©1997 M. Alfaro et al. (Eds.) Proceedings of the International Workshop on Orthogonal Polynomials in Mathematical Physics. Leganés, 24-26 June, 1996.

ORTHOGONAL POLYNOMIALS: CONNECTION AND LINEARIZATION COEFFICIENTS

A. Ronveaux

Abstract

Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_m(x)\}_{m=0}^{\infty}$ be two families of orthogonal polynomials. The linearization problem involves only one family via the relation:

$$P_i(x)\ P_j(x) = \sum_{k=|i-j|}^{i+j} L_{ijk}\ P_k(x)$$

and the connection problem mixes both families:

$$P_n(x) = \sum_{m=0}^{n} C_m(n)\ Q_m(x).$$

In many cases, it is possible to build a recurrence relation involving only m , satisfied by the linearization coefficients L_{ijk} and connection coefficients $C_m(n)$, in particular if the families $P_n(x)$ and $Q_m(x)$ are classical: continuous or discrete. We intend to examine these coefficients from the point of view of recurrence properties, emphasizing mainly the structure of these recurrence relations. This article contains some results already published, in print or in preparation, by A. Ronveaux, A. Zarzo, E. Godoy, I. Area, R. Álvarez-Nodarse, S. Belmehdi, N. M. Hounkonnou and S. Lewanowicz.

1 Connection problem.

Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_m(x)\}_{m=0}^{\infty}$ be two sequences of polynomials of degree exactly equal to n. The so-called Connection Problem asks to find the coefficients $C_m(n)$ in the expansion:

$$P_n(x) = \sum_{m=0}^{n} C_m(n)\ Q_m(x)\ , \tag{1}$$

and goes back to Stirling [35] as pointed out by Askey in his 1975 survey [4].

Let us represent the polynomials $P_n(x)$ and $Q_m(x)$ in the monomial basis $\{x^k\}_{k=0}^{\infty}$, and analyze the structure of the connection coefficients $C_m(n)$ as depending on several assumptions on families $P_n(x)$ and $Q_m(x)$

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k\ , \qquad Q_m(x) = \sum_{k=0}^{m} \overline{a}_{m,k}x^k\ , \qquad a_{n,k}, \overline{a}_{n,k} \in \mathbb{R}\ . \tag{2}$$

Other bases like $x^{[n]}$, $(x)_n$ or Bernstein basis can also be used, and polynomials $P_n(x)$ and $Q_m(x)$ can be defined by some properties or equations. So let us use in this article the following definitions:

1. **Falling factorial or Stirling polynomial** [1]
   $x^{[n]} = x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)} x^m$, where $S_n^{(m)}$ are Stirling numbers of first kind [35].

2. **Rising factorial or Pochhammer symbols** [1]
$(x)_n = x(x+1)\cdots(x+n-1)$, $(x)_0 = 1$.

3. **Appell family** $\{P_n(x)\}$ [7]
$P_n'(x) = \lambda(n)P_{n-1}(x)$

4. **Orthogonal family** $\{P_n(x)\}$ [7, 8]
Monic polynomial $P_n(x)$ of degree n generated by the three-term Recurrence Relation (R.R.)

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \ , \qquad n \geq 0 \ ,$$

$$P_{-1}(x) = 0 \ , \quad P_0(x) = 1 \ , \qquad \gamma_n \neq 0 \ , \ n \geq 0 \ .$$

5. **Classical orthogonal polynomials (continuous)** [25, 26]
Polynomial solution $P_n(x)$ of degree n of the second order Differential Equation (D.E.):

$$\mathcal{L}_{2,n}[P_n(x)] \equiv \sigma P_n'' + \tau P_n' - n\tau'_{(n-1)/2} P_n = 0 \ ,$$

$\tau_r = \tau + r\sigma'$, $\sigma = \sigma(x)$ polynomial of degree 2: Jacobi (J), Bessel (B); of degree 1: Laguerre (L); of degree 0: Hermite (He), and $\tau \equiv \tau(x)$ polynomial of first degree.

6. **Classical orthogonal polynomial (discrete)** [25, 26]
Polynomial solution $P_n(x)$ of degree n of the second order Difference Equation

$$\mathcal{D}_{2,n}[P_n(x)] \equiv \sigma \Delta \nabla P_n + \tau \Delta P_n - n\tau'_{(n-1)/2} P_n = 0$$

$\sigma \equiv \sigma(x)$ polynomial of degree 2: Hahn (H) and Hahn-Eberlein (H-E), and polynomial of degree 1: Krawtchouk (K), Meixner (M), Charlier (C). $\tau = \tau(x)$ polynomial of first degree. $(\Delta f(x) = f(x+1) - f(x) \ , \ \nabla f(x) = \Delta f(x-1))$

7. **Semi-classical orthogonal family (continuous and discrete)** [11, 24]
Family of orthogonal polynomial $P_n(x)$ which satisfies a structure relation (S.R.)

$$\sigma(x) L P_n(x) = \sum_{k=n-s-1}^{n+t-1} \alpha_{k,n} P_k(x) \qquad (\alpha_{k,n} \text{ constants})$$

where $L = d/dx$ (continuous) or $L = \Delta$ (discrete).
For $t = 0, 1, 2$ and $s = 0$ semi-classical orthogonal polynomials (O.P.) are classical O.P.

