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A CLASS OF JACOBI POLYNOMIALS ORTHOGONAL WITH RESPECT TO VARYING WEIGHTS.

M.X. He, P. Natalini, and P.E. Ricci

Abstract

We introduce a wide class of Jacobi Polynomials orthogonal with respect to varying weights, which are connected with the Relativistic Hermite polynomials introduced by V. ALDAYA - J. BISQUERT - J. NAVARRO-SALAS [1] for representing the wave functions of the quantum relativistic harmonic oscillator.

1 Introduction

In recent papers the Relativistic Hermite Polynomials (RHP) have been introduced by V. ALDAYA - J. BISQUERT - J. NAVARRO-SALAS [1], and studied by J.S. DEHESA, A. ZARZO, A. MARTINEZ, B. NAGEL [18], [19], [20], [21], [8].

These polynomials arise in the theory of the quantum relativistic harmonic oscillator, for representing the wave functions of multidimensional Schrödinger operators.

Some new sets of orthogonal polynomials, related to the RHP, and consequently called Relativistic polynomials, have been introduced and studied by the present authors, and by S. NOSCHESE, K. PAN.

These are: the Relativistic Laguerre Polynomials (RLP), the Relativistic Szegö Polynomials (RSP), the Relativistic Jacobi Polynomials (RJP), the Relativistic Bessel Polynomials (RBP).

Generally speaking all these polynomials satisfy the following properties:

they are connected with the RHP;

they depend on a parameter N (which is related with the speed of light, in case of the RHP) and approach to the classical sets of orthogonal polynomials when N tends to infinity;

they are solutions of hypergeometric type equations;

they are orthogonal with respect to varying weights in suitable supports;

the varying weights satisfy a Pearson type differential equation connecting all of them.

It is worth to note that the RJP, RLP, RBP have been called "relativistic" only in accordance with the first of the above mentioned properties, since no connection with the wave functions of some quantum relativistic harmonic oscillator has been proved until now.

We have studied

- Algebraic and differential properties, integral representations, generating functions, orthogonality [4], [9], [10], [5], [13], [16], [3], [2].

- Distribution of zeros (numerical approximation of the moments of the density of zeros) [17], [12], [14].

- Asymptotic distribution of zeros [11], [6], [7].

2 Recalling the Relativistic Hermite Polynomials

In the above mentioned paper V. ALDAYA - J. BISQUERT - J. NAVARRO-SALAS [1] introduced a set of orthogonal polynomials in order to represent the wave function of the quantum relativistic harmonic oscillator, i.e.

$$\tilde{\psi} = \tilde{\psi}_n(t, x) = e^{-in\omega t}\phi_n^N(x) = e^{-in\omega t}\alpha^{-(N+n)}H_n^N(\xi(x)) ,$$

where $N := \frac{mc^2}{\hbar\omega}$, c denotes the speed of light, $\hbar$ the Planck constant, m, ω, denote physical parameters (mass and frequency),

$$\alpha := \alpha(x) := \sqrt{1 + \frac{\omega^2}{c^2}\, x^2} , \qquad \xi := \xi(x) := \sqrt{\frac{m\omega}{\hbar}}\, x = \frac{\omega}{c}\sqrt{N}\, x ,$$

$$\phi_n^N(x) = \alpha^{-(N+n)}H_n^N(\xi(x)) , \qquad (\forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) ,$$

$H_n^N(\xi(x))$ are called Relativistic Hermite Polynomials, (RHP) and constitute a family of polynomials satisfying the orthogonality property (with respect to a varying weight function):

$$(\phi_n^N, \phi_m^N) = \int_{-\infty}^{+\infty} \alpha^{-(2N+n+m)} H_n^N\, H_m^N\, \mathrm{d}x = 2^n n! \frac{(2N+n-1)!}{(2N-1)!} \, \frac{1}{(2N)^n} \, \frac{(2N-3)!!}{(2N-2)!!}\sqrt{N}\, \pi\, \delta_{n,m} ,$$

(δ = Kronecker delta).

Further properties of the RHP have been studied in [18] - [19] - [20] - [8].

