

we get

$$\begin{aligned} E_n(H) &= 2^n n!\sqrt{\pi}\left(n\gamma - 2\epsilon\sum_{k=1}^{m+\epsilon}\frac{1}{2k-1} - 4\sum_{i=1}^{m} x_{n,i}^2\,{}_2F_2\left(1,1;\frac{3}{2},2;-x_{n,i}^2\right)\right. \\ &\left. + 2\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k}{k}\sum_{i=1}^{m} M\left(k;\frac{1}{2};-x_{n,i}^2\right)\right). \end{aligned} \tag{29}$$

In turn, from these results, using Eq. (4) we obtain for S_n the expressions

$$\begin{aligned} S_n &= \ln\left(2^n n!\sqrt{\pi}\,\right) + n + \frac{1}{2} + n\gamma - 2\sum_{i=1}^{n} x_{n,i}^2\,{}_2F_2\left(1,1;\frac{3}{2},2;-x_{n,i}^2\right) \\ &+ \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k}{k}\sum_{i=1}^{n} M\left(k;\frac{1}{2};-x_{n,i}^2\right), \end{aligned} \tag{30}$$

and

$$\begin{aligned} S_n &= \ln\left(2^n n!\sqrt{\pi}\,\right) + n + \frac{1}{2} + n\gamma - 2\epsilon\sum_{k=1}^{m+\epsilon}\frac{1}{2k-1} - 4\sum_{i=1}^{m} x_{n,i}^2\,{}_2F_2\left(1,1;\frac{3}{2},2;-x_{n,i}^2\right) \\ &+ 2\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k}{k}\sum_{i=1}^{m} M\left(k;\frac{1}{2};-x_{n,i}^2\right), \end{aligned} \tag{31}$$

respectively, which are the generalization of Eq. (7) to arbitrary values of n. For example, in the $n = 2$ case, the zeros of the polynomial $H_2(x) = 4x^2 - 2$ are $\pm\frac{1}{\sqrt{2}}$, so that we have

$$S_2 = \ln\left(8\sqrt{\pi}\,\right) + \frac{5}{2} + 2\gamma - 2\,{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{1}{2}\right) - 8\,M\left(1;\frac{1}{2};-\frac{1}{2}\right) + 4\,M\left(2;\frac{1}{2};-\frac{1}{2}\right),$$

while in the $n = 3$ case, the zeros of $H_3(x) = 8x^3 - 12x$ are $0, \pm\sqrt{\frac{3}{2}}$, and we have

$$\begin{aligned} S_3 &= \ln\left(48\sqrt{\pi}\,\right) + \frac{5}{6} + 3\gamma - 6\,{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{1}{2}\right) - 12\,M\left(1;\frac{1}{2};-\frac{3}{2}\right) \\ &+ 12\,M\left(2;\frac{1}{2};-\frac{3}{2}\right) - \frac{16}{3}\,M\left(3;\frac{1}{2};-\frac{3}{2}\right). \end{aligned}$$

3 Asymptotic expansion

The complete asymptotic expansion of the confluent hypergeometric function $M(a,c,z)$ for z real, $|z| \to \infty$ is given by [8],

$$\begin{aligned} M(a,c,z) &\sim \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}\,{}_2F_0\left(a, 1+a-c; -z^{-1}\right) \\ &+ \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}\,{}_2F_0\left(c-a, 1-a; z^{-1}\right). \end{aligned}$$

If $z = -t^2$ is a negative real number, the first term is dominant since the second one vanishes exponentially for $z \to -\infty$. Therefore, for $a = k, c = \frac{1}{2}$, we have

$$\begin{aligned} M\left(k, \frac{1}{2}, -t^2\right) &\sim \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-k\right)} t^{-2k} {}_2F_0\left(k, k+\frac{1}{2}; t^{-2}\right) \\ &= \frac{(2k-1)!}{2^{2k-1}(k-1)!}(-1)^k t^{-2k} {}_2F_0\left(k, k+\frac{1}{2}; t^{-2}\right). \end{aligned} \tag{32}$$