8. **Hildebrandt classical polynomial (continuous)** [12]
Polynomial solution $P_n(x,k)$ of degree n of the Differential Equation:

$$\sigma(x) P_n''(x,k) + \tau_{k-n}(x) P_n'(x,k) - n\tau'_{k-n/2-1/2} P_n(x,k) = 0 \ .$$

These polynomials: H-J, H-B, H-L, H-He, reduce to the family #5. (orthogonal) when $k = n$, and to the family $P_n^{(r)}(x)$, when $k - n = r$ ($P_n^{(r)}(x)$, r-th derivative of $P_n(x)$; $[P_{n+r}(x; n+r)]^{(r)} = P_n(x, n+r)$).
For fixed k , $P_n(x,k)$ is an Appell family.

9. **Bernstein Basis (continuous)** [34]

$$B_i^n(x) = \begin{pmatrix} n \\ i \end{pmatrix} x^i (1-x)^{n-i} = \sum_{j=i}^{n} (-1)^{j-i} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} x^j \ , \qquad 0 \leq i \leq n \ .$$

10. Bernstein Basis (discrete) [34]

$$b_i^n(N,x) = \binom{n}{i} \frac{x^{[i]}(N-x)^{[n-i]}}{N^{[i]}} \;, \quad 0 \le i \le n \le N$$

The bases $B_i^n(x)$ and $b_i^n(N,x)$ are of degree n ($0 \le i \le n \le N$) for all i , contrary to the previous polynomials $P_n(x)$ and $Q_n(x)$ of degree equal to n .

The vectors $\vec{P}_n^t = (P_0, \cdots, P_n)$ and $\vec{Q}_n^t = (Q_0, \cdots, Q_n)$ define now a lower triangular connection matrix $[C_{n,m}]$

$$\vec{P}_n = [C_{n,m}]\vec{Q}_n \qquad \text{with } C_{n,m} = C_m(n) \tag{3}$$

which has a specific structure (see Table 1) following the assumptions #1 to #10.

In general, we work with monic polynomials ($a_{n,n} = \overline{a}_{n,n} = 1$). In that case, the set of all connection matrix of size $n+1$ by $n+1$ forms a multiplicative group called $T(n+1, \mathrm{I\!R})$, subgroup of $SL(n+1, \mathrm{I\!R})$.

Let us emphasize the structure of the connection coefficients in two following cases: i) assumption #4 (families P_n and Q_n both orthogonal) and ii) assumption #7 (families P_n and Q_n both semi-classical).

It is important to realize first (and to prove easily by derivation of the S.R.) that the existence of a S.R. for the family $P_n(x)$ implies that $P_n(x)$ is solution of a second order differential (or difference) equation:

$$\mathcal{L}_{2,n}[P_n(x)] = 0 \;, \qquad (\; \mathcal{D}_{2,n}[P(x)] = 0 \;). \tag{4}$$

These operators $\mathcal{L}_{2,n}$ and $\mathcal{D}_{2,n}$ are of course more general than the operators considered in the assumptions #5 and #6.

i) $\{P_n(x)\}_{n=0}^{\infty}$ **and** $\{Q_m(x)\}_{m=0}^{\infty}$ **are both orthogonal families.** (assumption #4). Both families of polynomials involved in the connection problem (1) satisfy a three-term recurrence relation which, considering monic polynomials without loss of generality, are respectively:

$$\begin{aligned} x\,P_n(x) &= P_{n+1}(x) + \beta_n\,P_n(x) + \gamma_n\,P_{n-1}(x)\;, \qquad n \ge 0\;, \\ P_{-1}(x) &= 0\;, \qquad P_0(x) = 1\;, \end{aligned} \tag{5}$$

$$\begin{aligned} x\,Q_m(x) &= Q_{m+1}(x) + \overline{\beta}_m\,Q_m(x) + \overline{\gamma}_m\,Q_{m-1}(x)\;, \qquad m \ge 0\;, \\ Q_{-1}(x) &= 0\;, \qquad Q_0(x) = 1\;, \end{aligned} \tag{6}$$

where $\gamma_n \neq 0$ and $\overline{\gamma}_m \neq 0$. Then, multiplication of (1) by x , the use of (5) and (6) and expansion of P_{n+1} , P_n and P_{n-1} in terms of Q_m by means of (1), generates the "cross-rule":