3 New sets of Polynomials orthogonal with respect to varying weights

We have studied in the papers mentioned in section 1 some new sets of Polynomials orthogonal with respect to varying weights and connected with the RHP. The main properties of these polynomials, which will be denoted here by the symbol $G_n^{(N)}(x)$, can be easily summarized as follows.

They satisfy a differential equation (of hypergeometric type):

$$\sigma(x)y_n'' + \tau(n, x)y_n' + \lambda_n y_n = 0 , \tag{3.1}$$

where σ is a polynomial of degree ≤ 2, τ is a polynomial of degree ≤ 1, λ_n is a constant, depending on n, satisfying

$$\lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma''$$

(see the classical book of A.F. NIKIFOROV - V.B. UVAROV [15]).

By changing $\sigma(x)$ in $z(1-z)$, a canonical hypergeometric form can be obtained for each $G_n^{(N)}(x)$, (in particular the B. NAGEL representation formula for the RHP can be found in [8]):

$$G_n^{(N)}(x) = k_n\, {}_2F_1(a_n, b_{n,N}, c_{n,N}; z) , \quad (z = g(x)) ,$$

with a suitable choice of the constants a_n, $b_{n,N}$, $c_{n,N}$, depending on the class of polynomials under consideration.

The symmetrization factor $\mu_n^{(N)}(x)$ by means of which eq. (3.1) can be rewritten in the self-adjoint form:

$$\left(\mu_n^{(N)}(x)\sigma(x)y_n'\right)' - \mu_n^{(N)}(x)\lambda_n y_n = 0 , \tag{3.2}$$

is the (varying) weight with respect to which the polynomials are orthogonal.

The following Pearson-type equation connecting the weights holds true:

$$\left(\mu_n^{(N)}(x)\sigma(x)\right)' = \tau(n,x)\mu_n^{(N)}(x) \ . \tag{3.3}$$

A Rodrigues formula can always be obtained

$$G_n^{(N)}(x) = \frac{C_n}{\mu_n^{(N)}(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\psi(x)(\mu_n^{(N)}(x))^n\right] \ , \tag{3.4}$$

where C_n is a normalizing constant.

The orthogonality property reads

$$\int_I G_n^{(N)}(x)G_m^{(N)}(x)\mu_{n,m}^{(N)}(x)\,\mathrm{d}x = k_n\delta_{n,m} \ , \tag{3.5}$$

where I is a suitable support, and the varying weight $\mu_{n,m}^{(N)}(x)$ can be written

$$\mu_{n,m}^{(N)}(x) = \varphi_n(x)\varphi_m(x)\mu^{(N)}(x) \ , \tag{3.6}$$

with a suitable choice of functions φ_n $(n \in \mathbb{N}_0)$, so that the so called complete orthogonality becomes:

$$\int_I \left[\varphi_n(x)G_n^{(N)}(x)\right]\ \left[\varphi_m(x)G_m^{(N)}(x)\right]\mu^{(N)}(x)\,\mathrm{d}x = k_n\delta_{n,m} \ . \tag{3.7}$$

A three-term recurrence relation can always be written in the canonical form:

$$A_{n,N}G_{n+1}^{(N)}(x) = [B_{n,N} - x]\,G_n^{(N)}(x) + C_{n,N}G_{n-1}^{(N)}(x) \ , \tag{3.8}$$

where

$$G_{-1}^{(N)}(x) := 0 \ ; \ G_0^{(N)}(x) := 1 \ .$$

4 A class of Jacobi Polynomials orthogonal with respect to varying weights

In [5] a set of RJP $\{P_n^{(\alpha,\beta;N)}(x)\}_{n\in\mathbb{N}_0}$ has been considered, by assuming the following definition properties:

$$\lim_{N\to\infty} P_n^{(\alpha,\beta;N)}(x) = P_n^{(\alpha,\beta)}(x) \ , \qquad (\forall\, n \in \mathbb{N},\ \forall\, \beta > -1) \tag{4.1}$$

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta;N)}\left(1 - \frac{2x}{\beta}\right) = L_n^{(\alpha,N)}(x) \ , \qquad (\forall\, n \in \mathbb{N},\ \forall\, N > 1/2) \tag{4.2}$$

where $L_n^{(\alpha,N)}$ denote the corresponding RLP, defined in [9].