More generally, for ${}_pF_p$ hypergeometric functions, it has been shown [8] that, for $|z| \to \infty$, $\arg z = 0$,

$${}_pF_p(\alpha_1, \alpha_2, \ldots, \alpha_p; \rho_1, \rho_2, \ldots, \rho_p; -z) \sim \frac{\prod_{j=1}^p \Gamma(\rho_j)}{\prod_{j=1}^p \Gamma(\alpha_j)} \left(K_{p,p}(-z) + L_{p,p}(z)\right).$$

The function $K_{p,q}(z)$ is defined as

$$K_{p,q}(z) = \frac{(2\pi)^{(1-\beta)/2}}{\beta^{1/2}} \exp\left(\beta z^{1/\beta}\right) z^{\delta} \sum_{r=0}^{\infty} N_r\, z^{-r/\beta},$$

where

$$\beta = q + 1 - p, \quad \delta = \frac{1}{\beta}\left(\frac{\beta-1}{2} + \sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \rho_j\right), \quad N_0 = 1,$$

and for $r \geq 1$ the constants N_r are determined by means of recurrent formulas, while the function $L_{p,q}(z)$ can be formally written for arbitrary values of p and q in terms of Meijer's G function as

$$L_{p,q}(z) = G^{p,1}_{q+1,p}\left(z^{-1} \,\middle|\, \begin{array}{c} 1, \rho_1, \rho_2, \ldots, \rho_q \\ \alpha_1, \alpha_2, \ldots, \alpha_p \end{array}\right).$$

In the particular case $p = q = 2$, $\alpha_1 = \alpha_2 = 1$, $\rho_1 = \frac{3}{2}$, $\rho_2 = 2$, we have

$${}_2F_2\left(1, 1; \frac{3}{2}, 2; -t^2\right) \sim \frac{\sqrt{\pi}}{2}\left(K_{2,2}(-t^2) + L_{2,2}(t^2)\right),$$

where

$$K_{2,2}(z) = e^z z^{-3/2} \sum_{r=0}^{\infty} N_r\, z^{-r}, \quad L_{2,2}(z) = G^{2,1}_{3,2}\left(z^{-1} \,\middle|\, \begin{array}{c} 1, \frac{3}{2}, 2 \\ 1, 1 \end{array}\right).$$

We see that $K_{2,2}(-t^2)$ vanishes exponentially for $|t| \to \infty$. On the other hand, using results in Ref. [8], it can be shown [10] that

$$G^{2,1}_{3,2}\left(z^{-1} \,\middle|\, \begin{array}{c} 1, \frac{3}{2}, 2 \\ 1, 1 \end{array}\right) = \frac{1}{\sqrt{\pi} z}\left(\ln|z| + \gamma + 2\ln 2 - \frac{1}{2z}\, {}_3F_1\left(1, 1, \frac{3}{2}; 2; z^{-1}\right)\right),$$

so that we get

$${}_2F_2\left(1, 1; \frac{3}{2}, 2; -t^2\right) \sim \frac{1}{2t^2}\left(2\ln|t| + \gamma + 2\ln 2 - \frac{1}{2t^2}\, {}_3F_1\left(1, 1, \frac{3}{2}; 2; t^{-2}\right)\right). \tag{33}$$

Substituting Eqs. (32) and (33) into (25), we finally obtain

$$\begin{aligned} V_n(t) \sim\ & 2^n n! \sqrt{\pi}\left(-\ln|t| + \frac{1}{4t^2}\, {}_3F_1\left(1, 1, \frac{3}{2}; 2; t^{-2}\right)\right. \\ & \left. + \sum_{k=1}^{n} \binom{n}{k} \frac{(2k-1)!}{2^k k!} t^{-2k}\, {}_2F_0\left(k, k+\frac{1}{2}; t^{-2}\right)\right), \end{aligned} \tag{34}$$

which is the complete asymptotic expansion of $V_n(t)$ for $|t| \gg 1$.