$$(\overline{\beta}_m - \beta_n)\,C_m(n) + C_{m-1}(n) + \overline{\gamma}_{m+1}\,C_{m+1}(n) - C_m(n+1) - \gamma_n\,C_m(n-1) = 0 \tag{7}$$

which is valid for $m = 0, 1, \cdots, n+1$, with the convention $C_k(j) = 0$ when $k < 0$ or $k > j$. Notice that this "cross-rule" mixes the three adjacent coefficients ($m-1, m, m+1$) of the row n , crossing at the (m,n) position the column m of the aforementioned $[C_{n,m}]$–matrix. Any coefficient $C_m(n)$ can be computed from this "cross-rule" starting from the initial condition: $C_n(n) = 1$ (monic polynomials), $C_{n-1}(n) = a_n - \overline{a}_n$ and $C_{n-2}(n) = b_n - \overline{b}_n - (a_n - \overline{a}_n)\overline{a}_{n-1}$, where a_n and b_n (respectively $\overline{a}_n$ and $\overline{b}_n$) are the first coefficients in the expansions of $P_n(x)$ and $Q_n(x)$, $a_n = \frac{a_{n,n-1}}{a_{n,n}}$, $b_n = \frac{a_{n,n-2}}{a_{n,n}}$.

$$\boxed{P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x)}$$

$P_n(x)$	$Q_m(x)$	**Connection matrix** $[C_{n,m}] = [C_m(n)]$
arbitrary	arbitrary	no structure (except $C_n(n) \neq 0$)
x^n	$x^{[m]}$	hook structure $s_n^{(m)} = s_{n-1}^{(m-1)} + m s_{n-1}^{(m)}$ $s_n^{(m)}$ = Stirling numbers of second kind
$x^{[n]}$	x^m	hook structure $s_n^{(m)} = s_{n-1}^{(m-1)} - (n-1) s_{n-1}^{(m)}$ $s_n^{(m)}$ = Stirling numbers of first kind
Appell	$\begin{cases} x^m \\ \text{Appell} \end{cases}$	diagonal structure $C_{m+1}(n+1) = k(m,n) C_m(n)$
orthogonal	orthogonal	cross structure
semi-classical O.P.	semi-classical O.P.	row structure
x^n	semi-classical O.P.	row structure
Hildebrandt classical	semi-classical O.P.	row structure
$\begin{cases} \mathcal{L}_{r,n}[P_n(x)] = 0 \\ \mathcal{D}_{r,n}[P_n(x)] = 0 \end{cases}$	semi-classical O.P.	row structure
primitive of semi-classical	semi-classical O.P.	row structure
arbitrary	Bernstein basis (continuous or discrete)	no structure (full square matrix)

Table 1: Structure of the connection matrix.

ii) $\{P_n(x)\}_{n=0}^{\infty}$ **and** $\{Q_m(x)\}_{m=0}^{\infty}$ **are both semi-classical** (assumption #7) These assumptions transform the "cross-rule" given in (7) in a "row-rule", avoiding the mixing between m and n .

The connection coefficients $C_m(n)$ are now linked by the relation (n fixed):

$$\mathcal{L}_{2,n}[P_n(x)] = \sum_{m=0}^{n} C_m(n)\mathcal{L}_{2,n}[Q_m(x)] = 0 \qquad \text{(same with } \mathcal{D}_{2,n}\text{)} \tag{8}$$

from the Differential (or Difference) Equation satisfied by P_n . Each term $x^k\ L^j[Q_m(x)]$ can be expanded in $Q_m(x)$ using iteration of the S.R., the R.R., and also by multiplication by polynomials. This is the basis of our algorithm described and summarized in Section 3.

2 Linearization problems

Let $\{P_k(x)\}_{k=0}^{\infty}$ be a system of polynomials of degree exactly equal to k . The traditional linearization problem [4] is based in the expansion of the product P_iP_j in the $\{P_k(x)\}_{k=0}^{\infty}$ basis

$$P_iP_j = \sum_{k=0}^{i+j} L_{i,j,k}\ P_k\ . \tag{9}$$

The first attempt to find the coefficients $L_{i,j,k}$ goes back to Neumann [13] in the Legendre case using the 4-th order differential equation satisfied by a product of two Legendre polynomials. In the linear space of polynomials $P_i \equiv P_i(x)$, the problem of linearization of products of two polynomials occurs very naturally in a lot of situations ranging from combinatorics and statistical mechanics, to group theory in relation with Clebsch-Gordan coefficients [14, 10, 26, 15].