Consider the expression of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{1}{k!}\binom{n+\alpha}{n-k}(\alpha+\beta+n+1)_k\left(\frac{x-1}{2}\right)^k \ , \tag{4.3}$$

which can be easily obtained by the classical expression:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n}(\alpha+k+1)_{n-k}(\alpha+\beta+n+1)_k\binom{n}{k}\left(\frac{x-1}{2}\right)^k \ ,$$

taking into account

$$\frac{1}{k!}\binom{n+\alpha}{n-k} = \frac{1}{n!}(\alpha+k+1)_{n-k}\binom{n}{k} .$$

By putting $\frac{1-2x}{\beta}$ instead of x in formula (4.3), and letting $\beta \to \infty$, the classical limit property connecting the Jacobi to the Laguerre polynomials follows:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{(\alpha)}(x) , \qquad (\forall n \in \mathbb{N}) .$$

Conditions (4.1)-(4.2) are satisfied by considering the explicit expressions of the RLP (see [9])

$$L_n^{(\alpha,N)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-1)^k \left(-n-N+\frac{1}{2}\right)_k}{N^k} \frac{(-x)^k}{k!} , \tag{4.4}$$

and putting, by definition:

$$P_n^{(\alpha,\beta;N)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \left(\frac{N-\beta}{N}\right)^k \left[\frac{AN-B\beta}{N-\beta}\right]_k \frac{1}{k!}\left(\frac{x-1}{2}\right)^k , \tag{4.5}$$

where the parameters A, B must be chosen in such a way that conditions (4.1) and (4.2) are satisfied.

This can be done (see [9]) by imposing two limit conditions

$$\left[\frac{AN-B\beta}{N-\beta}\right]_k \overset{(N\to\infty)}{\longrightarrow} [\alpha+\beta+n+1]_k , \tag{4.6}$$

$$\left[\frac{AN-B\beta}{N-\beta}\right]_k \overset{(\beta\to\infty)}{\longrightarrow} \left[-n-N+\frac{1}{2}\right]_k . \tag{4.7}$$

To this aim, we put

$$A = \alpha + r\beta + n + 1 ; \qquad B = -n + \frac{1}{2} + sN ,$$

$$r - s = 1 + o(1) \quad (\text{for } \{N\to\infty\} \text{ or } \{\beta\to\infty\})$$

$$(\text{e.g. } o(1) = \rho_n(N,\alpha,\beta) = \rho(N,\beta) = \frac{p(n,\alpha)}{P(N,\beta)} ,$$

$$p, P \text{ polynomials and } \deg_N P(N,\beta) \geq 1 \text{ and } \deg_\beta P(N,\beta) \geq 1) .$$

Then it is easily shown that conditions (4.6)-(4.7) are satisfied.

Consequently we can introduce a general set of RJP generalizing the set introduced by M. HE - P. NATALINI, which corresponds to the simplest choice: $\rho(N,\beta) \equiv 0$, by letting:

$$P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \left(\frac{n-\beta}{N}\right)^k \frac{1}{k!} \times \tag{4.8}$$

$$\times \left[\frac{[\alpha(1+\rho(N,\beta))\beta+n+1]N + \left(n-\frac{1}{2}\right)\beta}{N-\beta}\right]_k \left(\frac{x-1}{2}\right)^k .$$

Then, the limit conditions (4.1)-(4.2) hold true, since

$$\lim_{N\to\infty} P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x) = P_n^{(\alpha,\beta)}(x) , \qquad (\forall n \in \mathbb{N},\ \forall \beta > -1) \tag{4.9}$$

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta;N)}\left(\rho(N,\beta);1-\frac{2x}{\beta}\right) = L_n^{(\alpha,N)}(x)\ ,\ (\forall n\in\mathbb{N},\ \forall N>1/2) \tag{4.10}$$