4 Alternative expressions

Using Eq. (21) for $W_0(t)$, and Eq. (23) for $W_{2k}(t)$, $k>0$, Eq. (25) can be written as

$$\begin{aligned} V_n(t) &= 2^n n!\sqrt{\pi}\left(\frac{\gamma}{2}+\ln 2+\frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^r(r-1)!}{(2r)!}(2t)^{2r}\right. \\ &\left.+\frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k}{k!}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(k+r)}{(2r)!}(2t)^{2r}\right), \end{aligned}$$

which, taking advantage of Eq. (26), simplifies to

$$\begin{aligned} V_n(t) &= 2^{n-1}n!\sqrt{\pi}\left(-\psi\left(m+\epsilon+\frac{1}{2}\right)+\sum_{r=1}^{\infty}\frac{(-1)^r(r-1)!}{(2r)!}(2t)^{2r}\right. \\ &\left.+\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k}{k!}\sum_{r=1}^{\infty}\frac{(-1)^r\Gamma(k+r)}{(2r)!}(2t)^{2r}\right). \end{aligned} \tag{35}$$

Taking into account Eq. (17), together with the identity [8]

$$\binom{n}{k}=\frac{(-1)^k(-n)_k}{k!},$$

the summation over k in the double series of Eq. (35) can be performed in terms of the Gauss hypergeometric function,

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k\Gamma(r+k)}{k!}=(r-1)!\,(F(r,-n;1;2)-1).$$

Substituting this result into (35), we get

$$V_n(t)=2^{n-1}n!\sqrt{\pi}\left(-\psi\left(m+\epsilon+\frac{1}{2}\right)+\sum_{r=1}^{\infty}\frac{(-1)^r 2^{2r}(r-1)!}{(2r)!}F(r,-n;1;2)t^{2r}\right), \tag{36}$$

which is an alternative expression for the logarithmic potential $V_n(t)$.

The entropy of the Hermite polynomials defined by Eq. (8) can thus be written in the form

$$\begin{aligned} E_n(H) &= 2^n n!\sqrt{\pi}\left(-\ln\left(2^{2n}\right)-n\,\psi\left(m+\epsilon+\frac{1}{2}\right)\right. \\ &\left.+n\sum_{r=1}^{\infty}\frac{(-1)^r 2^{2r}(r-1)!}{(2r)!}F(r,-n;1;2)\mu_{2r}(n)\right), \end{aligned} \tag{37}$$

where $\mu_r(n)$ $(r=0,1,2,\ldots)$ are the spectral moments around the origin of the Hermite polynomial $H_n(x)$, i.e., the quantities

$$\mu_r(n)=\frac{1}{n}\sum_{i=1}^{n}(x_{n,i})^r.$$

Eq. (4) then yields, taking into account that $m+\epsilon=(n+\epsilon)/2$,

$$\begin{aligned} S_n &= \ln\left(\frac{n!\sqrt{\pi}}{2^n}\right)+n+\frac{1}{2}-n\,\psi\left(\frac{n+\epsilon+1}{2}\right) \\ &+n\sum_{r=1}^{\infty}\frac{(-1)^r 2^{2r}(r-1)!}{(2r)!}F(r,-n;1;2)\mu_{2r}(n). \end{aligned} \tag{38}$$

This new expression for S_n is less useful than Eqs. (30) and (31) for practical calculations, since, unfortunately, there is no global and compact expression for the moments $\mu_{2r}(n)$, but they have to be recurrently generated [3]. However, Eq. (38) turns out to be more appropriate than Eqs. (30) and (31) to discuss the problem of the relation between our present exact results and the asymptotic approximation (6). Use of the well-known asymptotic expansions for the Gamma and Psi functions [8] gives

$$\ln(n!) \sim \left(n+\frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln(2\pi) + O(n^{-1}) ,$$
$$\psi\left(\frac{n+\epsilon+1}{2}\right) \sim \ln\left(\frac{n+\epsilon}{2}\right) + O(n^{-2}) \sim \ln n - \ln 2 + \frac{\epsilon}{n} + O(n^{-2}) ,$$

where the remaining terms of these expansions can be explicitly written in terms of Bernoulli numbers. Substituting these expressions into Eq. (38), we obtain the asymptotic formula

$$S_n \sim \ln\left(\pi\sqrt{2n}\right) + \frac{1}{2} - \epsilon + n\sum_{r=1}^{\infty}\frac{(-1)^r 2^{2r}(r-1)!}{(2r)!}F(r,-n;1;2)\mu_{2r}(n) + O(n^{-1}) ,$$

and comparison of this result with Eq. (6) yields

$$n\sum_{r=1}^{\infty}\frac{(-1)^r 2^{2r}(r-1)!}{(2r)!}F(r,-n;1;2)\mu_{2r}(n) \sim \epsilon - \frac{3}{2} + o(1) .$$

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