From the algebraic point of view, the coefficients $L_{i,j,k} = L_{j,i,k}$ are the structure constants C_{ij}^k of the basis $e_i \equiv P_i$ in the infinite dimensional associative and commutative algebra of polynomials:

$$e_ie_j = C_{ij}^k e_k\ , \tag{10}$$

using Einstein summation convention.
Again the following assumption changes drastically the relation satisfied by $L_{i,j,k}$.

i) At the level of this associative and commutative algebra, $L_{i,j,k}$ verify complicated **nonlinear algebraic** relation obtained by expanding relation (9) after multiplication by P_r

$$\sum_{s=0}^{r+i} L_{r,i,s}P_sP_j = \sum_{k=0}^{i+j} L_{i,j,k}P_rP_k\ , \tag{11}$$

and expanding in terms of P_t

$$\sum_{s=0}^{r+i} L_{r,i,s} \sum_{t=0}^{s+j} L_{s,j,t} = \sum_{k=0}^{i+j} L_{i,j,k} \sum_{t=0}^{r+k} L_{r,k,t} \qquad \forall\ i,j,r\ ; \tag{12}$$

ii) The orthogonality structure, inside the P_i family (making now the summation from $k = |i-j|$ to $k = i+j$), gives a three-term recurrence relation.

Multiplying the R.R. (assumption #4 in section 1) by P_i and using equation (9), we get

$$\sum_{k=|i-j|}^{i+j} L_{i,j,k}(P_{k+1} + \beta_kP_k + \gamma_kP_{k-1})$$

$$= \sum_{k=|i-j-1|}^{i+j+1} L_{i,j+1,k}P_k + \sum_{k=|i-j|}^{i+j} \beta_jL_{i,j,k}P_k + \sum_{k=|i-j+1|}^{i+j-1} \gamma_jL_{i,j-1,k}P_k \quad \forall\ i,j \tag{13}$$

from which, using the linear independence of P_k , we get a **linear relation** between the $L_{i,j,k}$

$$\begin{aligned}&\sum_k L_{i,j+1,k} + \beta_j L_{i,j,k} + \gamma_j L_{i,j-1,k} \\ &\quad = \sum_k L_{i,j,k-1} + \sum_k \beta_k L_{i,j,k} + \sum_k \gamma_k L_{i,j,k+1} \quad \forall\, i,j\ . \qquad (14)\end{aligned}$$

Even the fact that the orthogonality structure inside the P_i family transforms nonlinear relations into linear relations, equation (14) is practically intractable because **two** families of indices (i and j) are already involved in this recurrence.

iii) The restriction to **semi-classical** orthogonal polynomials allows to obtain again the linear relation between the $L_{i,j,k}$ but involving only **one** index k, using the fourth order differential (or difference) equation satisfied by the product P_iP_j and reducing therefore the linearization problem in this case to a connection problem (see section 1).

3 Algorithm building $R[C_{m+p}(n), C_{m-q}(n)] = 0$

In this work, we approach the connection and linearization coefficients as solution of a recurrence relation. Let us summarize the algorithm we have developed in [29, 31] building a **one index** recurrence relation for the connection coefficient $C_m(n)$. The assumptions are slightly more general than the assumptions used in the derivation of the "row-rule" in section 1.

Our method requires only two assumptions:

1. The $\{P_n(x)\}$-family in equation (1) has to be a solution of a linear differential or difference equation of any order r :
$$\sum_{j=0}^{r} A_j(x;n)\ L^j[P_n(x)] = 0 \qquad (15)$$
where the operator L can be either the derivative ($D = d/dx$) or the difference operator (Δ) and $A_j(x;n)$ are polynomials in the variable x , of **degree** independent of n .

2. The $\{Q_m(x)\}$–family in equation (1) should satisfy first a recurrence relation of type:
$$x\ Q_m(x) = \sum_{k=m-h}^{m+1} \beta_{k,m}\ Q_k(x) \qquad (16)$$
and also:
$$\sigma(x)\ LQ_m(x) = \sum_{k=n-1-s}^{n-1+t} \alpha_{k,m}\ Q_k(x) \qquad (17)$$
where $\sigma(x)$ is a polynomial of degree t , $\beta_{k,m}$ and $\alpha_{k,m}$, are constants. When $h = 1$, the $\{Q_m(x)\}_{m=0}^{\infty}$ family is orthogonal.

The relation (17) (also called "Structure Relation" in orthogonal context [24]) is satisfied not only by classical orthogonal polynomials continuous and discrete, but also by a very large class of orthogonal polynomials called semi-classical [11]. Any family of orthogonal polynomials which satisfy second order differential or difference equations of the form (15) is semi-classical and for such family a structure relation exists. All orthogonal polynomials with orthogonality measure

$\rho(x) = \rho_c(x) + \sum_i M_i \ \delta(x - c_i)$, $\rho(x) = \pi(x)\rho_c(x)$ or $\rho(x) = \theta_\lambda \rho_c(x)$, where ρ_c denotes a classical weight, are for example semi-classical ($\pi(x)$ rational function of x , $\theta_\lambda(x)$ the step function: $\theta_\lambda(x) = 0$, $x < \lambda$; $\theta_\lambda(x) = 1$, $\lambda \le x$).