By recalling the combinatorial identity

$$\binom{n+\alpha}{n-k} = (-1)^k \frac{(\alpha+1)_n(-n)_k}{n!(\alpha+1)_k}$$

we have the following

Proposition I *The explicit representation for the RJP $P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)$ holds true:*

$$P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x) :=$$

$$= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{1}{k!}\,\frac{(-n)_k\left[\frac{[\alpha+(1+\rho(N,\beta))\beta+n+1]N+\left(n-\frac{1}{2}\right)\beta}{N-\beta}\right]_k}{(\alpha+1)_k}\left(\frac{(\beta-N)(x-1)}{2N}\right)^k =$$

$$= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[-n,\frac{[\alpha+(1+\rho(N,\beta))\beta+n+1]N+\left(n-\frac{1}{2}\right)\beta}{N-\beta},\alpha+1;\frac{(\beta-N)(x-1)}{2N}\right]$$

Proposition II *The RJP $P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)$ are solutions of the hypergeometric differential equation:*

$$\sigma(x,N)\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} + \tau(x;n;N)\frac{\mathrm{d}y_n}{\mathrm{d}x} + \lambda_n(N)y_n = 0\ , \tag{4.11}$$

where

$$\sigma(x;N) = \left[\frac{N+\beta}{N-\beta} - \frac{2\beta}{N-\beta}x - x^2\right]\ ,$$

$$\tau(x;n;N) = \frac{[(1+\rho(N,\beta))\beta-\alpha]N+\left(2n-\frac{3}{2}\right)\beta}{N-\beta} - \frac{[\alpha+(1+\rho(N,\beta))\beta+2]N+\left(2n-\frac{3}{2}\right)\beta}{N-\beta}\,x\ ,$$

$$\lambda_n(N) = \frac{n}{N-\beta}\left[[\alpha+(1+\rho(N,\beta))\beta+n+1]N+\left(n-\frac{1}{2}\right)\beta\right]\ .$$

Proposition III *The symmetrization factor for eq.* (4.11) *is given by*

$$\mu_n(x;N;\rho(N,\beta)) := (1-x)^{\alpha}\left(x+\frac{N+\beta}{N-\beta}\right)^{\frac{\beta[(1+\rho(N,\beta))N+2n+\frac{1}{2}+\alpha]}{N-\beta}}$$

so that, if $N>\beta$, the support of the orthogonality measure is

$$I := \left[-\frac{N+\beta}{N-\beta},1\right]\ . \tag{4.12}$$

Proposition IV *The RJP $P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)$ can be defined by the Rodrigues formula:*

$$P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x) :=$$

$$= \frac{(-1)^n}{2^n n!}\left(\frac{N-\beta}{N}\right)^n (1-x)^{-\alpha}\left(x+\frac{N+\beta}{N-\beta}\right)^{-\frac{\beta[(1+\rho(N,\beta))N+2n+\frac{1}{2}+\alpha]}{N-\beta}} \times$$

$$\times\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{(1-x)^{\alpha+n}\left(x+\frac{N+\beta}{N-\beta}\right)^{\frac{\beta[(1+\rho(N,\beta))N+2n+\frac{1}{2}+\alpha]}{N-\beta}+n}\right\}$$

Proposition V *The RJP $P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)$ verify the three-term recurrence relation:*

$$A_{n,N}P_{n+1}^{(\alpha,\beta;N)}(\rho(N,\beta);x) = \tag{4.13}$$

$$= [B_{n,N}-x]P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x) + C_{n,N}P_{n-1}^{(\alpha,\beta;N)}(\rho(N,\beta);x) \ ,$$

$$\left(P_{-1}^{(\alpha,\beta;N)}(\rho(N,\beta);x) := 0 \ ; \quad P_0^{(\alpha,\beta;N)}(\psi(\alpha);\rho(N,\beta);x) := 1\right) \ ,$$

where:

$$A_{n,N} = (n+1)\frac{2N}{N-\beta}\frac{\left[\frac{[\alpha+(1+\rho(N,\beta))\beta+n+1]N+\left(n-\frac{1}{2}\right)\beta}{N-\beta}\right]_n}{\left[\frac{[\alpha+(1+\rho(N,\beta))\beta+n+2]N+\left(n+\frac{1}{2}\right)\beta}{N-\beta}\right]_{n+1}}$$

$$B_{n,N} \quad = 1 + \frac{2nN(\alpha+n)}{N[\alpha+(1+\rho(N,\beta))\beta+2n]+\frac{1}{2}\beta}$$

$$+\frac{2N(n+1)(\alpha+n+1)}{N[\alpha+(1+\rho(N,\beta))\beta+2n+2]+\frac{1}{2}\beta}$$