The algorithm runs in the following steps

1. Apply relation (15) to the definition (1)

$$\sum_{m=0}^{n} \sum_{j=0}^{r} C_m(n) A_j(x;n) L^j[Q_m(x)] = 0 \ .$$

2. Expand all terms in $A_j(x;n)L^j[Q_m(x)]$ in a constant coefficient relation between the polynomials $Q_m(x)$. In order to do that, all power in $A_j(x;n)$ and all derivatives (or differences) in $L^j[Q_m(x)]$ are expanded using relation (16) and (17), iterates a number of times depending on the highest degree in $A_j(x;n)$ and the order j in $L^j[Q_m(x)]$.

3. From the linear independence of polynomial $Q_m(x)$ (other basis like $Q'_m(x)$ or $Q''_m(x)$ are possible and sometimes more appropriate [9, 3]), a shift of indices generates the search recurrence

$$R[C_{m+p}(n), \cdots, C_{m-q}(n)] = 0 \ .$$

For problems involving classical orthogonal continuous or discrete (sect. 1, assumptions #5 and #6), all the data contained in (15), (16), (17) are computed via relations given in [38] from the polynomials $\sigma(x)$ and $\tau(x)$ appearing in assumptions #5 and #6 of section 1. These $\sigma(x)$ and $\tau(x)$ give also the orthogonality weight $\rho = \rho(x)$ via the Pearson equations

$$(\sigma\rho)' = \tau\rho \ , \qquad \Delta(\sigma\rho) = \tau\rho \ .$$

These two polynomials σ and τ are the basic input in the program which follows automatically steps 1 to 3.

Among the many results partly published, let us select the connection between two Meixner polynomials (assumption #6).

Meixner-Meixner

$$\sigma(x) = x \ , \quad \tau(x) = \gamma\mu - x(1-\mu) \qquad \gamma > 0 \ , \ 0 < \mu < 1 \ .$$

The coefficients for the connection problem

$$M_n^{(\gamma,\mu)}(x) = \sum_{m=0}^{n} C_m(n) M_m^{(\delta,\nu)}(x)$$

satisfy the **three-terms** recurrence relation

$$\begin{aligned} &(1-\mu)(1-m+n)(-1+\nu)^2 C_{m-1}(n) \\ &+m(-1+\nu)[(-1+\delta+2m-n)\nu + \mu(1-\gamma-m+(-\delta+\gamma-m+n)\nu)]C_m(n) \\ &+m(1+m)(\delta+m)(\mu-\nu)\nu C_{m+1}(n) = 0 \end{aligned} \tag{18}$$

valid for $1 \le m \le n$, with initial conditions

$$\begin{aligned} C_n(n) &= 1 \\ C_{n-1}(n) &= \frac{n[\mu(1-\gamma-n) + (-1+\delta+(-\delta+\gamma)\mu+n)\nu]}{(1-\mu)(1-\nu)} \ . \end{aligned}$$

In case $\mu = \nu$, it reduces to a two-terms recurrence relation, whose solution is [31, 21]:

$$C_m(n) = \left(\begin{array}{c} n \\ m \end{array} \right) \left(\frac{\mu}{1-\mu} \right)^{n-m} (\gamma - \delta)_{n-m} \ . \tag{19}$$

In case $\gamma = \delta$, the (new) result is [3]

$$C_m(n) = \left(\begin{array}{c} n \\ m \end{array} \right) \left(\frac{\Gamma(n+\delta)}{\Gamma(m+\delta)} \right) \left(\frac{\nu - \mu}{(1-\mu)(1-\nu)} \right)^{n-m} \ . \tag{20}$$

Table 2 presents informations on explicit results or on the structure of the recurrence relation satisfied by $C_m(n)$ connecting essentially Classical Orthogonal Polynomials. Many results are already published, in print or in preparation. We refer mainly to works related to the recurrence relation approach in which, of course, other approaches and results are mentioned.

4 Group properties

The Lie group $T(n+1, \mathbb{R})$ of lower triangular $n+1$ matrices with diagonal elements being equal to 1 contains $\frac{n(n+1)}{2}$ parameters. If for a family of polynomials $\vec{P}_n(x, \alpha_i)$ with r parameters $i = 1, \cdots, r$, we can solve explicitly the connection problem, we generate automatically a r dimensional subgroup of $T(n, \mathbb{R})$ as a realization of this.

The group property generates the following multiplicative law between the matrices connecting parameters $\alpha, \overline{\alpha}, \overline{\overline{\alpha}}$:

$$[C_{n,m}]^{\alpha,\overline{\alpha}}[C_{m,k}]^{\overline{\alpha},\overline{\overline{\alpha}}} = [C_{n,k}]^{\alpha,\overline{\overline{\alpha}}}$$

with

$$\vec{P}_n(x, \alpha_i) = [C_{n,m}]^{\alpha,\overline{\alpha}} \vec{P}_n(x, \overline{\alpha}_i)$$

$$[C_{n,m}]^{\alpha,\overline{\alpha}}[C_{m,k}]^{\overline{\alpha},\alpha} = \mathbb{1} = [C_{n,k}]^{\alpha,\alpha} \ .$$

As an example of one dimensional subgroup, let us consider the Charlier-Charlier connection problem:

$$C_n^{(\alpha)}(x) = \sum_{m=0}^{n} C_m(n) C_m^{(\overline{\alpha})}(x)$$

for which $C_m(n) = \left(\begin{array}{c} n \\ m \end{array} \right) (\overline{\alpha} - \alpha)^{n-m}$.

With $\lambda = \overline{\alpha} - \alpha$, the one dimensional subgroup appears as

$$[C_{n,m}]^{\lambda} = \left[\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ \lambda & 1 & 0 & \cdots & 0 \\ \lambda^2 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda^k & \binom{k}{1} \lambda^{k-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda^n & \binom{n}{1} \lambda^{n-1} & \binom{n}{2} \lambda^{n-2} & \cdots & 1 \end{array} \right]$$

which gives the matrix property

$$[C_{n,m}]^{\lambda}[C_{m,k}]^{\overline{\lambda}} = [C_{n,k}]^{\lambda+\overline{\lambda}}$$

$$\{[C_{n,m}]^{\lambda}\}^{-1} = [C_{n,m}]^{-\lambda}$$

$P_n(x)$	$Q_m(x)$	$R[C_{m+p}(n), \cdots, C_{m-q}(n)] = 0$
x^n	(J,B,L,He)	max 3 terms, solved, [39, 23]
$x^{[n]}$	(H,H-E,,K,M,C))	max 3 terms, solved, [39]
(J,B,L,He)	(J,B,L,He)	max 3 terms, partly solved, [9]
(H,H-E,K,M,C))	$\begin{cases} \text{(H,H-E,K,M,C)} \\ x^{([m]} \end{cases}$	max 3 terms, partly solved, [31, 3, 21] max 3 terms, partly solved, [39]
$\begin{cases} P_n(x+b), P_n(ax) \\ P_n = \text{(J,B,L,He)} \end{cases}$	(J,B,L,He)	$\begin{cases} Q_m(x) \equiv P_m(x) \\ \text{max 5 terms, partly solved, [9]} \end{cases}$
$\begin{cases} P_n(x+y), \\ P_n = \text{(H,H-E,K,M,C)} \end{cases}$	(H,H-E,K,M,C)	$\begin{cases} Q_m(x) \equiv P_m(x) \\ \text{max 3 terms, partly solved, [31, 3]} \end{cases}$
Bernstein basis - continuous - discrete	shifted Jacobi Hahn polynomial	5 terms, solved, [33]
(H-J,H-B,H-L,H-He)	(J,B,L,He)	no mixture max 3 terms, in progress
$\begin{cases} P_iP_j \\ P_i, P_j = \text{(J,B,L,He)} \\ P_i, P_j = \text{(H,H-E,K,M,C)} \end{cases}$	(J,B,L,He) (H,H-E,K,M,C)	$\begin{cases} \text{JJ,BB,LL,HeHe 3 terms [28, 20]} \\ \text{mixture, max 5 terms, [30]} \end{cases}$ no mixture, max 3 terms, [6]
$\begin{cases} [P_i]^N \\ P_i = \text{(J,B,L,He)} \\ P_i = \text{(H,H-E,K,M,C)} \end{cases}$	(J,B,L,He) (H,H-E,K,M,C)	in progress ($N = 2$, max 5 terms) [28] in progress ($N = 2$, max 9 terms) [22]
$\begin{cases} P_n^{(r)}, \ r \text{ associated} \\ P_n = \text{(J,B,L,He)} \\ P_n = \text{(H,H-E,K,M,C)} \end{cases}$	(J,B,L,He) (H,H-E,K,M,C)	$\begin{cases} r = 1 \text{ , [29, 18, 17]} \\ \text{arbitrary } r \text{ , [27, 18, 17]} \end{cases}$ $\begin{cases} r = 1 \text{ , [5, 32]} \\ \text{arbitrary } r \text{ , in progress} \end{cases}$
$\begin{cases} \text{(J,B,L,He)} \\ \text{(H,H-E,K,M,C)} \end{cases}$	$\begin{cases} D^m\text{(J,B,L,He)} \\ \Delta^m\text{(H,H-E,K,M,C)} \end{cases}$	Gram-Charlier series in progress
$\begin{cases} \text{even part of (J,B,L)} \\ \text{odd part of (J,B,L)} \end{cases}$	(J,B,L,He)	in progress
reciprocal of (J,B,L,He)	(J,B,L,He)	in progress

Table 2: Applications of the algorithm involving classical O.P.