$$C_{n,N} = (n+\alpha)\left(\frac{2N}{N-\beta}\right)^{\frac{N+\beta}{N-\beta}} \frac{N[\alpha+(1+\rho(N,\beta))\beta+2n-1]-\frac{1}{2}\beta}{N[\alpha+(1+\rho(N,\beta))\beta+2n+1]-\frac{1}{2}\beta}\times$$

$$\times\frac{\left[\frac{[(1+\rho(N,\beta))\beta+n]N+\left(\alpha+n-\frac{3}{2}\right)\beta}{N-\beta}\right]_{\alpha+n-1}}{\left[\frac{[(1+\rho(N,\beta))\beta+n+1]N+\left(\alpha+n-\frac{1}{2}\right)\beta}{N-\beta}\right]_{\alpha+n}}$$

Proposition VI *The RJP $P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)$ satisfy the varying orthogonality property:*

$$\int_I P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)P_m^{(\alpha,\beta;N)}(\rho(N,\beta);x)\mu_{n,m}^{(N)}(x)\,\mathrm{d}x = k_n\delta_{n,m} \ ,$$

where I is defined by (4.12)*, k_n denotes the square of the weighted L^2 norm of the n-th polynomial $P_n^{(\alpha,\beta;N)}(\rho(N,\beta);x)$, which is obviously depending on $\rho(N,\beta)$,*

$$\mu_{n,m}^{(N)}(x) = (1-x)^\alpha\left(x+\frac{N+\beta}{N-\beta}\right)^{\frac{\beta[(1+\rho(N,\beta))N+2n+2m+\frac{1}{2}+\alpha]}{N-\beta}} =$$

$$= \varphi_{n,N}(x)\varphi_{m,N}(x)(1-x)^\alpha\left(x+\frac{N+\beta}{N-\beta}\right)^{\frac{\beta[(1+\rho(N,\beta))N+\frac{1}{2}+\alpha]}{N-\beta}} \ ,$$

so that, by putting

$$\varphi_{n,N}(x) := \left(x+\frac{N+\beta}{N-\beta}\right)^{\frac{2n\beta}{N-\beta}} \ ,$$

$$\mu^{(\alpha,\beta;N)}(\rho(N,\beta);x) := (1-x)^\alpha\left(x+\frac{N+\beta}{N-\beta}\right)^{\frac{\beta[(1+\rho(N,\beta))N+\frac{1}{2}+\alpha]}{N-\beta}} \ ,$$

the complete orthogonality condition can be written in the form (3.7)*:*

$$\int_I \left[\varphi_{n,N}P_n^{(\alpha,\beta;N)}\right]\ \left[\varphi_{m,N}P_m^{(\alpha,\beta;N)}\right]\mu^{(\alpha,\beta;N)}\,\mathrm{d}x = k_n\delta_{n,m} \ .$$

Conclusion

The extremely wide possibility to construct Relativistic Jacobi Polynomials, simply by choosing in the preceding section function $\rho(N,\beta)$ satisfying the prescribed limit conditions, seems to give relevant importance to analyze the physical meaning of such sets of polynomials orthogonal with respect to varying weights.

Note added in proofs

In a recent paper Prof. Mourad E.H. ISMAIL (J. Phys. A: Math. Gen. **29** (1996), 3199-3202) expressed a severe criticism about relativistic orthogonal polynomials, since they can be obtained from the classical Jacobi polynomials by means of suitable changes of variable. We want to remind that in every paper on this subject we have pointed out the existence of a Nagel type formula, expressing relativistic polynomials in terms of the Gauss hypergeometric function. However, in our opinion, the physical meaning of the relativistic Hermite polynomials and the possibility to construct families of polynomials (depending on N) related to each classical weight function deserve a particular attention. It was not probably clear, *a priori*, how to modify the classical Jacobi polynomials, by suitable changes of variable, in order to obtain the considered polynomial families.

References

[1] V. Aldaya, J. Bisquert, and J. Navarro-Salas: *The quantum relativistic harmonic oscillator: The relativistic Hermite polynomials*, PHYS. LETT. A, **156** (1991), 381-385.