This one-dimensional subgroup is, of course, abelian, but it should be interesting to investigate two dimensional subgroups coming from Jacobi-Jacobi, Meixner-Meixner, Hahn-Hahn, ... From the fact that there exists only two real Lie algebras of dimension two (nine of dimension three, ...), which family of two parameter orthogonal polynomials corresponds to the abelian or solvable case?

Remarks: This short survey presents in some way a new approach in order to study systematically connection and linearization problems involving polynomials orthogonal and also not orthogonal. An one variable recurrence relation is the basic tool which allows to compute or to study these connection and linearization coefficients. For a large class of polynomials these recurrence relations are built almost automatically by a very simple algorithm which applies in all cases.

In many situations, these recurrences are solved by the computer using Mathematica, Maple, Reduce, ... Even if the explicit solution is not given by the computer, properties of these coefficients can be obtained: positivity, asymptotics, limit relations, [36, 37, 16].

This global approach, which applies already for some q-orthogonal polynomials [2], is quite different from other more traditional approaches using for each family an ad hoc technique: generating functions, hypergeometric series properties (se e.g. [4]).

ACKNOWLEDGEMENTS It is a pleasure to thank the Universidad Carlos III de Madrid for the kind invitation and for the financial support which allowed me to present this material in Leganés with the ocasion of the International Workshop on Orthogonal Polynomials and Mathematical Physics (24-26 June 1996). Parts of the published results presented in this work were supported by a NATO grant (CRG-960213).

References

[1] M. Abramowitz and I.A. Stegun: HANDBOOK OF MATHEMATICAL FUNCTIONS. Dover, 1965.

[2] R. Álvarez-Nodarse and A. Ronveaux: *Recurrence relations for connection coefficients between q-orthogonal polynomials of discrete variables in the non-uniform lattice* $X(s) = q^{2s}$. J. PHYS. A: MATH. GEN. **29** (1996), 1–11.

[3] I. Area, E. Godoy, A. Ronveaux and A. Zarzo: *Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Discrete case.* 1996 (submitted).

[4] R. Askey: *Orthogonal Polynomials and Special Functions.* REGIONAL CONFERENCE IN APPLIED MATHEMATICS. **21**, SIAM, Philadelphia, Pennsylvania, 1975.

[5] N. M. Atakishiyev, A. Ronveaux and K.B. Wolf: *Difference equations for the associated polynomials on the linear lattice.* J. THEOR. MATH. PHYS. (Moscow) **106** (1996), 76–83.

[6] S. Belmehdi, S. Lewanowicz and A. Ronveaux: *Linearization of product of orthogonal polynomials of a discrete variable.* (1996), (submitted).

[7] T. S. Chihara: *An Introduction to Orthogonal Polynomials.* Gordon and Breach Publ., New York, 1978.

[8] A. Erdélyi Ed.: *Higher Transcendental Functions.* **Vol. I**, McGraw-Hill, New York, 1953.

[9] E. Godoy, I. Area, A. Ronveaux and A. Zarzo: *Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Continuous case.* (1996), (submitted).

[10] C. D. Godsil: ALGEBRAIC COMBINATORICS. Chapman and Hall, New York, 1993.

[11] E. Hendriksen and H. van Rossum: *Semiclassical orthogonal polynomials.* LECTURE NOTES IN MATH. **1171**, Springer, Berlin (1984), 354–361.

[12] E. H. Hildebrandt: *Systems of polynomials connected with the Charlier expansions and the Pearson differential and difference equation.* ANN. MATH. STATIST. **2**, (1931), 379–439.

[13] W. Hobson: THE THEORY OF SPHERICAL AND ELLIPSOIDAL HARMONICS. Chelsea, New York, 1965.

[14] E. Hylleraas: *Linearization of products of Jacobi polynomials.* MATH. SCAND. **10** (1962), 189–200.

[15] J. LABELLE AND Y.N. YEH: *The combinatorics of Laguerre, Charlier and Hermite polynomials.* STUDIES IN APPL. MATH. **80** (1980), 25–36.

[16] R. Lasser: *Linearization of the product of associated Legendre polynomials.* SIAM J. MATH. ANAL. **14** (1983), 403–408.

[17] S. Lewanowicz: *Quick construction of recurrence relations for the Jacobi coefficients.* J. COMPUT. APPL. MATH. **43** (1992), 355–372.

[18] S. Lewanowicz: *Results on the associated classical orthogonal polynomials.* J. COMPUT. APPL. MATH. **65** (1995), 215–231.

[19] S. Lewanowicz: *Results on the associated Jacobi and Gegenbauer polynomials.* J. COMPUT. APPL. MATH. **49** (1993), 137–143.

[20] S. Lewanowicz: *Second order recurrence relations for the linearization coefficients of the classical orthogonal polynomials.* J. COMPUT. APPL. MATH. **69** (1996), 159–170.

[21] S. Lewanowicz: *Recurrence relations for the connection coefficients of orthogonal polynomials of a discrete variable.* J. COMPUT. APPL. MATH. **76** (1996), 213–229.

[22] S. Lewanowicz and A. Ronveaux: *Linearization of powers of an orthogonal polynomial of a discrete variable.* (1996) (submitted).

[23] Y. L. Luke: THE SPECIAL FUNCTIONS AND THEIR APPROXIMATIONS. Academic Press, New York, 1969.

[24] P. Maroni: *Prolégomènes à l'étude des polynômes orthogonaux semi-classiques.* ANNAL. DI MAT. PURA ET APPL. **149** (4) (1987), 165–184.

[25] A.F. Nikiforov and V.B. Uvarov: FONCTIONS SPÉCIALES DE LA PHYSIQUE MATHÉMATIQUE. MIR, Moscou, 1983.

[26] A.F. Nikiforov, S.K. Suslov and V.B. Uvarov: *Classical Orthogonal Polynomials of a Discrete Variable.* Springer-Verlag, Berlin, 1991.

[27] A. Ronveaux: *4th order differential equations and orthogonal polynomials of the Laguerre-Hahn class.* In: ORTHOGONAL POLYNOMIALS AND THEIR APPLICATION (C. Brezinski, L. Gori and A. Ronveaux eds.) IMACS Annals on Computing and Applied Math., **Vol. 9** (1–4); CIV (1991), J.C. Baltzer A.G. Basel, 379–385.

[28] A. Ronveaux, M.N. Hounkonnou and S. Belmehdi: *Recurrence relations between linearization coefficients of orthogonal polynomials.* Report, Laboratoire de Physique Mathématique (1993), FUNDP Namur.

[29] A. Ronveaux, A. Zarzo and E. Godoy: *Recurrence relations for connection coefficients between two families of orthogonal polynomials.* J. COMPUT. APPL. MATH. **62** (1995), 67–73.

[30] A. Ronveaux, M.N. Hounkonnou and S. Belmehdi: *Generalized linearization problems.* J. PHYS. A: MATH. GEN. **28** (1995), 4423–4430.

[31] A. Ronveaux, S. Belmehdi, E. Godoy and A. Zarzo: *Recurrence relations approach for connection coefficients. Applications to classical discrete orthogonal polynomials*, CRM. PROCEEDINGS AND LECTURE NOTES SERIES. **Vol.9**, AMS, Providence, RI, 1996, 321–337.

[32] A. Ronveaux, E. Godoy and A. Zarzo: *First associated and co-recursive of classical discrete orthogonal polynomials. Difference equations and connection problems.* (1996), (submitted).

[33] A. Ronveaux, A. Zarzo, I. Area and E. Godoy: *Bernstein bases and Hahn-Eberlein orthogonal polynomials.* (1996), (submitted).

[34] P. Sablonnière: *Discrete Bernstein bases and Hahn polynomials.* J. COMPUT. APPL. MATH. **49** (1993), 233–241.

[35] J. Stirling: METHODUS DIFFERENTIALIS: SIVE, TRACTATUS DE SUMMATIONE ET INTERPOLATIONE SERIERUM INFINITARUM. London, 1730.

[36] R. Szwarc: *Connection coefficients of orthogonal polynomials.* CANAD. MATH. BULL. **35** (1970), 100–102.

[37] R. SZWARC: *Linearization and connection coefficients of orthogonal polynomials.* MONAT. MATH. **113** (1992), 319–329.

[38] R. J. Yáñez, J.S. Dehesa and A.F. Nikiforov: *The three term recurrence relations and differential formulas for hypergeometric-type functions.* J. MATH. ANN. APPL. **188** (1994), 855–866.

[39] A. Zarzo, I. Area, E. Godoy and A. Ronveaux: *Results on some inversion problems for classical continuous and discrete orthogonal polynomials.* J. PHYS. A: MATH. GEN. (1997) (In print).

A. RONVEAUX
FACULTÉS UNIV. N.D. DE LA PAIX, NAMUR, BELGIUM
E-mail: Andre.Ronveaux@fundp.ac.be