[2] C. Belingeri and P.E. Ricci: *General Sets of Relativistic Jacobi Polynomials*, (submitted).

[3] B. Germano, S. Noschese, and P.E. Ricci: *General Sets of Relativistic Laguerre Polynomials*, (submitted).

[4] M. He, S. Noschese, and P.E. Ricci: *The Relativistic Szegö Polynomials*, INTEGRAL TRANSF. AND SPEC. FUNCT. (in print).

[5] M. He and P. Natalini: *The Relativistic Jacobi Polynomials*, INTEGRAL TRANSF. AND SPEC. FUNCT. (in print).

[6] M. He, K. Pan, and P.E. Ricci: *Asymptotics of the zeros of Relativistic Hermite Polynomials*, SIAM J. MATH. ANAL. (in print).

[7] M. He and K. Pan: *Distribution of the Zeros of Relativistic Jacobi Polynomials*, (submitted).

[8] B. Nagel: *The relativistic Hermite polynomial is a Gegenbauer polynomial*, J. MATH. PHYS., **35** (1994), 1549-1554.

[9] P. Natalini: *The Relativistic Laguerre Polynomials*, REND. MAT. APPL., S. VII, **16** (1996), 299-313.

[10] P. Natalini and S. Noschese: *Some properties of the Relativistic Laguerre Polynomials*, ATTI SEM. MAT. FIS. UNIV. MODENA (in print).

[11] P. Natalini and S. Noschese: *Asymptotic for the RHP Zeros*, INTEGRAL TRANSF. AND SPEC. FUNCT. (in print).

[12] P. Natalini and S. Noschese: *On the Moments of the Density of Zeros for the Relativistic Jacobi Polynomials*, REND. IST. MAT. UNIV. TRIESTE (in print).

[13] P. Natalini and S. Noschese: *The Relativistic Bessel Polynomials*, D.S. MITRINOVIC MEMORIAL VOLUME (in print).

[14] P. Natalini, S. Noschese and P.E. Ricci: *On the moments of the density of zeros for the Relativistic Bessel Polynomials*, J. APPLIED MATH. AND INFORMATICS (in print).

[15] A.F. Nikiforov and V.B. Uvarov: **Special Functions of Mathematical Physics**, Birkhäuser, Basel, 1988.

[16] S. Noschese: *A Vyskov-type formula for the Relativistic Laguerre Polynomials*, J. APPLIED MATH. AND INFORMATICS (in print).

[17] P.E. Ricci and S. Noschese: *On the moments of the density of zeros for the Relativistic Hermite and Laguerre polynomials*, COMPUTERS MATH. APPLIC. (in print).

[18] A. Zarzo and A. Martinez: *The quantum relativistic harmonic oscillator: Spectrum of zeros of its wave functions*, J. MATH. PHYS., **34** (1993), 2926-2935.

[19] A. Zarzo and A. Martinez: *Varying orthogonality for a class of hypergeometric type polynomials*, II INTERNATIONAL SEMINAR ON APPROXIMATION AND OPTIMIZATION, La Habana, Cuba, Sept. 1993. Approximation and Optimization series, Peter Lang Verlag, Berlin (1994) (in print).

[20] A. Zarzo, J.S. Dehesa, and A. Martinez: *On a new set of polynomials representing the wave functions of the quantum relativistic harmonic oscillator*, J. PHYS. A: MATH. GEN. (in print).

[21] A. Zarzo: *Ecuaciones Diferenciales de Tipo Hipergeométrico*, TESIS DOCTORAL, Universidad de Granada, Spain, 1995.

PIERPAOLO NATALINI - PAOLO EMILIO RICCI
DIPARTIMENTO DI MATEMATICA "GUIDO CASTELNUOVO"
UNIVERSITÀ DEGLI STUDI DI ROMA "LA SAPIENZA"
P.LE A. MORO, 2
00185 – ROMA
E-mail: RICCIP@SCI.UNIROMA1.IT
FAX #: (396) - 44701007

MATTHEW XISHENG HE
NOVA SOUTHEASTERN UNIVERSITY - FARQUHAR CENTER
3301 COLLEGE AVENUE
FORT LAUDERDALE - FLORIDA 33314
(U.S.A.)
E-mail: HEM@POLARIS.NCS.NOVA.EDU